Hausdorff Convergence of Julia Sets

Bernd Krauskopf   Hartje Kriete

Abstract

Consider a sequence $\{g_d\}_{d\in\mathbb{N}}$ converging uniformly on compact sets to g, where g and g_d are meromorphic functions on $\mathbb{C}$. We show that the Julia sets $J(g_d)$ converge to the Julia set $J(g)$ in the Hausdorff metric, if the Fatou set $F(g)$ is the union of basins of attracting periodic orbits and $\infty \in J(g)$. This result is discussed for families of finite type depending on a parameter, which is illustrated with the polynomials $\lambda(1+\frac{z}{d})^d$ converging to λe^z.

1 Introduction

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere and let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function. (All functions in this paper are of this type.) Recall that the *Fatou set* $F(f)$ is the set of $z \in \mathbb{C}$, such that the family of iterates $\{f^{\circ n}\}$ is defined and normal in a neighborhood of z. The *Julia set* $J(f)$ is the complement of $F(f)$ in $\widehat{\mathbb{C}}$. In other words, the Julia set $J(f)$ is the set of points near which f does not behave in a converging fashion. It is a non-empty, perfect and invariant set, in which the repelling periodic points of f are dense; see for example [Ber93, Bla84].

We are concerned with a sequence of functions $\{g_d\}_{d\in\mathbb{N}}$ converging uniformly on compact sets of $\mathbb{C}$ to a function g, and are interested in the dynamical consequences of this convergence. In particular, we study when the Julia sets $J(g_d)$ converge to the Julia set $J(g)$ as sets in the Hausdorff metric as $d \to \infty$.

Theorem 1.
If $F(g)$ is the union of basins of attracting periodic orbits and $\infty \in J(g)$, then $\{J(g_d)\}_{d\in\mathbb{N}}$ converges to $J(g)$ in the Hausdorff metric. The union may be empty, in which case $J(g) = \widehat{\mathbb{C}}$.

Communicated by M. Van Den Bergh.
1991 *Mathematics Subject Classification :* 58Fxx 30D05 54H20.
Key words and phrases : Julia set, Hausdorff convergence, exponential family.

Remark 2. The condition $\infty \in J(g)$ is necessary to avoid pathological cases, like the approximation of a polynomial by transcendental functions; see [KK96, KK98]. For the most interesting case that g is transcendental, it is automatically fulfilled. If the functions g and g_d are rationals of the same degree, it can be dropped; compare [Kri89].

In this paper we sketch the proof of Theorem 1; compare [Dou94, Kra93, Kri89, Kri95]. We discuss its application to functions of finite type, which gives a continuity result for hyperbolic functions. Finally the case of uniform convergence of families is discussed and illustrated with the example from [DGH86] of $P_{d,\lambda} := \lambda(1 + \frac{z}{d})^d$ converging to $E_\lambda := \lambda e^z$; see Figures 1 and 2 and compare [Kra93, KK97b] for information on how the figures were computed.

This paper was accepted for publication in the proceedings of the International Conference on Bifurcations in Differentiable Dynamics, held in Diepenbeek in June 1992. It is an updated version of a revised manuscript from February 1995. Related material has appeared in [Kis95, KK96, KK97a, KK97b, KK98].

