A cancellation theorem for 2-cones

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Abstract

The cone-length in spheres of a space X, $\mathrm{cl}_S(X)$, is the least integer n such that there are homotopy cofibrations:

$$\vee_{r\in R} S^{n_r} \longrightarrow X_i \longrightarrow X_{i+1} \qquad 0 \leq i < n$$

with $X_0 \sim *$ and $X_n \sim X$. We first prove a cancellation phenomenom for this notion. Let p be a prime and X be a 1-connected p-local space. If $X \vee S^n$ is the cofibre of a map between two wedges of spheres, then X is such a cofibre as well. In particular $\mathrm{cl}_S(X \vee S^n) \leq 2$ is equivalent to $\mathrm{cl}_S(X) \leq 2$. From this property, we deduce two extensions of the result of Félix and Thomas on spaces of Lusternik-Schnirelmann category two, first, for p-local spaces whose loop space has a decomposition and second, for rational spaces of infinite type.

In this paper spaces and maps are always based. The Lusternik-Schnirelmann category of a space X, $\mathrm{cat}X$, is the least integer $n \in \mathbf{N} \cup \{\infty\}$ with the property that X can be covered by $n+1$ open subsets contractible in X. This homotopy invariant is hard to determine and several approximations have been introduced. Among them, the strong category, $\mathrm{Cat}X$ of a space X [7]: $\mathrm{Cat}X$ is the least integer $n \in \mathbf{N} \cup \{\infty\}$ with the property that X has the homotopy type of a CW-complex which may be covered by $n+1$ self-contractible subcomplexes. Strong category is an upper bound for LS category; indeed, if X is path-connected, one has [15]:

$$\mathrm{cat}X \leq \mathrm{Cat}X \leq \mathrm{cat}X + 1\,.$$

We use Ganea and Cornea's characterizations of strong category. For a path connected space, Ganea proved [7] that $\mathrm{Cat}X$ is the least integer n such that there are n cofibrations:

$$L_i \to X_i \to X_{i+1} \qquad 0 \leq i < n\,,$$

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with $X_0 = *$ and $X_n \sim X$. In [3] Cornea improves this construction: He shows that one obtains the same invariant by requiring the spaces L_i to be i-suspensions, $L_i = \Sigma^i Z_i, \quad i \geq 1$. In particular, for the spaces we are concerned with, one has $\mathrm{Cat}X \leq 2$ iff there exists a cofibration $\Sigma Z_1 \to \Sigma Z_0 \to Y$, with $Y \sim X$. If we require that the spaces L_i are wedges of spheres, we get a homotopy invariant called the cone-length in spheres which is denoted by cl_S. One has $\mathrm{Cat}X \leq \mathrm{cl}_S X$, and there are examples where the inequality is strict. First we prove a cancellation phenomenon for the cone-length in p-local spheres:

Theorem 1: *Let p be a prime and X be a p-local, 1-connected, space of finite type. We denote by S^n the p-local sphere. If $X \vee S^n$ is the cofibre of a map between two wedges of p-local spheres, then X is such a cofibre as well. In particular, $cl_S(X \vee S^n) \leq 2$ is equivalent to $cl_S X \leq 2$.*

It is an open question how to characterize spaces X for which one has $\mathrm{cat}X = \mathrm{Cat}X$. For instance, the spaces of LS-category one are the co-H spaces and the spaces of strong category one are the suspensions. The existence of co-H-spaces that are not suspensions implies thus that there are spaces X with $\mathrm{cat}X = 1$ and $\mathrm{Cat}X = 2$.

Rational 1-connected co-H-spaces are wedges of rational spheres, [8], [13]. Therefore, the previous class of examples is not valid in the rational setting. In [10], Lemaire and Sigrist conjectured that for rational 1-connected spaces, X_0, one has $\mathrm{cat}X_0 = \mathrm{Cat}X_0$. Recently, Dupont [5] found a rational space with $\mathrm{cat}X_0 = 3$ and $\mathrm{Cat}X_0 = 4$. For rational spaces of LS-category 2, there is no counterexample to the Lemaire-Sigrist conjecture, since in [6] Félix and Thomas proved that if $\mathrm{cat}X_0 = 2$ and $H^k(X_0; \mathbf{Q})$ is finite dimensional for all k then $\mathrm{Cat}X_0 = 2$. From Theorem 1, we deduce some extensions of this result of Félix and Thomas.