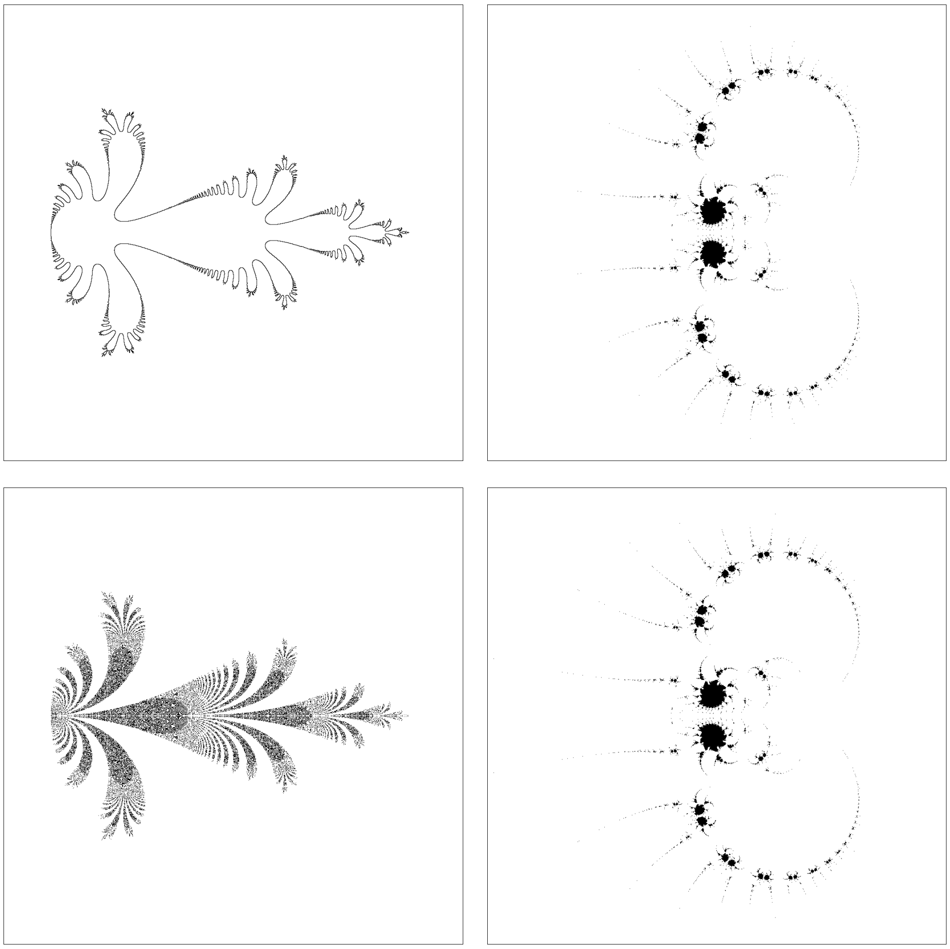

Figure 1: Convergence of $J(P_{d,\lambda})$ to $J(E_\lambda)$ in a chart near infinity. Top left: the quasi-circle $J(P_{65536,0.2})$, bottom left: the Cantor set of curves $J(E_{0.2})$, top right: the Cantor set $J(P_{65536,0.4})$, bottom right: slowly repelling periodic points of $J(E_{0.4}) = \mathbb{C}$.

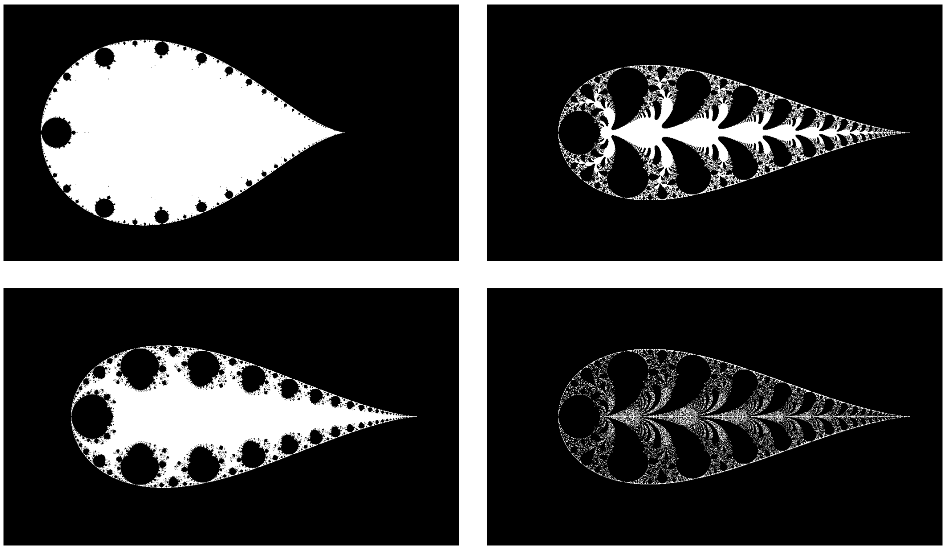

Figure 2: Convergence of hyperbolic components (dark domains) in a chart near infinity. Top left: $P_{2,\lambda}$, bottom left: $P_{16,\lambda}$, top right: $P_{65536,\lambda}$, and bottom right: E_λ.

2 Convergence of Julia Sets

On the Riemann sphere $\widehat{\mathbb{C}}$ with the chordal metric, the Hausdorff distance h between two closed sets $A, B \subset \widehat{\mathbb{C}}$ is defined in the usual way as

$$h(A,B) := \inf\{\varepsilon > 0 | A \subset V_\varepsilon(B) \text{ and } B \subset V_\varepsilon(A)\},$$

where $V_\varepsilon(X)$ is the ε-neighbourhood of the closed set $X \subset \widehat{\mathbb{C}}$. For the proof of Theorem 1 we need the following.

Lemma 3.
Let $O(z_0) \subset \mathbb{C}$ be an attracting (resp. repelling) k-periodic orbit of g, that is, $g^{\circ k}(z_0) = z_0$ and $|(g^{\circ k})'(z_0)| < 1$ (resp. $|(g^{\circ k})'(z_0)| > 1$).

(a) Then there is an $N \in \mathbb{N}$, such that g_d has an attracting (resp. repelling) k-periodic orbit $O(z_d)$ for all $d > N$. Moreover, $O(z_d)$ converges to $O(z_0)$ in the Hausdorff metric.

(b) Let $A(z_0)$ denote the basin of $O(z_0)$. Then for every compact $K \subset A(z_0)$ there is an $N \in \mathbb{N}$, such that $K \subset A(z_d)$ for all $d > N$, where $A(z_d)$ is the basin of $O(z_d)$ converging to $O(z_0)$.

Sketch of Proof: It is well known that a k-periodic orbit and, if it is attracting, its basin persists under small C^1-perturbations, see for example [HS74].

(a) In this complex analytic setting a proof can be given by a Rouché-type argument comparing the zeros of $(g_d^{\circ k}(z) - z)$ and $(g^{\circ k}(z) - z)$.

(b) According to (a) there is an orbit $O(z_d)$ converging to $O(z_o)$. One first shows the result for K from a small neighborhood of z_0, and then uses a kind of pull-back argument to prove it for general $K \subset A(z_0)$. Details can be found in [Kra93, Kri89, Kri95].

Proof of Theorem 1:
Since $\infty \in J(g)$, the Fatou set $F(g)$ consists of the basins $A(z_i)$ of attracting periodic orbits $O(z_i) \subset \mathbb{C}$. Fix $\varepsilon > 0$ and consider the set $F^\varepsilon(g) := \widehat{\mathbb{C}} \setminus V_\varepsilon(J(g))$, which is compact in $\widehat{\mathbb{C}}$. The set $F^\varepsilon(g)$ intersects only finitely many $A(z_i)$, and by Lemma 3(b), we get an $N \in \mathbb{N}$ such that $F^\varepsilon(g) \subset F(g_d)$ for all $d > N$. Hence, $J(g_d) \subset V_\varepsilon(J(g))$.
To show that $J(g) \subset V_\varepsilon(J(g_d))$ for sufficiently large d, let $\widehat{z} \in J(g)$. Since repelling periodic points are dense in $J(g)$, we take $z_0 \in U_\varepsilon(\widehat{z})$ to be a repelling periodic point of g, say, of period k. By Lemma 3(a), g_d has a repelling k-periodic orbit $O(z_d)$ for sufficiently large d, converging to $O(z_0)$, that is, $z_d \in U_\varepsilon(\widehat{z})$. Since $O(z_d)$ lies in $J(g_d)$ the arbitrarily chosen point $\widehat{z}$ lies in $V_\varepsilon(J(g_d))$. The compactness of $J(g)$ gives the result by choosing a finite subcovering. ∎

3 Functions of Finite Type

Let f be a meromorphic function and consider its set of *singular values* $\Sigma(f)$, which is the set of points in $\mathbb{C}$, for which there is no neighborhood where all local inverses of f are defined. $\Sigma(f)$ consists of critical values and is finite if f is rational, it consists of critical and asymptotic values and may be infinite if f is transcendental. In the sequel we study meromorphic functions of *finite type*, that is, for which $\Sigma(f) < \infty$. This class of functions is important because Sullivan's classification of the components of the Fatou set holds just as for rational functions; compare [Ber93, EL90, GK86]. ($F(f)$ cannot have wandering or Baker domains.) This means that $F(f)$ is the union of basins of attracting or parabolic periodic orbits, or of cycles of Siegel disks and Herman rings and all of their pre-images; see [Bla84]. It is known that this union may be empty if f is rational or transcendental, in which case $J(f) = \widehat{\mathbb{C}}$; compare [Ber93].