If p is a prime, we denote by $X_{(p)}$ the p-localisation of X (see [9]). As usual, set:

$$\Omega^k = \begin{cases} S^k & k = 2n-1 \\ \Omega S^{k+1} & k = 2n \end{cases}$$

Definition: Let p be a prime. A space Y is *weakly p-decomposable* if there exists an integer m (possibly ∞) such that Y has the homotopy type of an m-dimensional CW-complex and $\Omega Y \sim_{(p)} \prod_{i \in J} \Omega^{n_i} \times E$, with E m-connected.

Note that a decomposable space in the sense of [1], [14] is weakly p-decomposable for any prime p.

Theorem 2: *Let p be a prime. If X is a weakly p-decomposable 1-connected space of finite type such that $catX_{(p)} \leq 2$, then $CatX_{(p)} \leq cl_S X_{(p)} \leq 2$.*

In [12] McGibbon and Wilkerson proved that if X is a 1-connected finite CW-complex, with $\Sigma_r \mathrm{rank}\ \pi_r(X) < \infty$, then $\Omega X \sim_p \prod_{i \in J} \Omega^{n_i}$ for almost all primes p, that is, X is weakly p-decomposable for p large enough. The following corollary comes directly from their result and Theorem 2:

Corollary : *Let X be a* 1*-connected finite CW-complex, with* $catX \leq 2$ *and such that $\Sigma_r rank\ \pi_r(X) < \infty$, then for almost all primes p, one has $CatX_{(p)} \leq cl_S X_{(p)} \leq 2$.*

In the rational case, we may prove more:

Theorem 3: *Let X be a 1-connected rational space.*

1) If $X \vee \bigvee_{i\in L} S^{n_i}$ is the cofibre of a map between wedges of rational spheres, then X is as well.

2) If $catX \leq 2$ then $CatX \leq 2$.

Remark: Theorems 1 and 2 give obstructions to the decomposability of a space X in the sense of [1], [14]. For that, remark that the existence of cohomology operations $\mathcal{P}^{i_1}$, $\mathcal{P}^{i_2}$, $\mathcal{P}^{i_3}$ such that $\mathcal{P}^{i_1} \circ \mathcal{P}^{i_2} \circ \mathcal{P}^{i_3}(\alpha) \neq 0$, $\alpha \in H^*(X)$, implies $3 \leq cl_S(X)$. Therefore if $\mathrm{cat}X \leq 2$, the existence of such cohomology operations implies the non-decomposability of X. For instance, one can deduce easily that $\Sigma\mathbf{C}P^n$ $(n \geq 3)$ is not decomposable.

One of the techniques used in the proofs is the notion of homotopy pullback and pushout. We refer to [11], [2] and [4] for the definitions and main properties. By definition, a *cofibration sequence* is a couple of pointed maps $A \xrightarrow{f} B \xrightarrow{g} C$ such that the following diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & g\downarrow \\ * & \longrightarrow & C \end{array}$$

is a homotopy pushout. A *fibration sequence* is defined similarly.

In order to prove these theorems, we need some lemmas.

Lemma 1: [11]*Consider the following homotopy commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \gamma\downarrow & & \beta\downarrow \\ X & \xrightarrow{\delta} & Y \end{array}$$

Denote by C_α, C_β, C_γ and C_δ the homotopy cofibres of α, β, γ and δ, respectively, and by $\tilde{\beta} : C_\alpha \longrightarrow C_\delta$ and $\tilde{\delta} : C_\gamma \longrightarrow C_\beta$ the induced maps. We get a homotopy commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & \longrightarrow & C_\alpha \\ \gamma\downarrow & & \beta\downarrow & & \tilde{\beta}\downarrow \\ X & \xrightarrow{\delta} & Y & \longrightarrow & C_\delta \\ \downarrow & & \downarrow & & \downarrow \\ C_\gamma & \xrightarrow{\tilde{\delta}} & C_\beta & \longrightarrow & C \end{array}$$

in which C is the common homotopy cofibre of $\tilde{\delta}$ and of $\tilde{\beta}$.

The following cofibration sequence is obtained by applying Proposition 2.1. of [2] to the second Ganea fibration $*^2\Omega X \longrightarrow B_2\Omega X \longrightarrow X$.

Lemma 2: *If X has LS-category 2, one has a cofibration sequence:*

$$*^2\Omega X \vee (\wedge^2\Omega X \wedge \Sigma H) \xrightarrow{j} \Sigma\Omega X \vee (\Omega X \wedge \Sigma H) \xrightarrow{\rho} X \vee \Sigma H\,,$$

with $H = \Omega(*^3\Omega X)$.

Lemma 3: *Let p a prime number and let q be a number prime to p. Let S^n be a p-local sphere and denote by ι_n a generator of $H_n(S^n)$. Let $\psi : S^n \longrightarrow W \vee S^n$ be a continuous map such that $H_n(\psi)(\iota_n) = q\iota_n + \omega_n$, where $\omega_n \in H_n(W)$.*

Then the homotopy cofibre of ψ has the homotopy type of W.