A meromorphic function f of finite type is called *hyperbolic* if the forward orbit $O^+(\Sigma(f))$ of the set of singular values is relatively compact in $F(f)$. Since there are no wandering or Baker domains, all singular values are attracted to attracting periodic orbits. It is well-known that there has to be a singular value in the basin of each attracting periodic orbit. Consequently, the Fatou set $F(f)$ of a hyperbolic function f is the union of finitely many basins of attracting periodic orbits.

Corollary 4.

(a) If g is of finite type and hyperbolic and $\infty \in J(g)$, then $J(g_d)$ converges to $J(g)$ in the Hausdorff metric.

(b) If g is of finite type and $J(g) = \widehat{\mathbb{C}}$, then $J(g_d)$ converges to $J(g)$ in the Hausdorff metric.

This corollary is a continuity result. The Julia set $J(g)$ changes continuously under small perturbations if g is hyperbolic. Again, when g and g_d are rationals of the same degree, the condition $\infty \in J(g)$ can be dropped. It is interesting that in the case $J(g) = \widehat{\mathbb{C}}$, which could be described as being quite non-hyperbolic, one also has continuity. We remark that the right concept for continuity of Julia sets is not hyperbolicity (or the more general concept of subhyperbolicity not used here) but the notion of *repellor*, as is discussed in [Kri95].

4 Families of Functions

We specialize further and consider a sequence of families $\{g_{d,\lambda}\}_{d\in\mathbb{N}}$ converging to a family g_λ, depending on some parameter $\lambda \in \mathbb{C}^l$, and assume that $|\Sigma(g_\lambda)| = |\Sigma(g_{d,\lambda})| = r \in \mathbb{N}$, independently of d and λ. Corollary 4 can be applied for values of λ from the sets

$$\mathcal{H}(g_\lambda) := \{\lambda \in \mathbb{C}^l \,|\, g_\lambda \text{ is hyperbolic}\} \quad \text{and} \quad \mathcal{M}(g_\lambda) := \{\lambda \in \mathbb{C}^l \,|\, J(g_\lambda) = \widehat{\mathbb{C}}\}$$

in the parameter space $\mathbb{C}^l$ of (f_λ). The set $\mathcal{H}(g_\lambda)$ is shown to be open in [Kri95], a connected component of it is called a *hyperbolic component*. From Lemma 3 we get the following (with the continuity of the sets $\Sigma(g_\lambda)$ and $\Sigma(g_{d,\lambda})$; see [Kri95]).

Corollary 5.
If $\lambda \in \mathcal{H}(g_\lambda)$, then there is an $N \in \mathbb{N}$ such that $g_{d,\lambda}$ is hyperbolic for all $d > N$.

Remark 6. This is one direction of the proof of the pointwise convergence of hyperbolic components. It is shown in [KK97a] for the case $\lambda \in \mathbb{C}$, that they converge as kernels in the sense of Carathéodory, which is a stronger notion. In general, hyperbolic components cannot be expected to converge in the Hausdorff metric as the Julia sets do.

Example:
We use $P_{d,\lambda}(z) := \lambda(1 + \frac{z}{d})^d$ converging to $E_\lambda(z) := \lambda e^z$ to illustrate the concepts; compare [BDHRGH97, Dev91, DGH86, Kra93]. (See [KK97b, KK98] for other examples.) The family of entire transcendental functions E_λ is approximated by the polynomials $P_{d,\lambda}$, where we have $\Sigma(P_{d,\lambda}) = \Sigma(E_\lambda) = \{0\}$, independently of d and λ. (Note that we do not consider ∞ a singular value of $P_{d,\lambda}$.) For any polynomial, ∞ is an attracting fixed point and its basin is non-empty and lies in the Fatou set. Thus, $J(P_{d,\lambda})$ can never be the whole sphere $\widehat{\mathbb{C}}$. On the other hand, $J(E_1) = J(e^z) = \widehat{\mathbb{C}}$ as was shown in [Mis81], proving a conjecture by Fatou. Despite the principal differences, there is a remarkable convergence of the dynamics of $P_{d,\lambda}$ to that of E_λ, which was first studied in [DGH86].

In order to see for which λ the Julia sets converge, we need to locate the sets $\mathcal{H}(E_\lambda)$ and $\mathcal{M}(E_\lambda)$ in the parameter plane $\mathbb{C}$ of E_λ. The set $\mathcal{M}(E_\lambda)$ is a Cantor set of curves and can be described by symbolic dynamics; see [BDHRGH97, Dev91, DGH86]. Since there is only the singular value 0, one can subdivide the set $\mathcal{H}(E_\lambda)$ into the sets $C_k(E_\lambda) := \{\lambda \in \mathbb{C} \,|\, E_\lambda \text{ has an attracting } k\text{-periodic orbit}\}$, which are open and mutually disjoint; see Figure 2 (bottom right). Moreover, the following is known; see [BR84].

- $C_1(E_\lambda)$ is the cardioid-shaped region $\{\xi e^{-\xi} \;:\; |\; \xi \;|< 1\}$. A component of $C_n(E_\lambda)$ is tangent to $C_1(E_\lambda)$ at $\omega e^{-\omega}$ for every n-th root of unity ω.
- $C_k(E_\lambda)$ is unbounded for $k \geq 2$ and has infinitly many components for $k \geq 3$.

- On any hyperbolic component the unique attracting periodic orbit depends analytically on λ.

It is conjectured that $\mathcal{H}(E_\lambda) = \bigcup C_k(E_\lambda)$ is dense in $\mathbb{C}$. For $\lambda \in \mathcal{H}(E_\lambda)$ the Julia set $J(E_\lambda)$ is a Cantor set of curves; see [BDHRGH97, Dev91]. According to Corollary 4, the Julia sets $J(P_{d,\lambda})$ converge to $J(E_\lambda)$ for $\lambda \in \mathcal{H}(E_\lambda) \cup \mathcal{M}(E_\lambda)$ as is illustrated in Figure 1.

In order to illustrate the convergence of hyperbolic components consider the parameter planes of the polynomials $P_{d,\lambda}$. In analogy to the $C_k(E_\lambda)$ we define $C_k(P_{d,\lambda}) := \{\lambda \in \mathbb{C} \mid P_{d,\lambda}$ has a finite attracting k-periodic orbit$\}$. With $B_d := \{\lambda \in \mathbb{C} \mid J(g_d)$ is connected$\}$ it can easily be seen that $\bigcup C_k(P_{d,\lambda}) \subset B_d$. Note that B_2 is the well-known Mandelbrot set, up to an affine change of coordinates, and that the $C_k(P_{2,\lambda})$ consist of hyperbolic components in it. It is a conjecture that $\bigcup C_k(P_{d,\lambda})$ is dense in B_d.

According to [KK97a] each connected components of $\mathcal{H}(E_\lambda)$ is the kernel of connected components of $\mathcal{H}(P_{d,\lambda})$. Furthermore, the sets $C_k(P_{d,\lambda})$ converge to $C_k(E_\lambda)$ in the Hausdorff metric as $d \to \infty$; see [KK97b]. This convergence is illustrated in Figure 2, and it gives a connection between the Mandelbrot set and the parameter plane of E_λ. An open question for future research is the following stronger statement.

Conjecture 7.
For fixed $k \in \mathbb{N}$, connected components of $C_k(P_{d,\lambda})$ converge to connected components of $C_k(E_\lambda)$ in the Hausdorff metric as $d \to \infty$.