Proof of Lemma 3:

Denote by $\Psi : W \vee S^n \longrightarrow W \vee S^n$ the sum of ψ and of the canonical inclusion $W \longrightarrow W \vee S^n$. The map $\Psi : W \vee S^n \longrightarrow W \vee S^n$ is an isomorphism in homology, and thus Ψ is a homotopy equivalence. Therefore the homotopy cofibres of the canonical inclusion $\iota : S^n \longrightarrow W \vee S^n$ and $\psi = \Psi \circ \iota$ have the same homotopy type.

■

Proof of Theorem 1:

In this proof all spaces are p-local. We start from a cofibration sequence

$$\bigvee_{i\in I} S^{n_i} \xrightarrow{j} \bigvee_{k\in K} S^{n_k} \xrightarrow{\rho} X \vee S^n \quad (**)$$

Let ι_n be a generator of $H_n(S^n)$. Remark that in the homology long exact sequence of $(**)$,

$$\dots H_n\left(\bigvee_{i\in I} S^{n_i}\right) \xrightarrow{H_n(j)} H_n\left(\bigvee_{k\in K} S^{n_k}\right) \xrightarrow{H_n(\rho)} H_n(X) \oplus H_n(S^n) \xrightarrow{\delta} H_n\left(\bigvee_{i\in I} S^{n_i+1}\right) \dots$$

there are only two possibilities:

Case 1: There exists $\alpha_n \in H_n\left(\bigvee_{k\in K} S^{n_k}\right)$ such that $H_n(\rho)(\alpha_n) = \iota_n$ and, by construction, the element α_n is not divisible by p.

Case 2: $\delta(\iota_n) \neq 0$. Thus one can write $\delta(\iota_n) = p^r\beta_n \in H_n(\bigvee_i S^{n_i+1})$, where β_n is not divisible by p.

Case 1: Let $\alpha : S^n \longrightarrow \bigvee_{k\in K} S^{n_k}$ be a map such that $H_n(\alpha)(\iota_n) = \alpha_n$. Since α_n is not divisible by p we have $\bigvee_{k\in K} S^{n_k} = S^n_{\alpha_n} \vee \bigvee_{k\in K'} S^{n_k}$. By Lemma 3, the homotopy cofibre of α is the wedge of spheres $\bigvee_{k\in K'} S^{n_k}$.

Consider now the following homotopy diagram, constructed in the same way as in Lemma 1,

$$\begin{array}{ccccc}
* & \longrightarrow & S^n & = & S^n \\
\downarrow & & \alpha\downarrow & & \downarrow\rho\circ\alpha \\
\bigvee_{i\in I} S^{n_i} & \xrightarrow{j} & \bigvee_{k\in K} S^{n_k} & \xrightarrow{\rho} & X \vee S^n \\
\| & & \downarrow & & \downarrow \\
\bigvee_{i\in I} S^{n_i} & \xrightarrow{j'} & \bigvee_{k\in K'} S^{n_k} & \xrightarrow{\rho'} & X'
\end{array}$$

Note that $H_n(\rho\circ\alpha)(\iota_n) = H_n(\rho)(\alpha_n) = \iota_n$. Using again Lemma 3, for $q = 1$ we obtain $X' \sim X$, so $\mathrm{cl}_S X \le 2$.

Case 2: $\delta(\iota_n) = p^r\beta_n \neq 0$, β_n not divisible by p.

2.1) If $r = 0$ we can decompose $\bigvee_{i\in I} S^{n_i+1} = S^n_{\beta_n} \vee \bigvee_{i\in I'} S^{n_i+1}$.

By Lemma 1, we may construct the following diagram:

$$\begin{array}{ccccccc}
\bigvee_{i\in I'} S^{n_i} & \longrightarrow & \bigvee_{k\in K} S^{n_k} & \longrightarrow & X' & & \\
\downarrow & & \| & & \downarrow\tilde\iota & & \\
\bigvee_{i\in I} S^{n_i} & \xrightarrow{j} & \bigvee_{k\in K} S^{n_k} & \xrightarrow{\rho} & X\vee S^n & \xrightarrow{\delta} & \bigvee_{i\in I} S^{n_i+1} \\
\downarrow & & \downarrow & & \downarrow\pi' & & \downarrow \\
S^{n-1}_{\beta_n} & \longrightarrow & * & \longrightarrow & S^n_{\beta_n} & \xrightarrow{\cong} & S^n_{\beta_n}
\end{array}$$