Acknowledgements

The authors thank J.M. Aarts, W. Bergweiler, H.W. Broer, R.L. Devaney, W. Fischer, F. von Haeseler, G. Jank, H.Th. Jongen, W. Oberschelp, S.J. Patterson and F. Takens for fruitful discussions and encouragement. B.K. acknowledges an Erasmus grant from the EC and financial support of the Fields Institute, Canada. H.K. was supported by a post-doc grant from Deutsche Forschungsgemeinschaft and acknowledges a visiting grant from SFB Geometrie und Analysis, Universität Göttingen.

References

[Ber93] W. Bergweiler, "Iteration of meromorphic functions", *Bulletin AMS* **29**(2)(1993), 151–188.

[BR84] I.N. Baker and P.J. Rippon, "Iteration of Exponential Functions", *Ann. Acad. Sci. Fen., Series A. I. Mathematica* **9**(1984), 49–77.

[Bla84] P. Blanchard, "Complex analytic dynamics on the Riemann sphere", *Bull. AMS* **11**(1)(1984), 85–141.

[BDHRGH97] C. Bodelón, R.L. Devaney, M. Hayes, G. Roberts, L.R. Goldberg and J.H. Hubbard, "Hairs for the complex exponential family", preprint Boston University 1997. (`http://math.bu.edu/people/bob/papers.html`)

[Dev91] R.L. Devaney, "e^z: dynamics and bifurcations", *Int. J. of Bifurcation and Chaos*, **1**(2)(1991), 287–308.

[DGH86] R.L. Devaney, L.R. Goldberg and J.H. Hubbard, "A dynamical approximation to the exponential map by polynomials", Berkley 1986, preprint.

[Dou94] A. Douady, "Does the Julia set depend continuously on the polynomial?" in R.L. Devaney (Ed.), *Proceedings of Symposia in Applied Mathematics*, **49**(1994), pp. 91–138.

[EL90] A. Eremenko and M.Yu. Lyubich, "The dynamics of analytic transformations", *Leningrad Math. Journal* **1**(1990), 563–634.

[GK86] L.R. Goldberg and L. Keen, "A finiteness theorem for a dynamical class of entire functions", *Erg. Th. & Dynam. Syst.* **6**(1986), 183–192.

[HS74] M.W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra* (Academic Press, New York, 1974).

[Kis95] M. Kisaka. Local uniform convergence and convergence of Julia sets. *Nonlinearity* **8**(1995), 273–281.

[Kra93] Krauskopf, B. "Convergence of Julia set in the Approximation of λe^z by $\lambda(1+\frac{z}{d})^d$", *Int. J. of Bifurcation and Chaos* **3**(2)(1993), 257–270.

[KK96] B. Krauskopf and H. Kriete, "A note on non–converging Julia sets", *Nonlinearity* **9**(1996), 601–603.

[KK97a] B. Krauskopf and H. Kriete, "Kernel convergence of hyperbolic components", *Erg. Th. & Dynam. Syst.* **17**(5)(1997), 1137–1146.

[KK97b] B. Krauskopf and H. Kriete, "Hausdorff convergence and the limit shape of the unicorn", *Experimental Mathematics* **6**(2)(1997), 117–135.

[KK98] B. Krauskopf and H. Kriete, "Approximating transcendental Julia sets", in H. Kriete (Ed.), *Progress in Holomorphic Dynamics*, Pitman Research Notes in Mathematics Series **387**, Longman (1998), pp. 116-138.

[Kri89] H. Kriete, "Über die Stabilität von Julia Mengen", Doctoral Thesis, Göttingen 1989.

[Kri95] H. Kriete, "Repellors and the stability of Julia sets", *Mathematica Gottingensis* **4**(1995).

[Mis81] M. Misiurewicz, "On iterates of e^z", *Erg. Th. & Dynam. Syst.* **1**(1981), 103–106.

Bernd Krauskopf
Vakgroep Wiskunde
Rijksuniversiteit Groningen
9700 AV Groningen
The Netherlands
e-mail : berndk@nat.vu.nl
Current address: Natuur- en Sterrenkunde,
Vrije Universiteit, 1081 HV Amsterdam, The Netherlands

Hartje Kriete
Mathematisches Institut Georg–August–Universität
37073 Göttingen
Germany
e-mail : kriete@uni-math.gwdg.de