Let $\tilde\iota$ be the map induced between the cofibres, we will see that the composite $\pi\circ\tilde\iota$ is an homological isomorphism, where $\pi : X\vee S^n \longrightarrow X$ is the canonical projection. Consider the long exact sequence associated to the cofibration $X' \xrightarrow{\tilde\iota} X\vee S^n \xrightarrow{\pi'} S^n$

$$0 \to H_n(X') \xrightarrow{H_n(\tilde\iota)} H_n(X)\oplus H_n(S^n) \xrightarrow{H_n(\pi')} H_n(S^n) \longrightarrow H_{n-1}(X') \longrightarrow H_{n-1}(X) \to 0$$

Since $\delta(\iota_n) = \beta_n$, we deduce that $H_n(\pi')(\iota_n)$ is a generator of $H_n(S^n_{\beta_n})$, so $H_n(\pi')$ is surjective and $H_{n-1}(\pi\circ\tilde\iota)$ is an isomorphism between $H_{n-1}(X')$ and $H_{n-1}(X)$.

We can consider the following diagram:

$$\begin{array}{ccccc}
* & \longrightarrow & X' & \longrightarrow & X' \\
\downarrow & & \tilde\iota\downarrow & & \downarrow\pi\circ\tilde\iota \\
S^n & \xrightarrow{\iota} & X\vee S^n & \xrightarrow{\pi} & X \\
\downarrow & & \pi'\downarrow & & \downarrow \\
S^n & \xrightarrow{\pi'\circ\iota} & S^n & \longrightarrow & *
\end{array}$$

Since $\pi'\circ\iota$ is a homotopy equivalence so is $\pi\circ\tilde\iota$ by Lemma 1.

2.2) $\delta(\iota_n) = p^r\beta_n$, with $r > 0$. Looking at the long exact sequence we note that $H_n(j)(\beta_n) = 0$, because $H_n\left(\bigvee_{i\in I} S^{n_i+1}\right)$ is a free $\mathbf{Z}_{(p)}$-module. Therefore there exists $\upsilon_n \in H_n(X)\oplus H_n(S^n)$ such that $\delta(\upsilon_n) = \beta_n$. The element υ_n can be written $\upsilon_n = s\iota_n + \omega_n$, $\omega_n \in H_n(X)$ (note that υ_n is a torsion free element). We decompose $\iota_n - p^r\upsilon_n = (1-p^r s)\iota_n + p^r\omega_n$. Set $q = 1 - p^r s$; remark that $(p,q) = 1$. From $\delta(\iota_n - p^r\upsilon_n) = 0$, we deduce the existence of $\alpha_n \in H_n\left(\bigvee_{k\in K} S^{n_k}\right)$ such that $H_n(\rho)(\alpha_n) = q\iota_n - p^r\omega_n$.

Let $\beta : S^n \longrightarrow \bigvee_{k\in K} S^{n_k}$ be a map such that $H_n(\beta)(\iota_n) = \alpha_n$. Because $(p,q) = 1$, α_n is not divisible by p. As before, the cofibre of β is a wedge $\bigvee_{k\in K'} S^{n_k}$ and the following diagram

$$\begin{array}{ccccc}
* & \longrightarrow & S^n & = & S^n \\
\downarrow & & \beta\downarrow & & \downarrow\rho\circ\beta \\
\bigvee_{i\in I} S^{n_i} & \xrightarrow{j} & \bigvee_{k\in K} S^{n_k} & \xrightarrow{\rho} & X\vee S^n \\
\| & & \downarrow & & \downarrow \\
\bigvee_{i\in I} S^{n_i} & \xrightarrow{j'} & \bigvee_{k\in K'} S^{n_k} & \xrightarrow{\rho'} & X'
\end{array}$$

together with Lemma 3 implies that $X \sim X'$ and thus $\mathrm{cl}_S X \le 2$.

■

Proof of Theorem 2: Recall that all spaces are p-local. From Lemma 2, we get a cofibration sequence:

$$*^2\Omega X_{(p)} \vee (\wedge^2\Omega X_{(p)} \wedge \Sigma H) \xrightarrow{j} \Sigma\Omega X_{(p)} \vee (\Omega X_{(p)} \wedge \Sigma H) \xrightarrow{\rho} X_{(p)} \vee \Sigma H \quad (*)$$

with $H = \Omega(*^3\Omega X_{(p)})$. As X is weakly p-decomposable, we may suppose that X has the homotopy type of a m-dimensional CW-complex such that:

$$\Omega X \sim_{(p)} \prod_{i\in J} \Omega^{n_i} \times E$$

with E m-connected. By using the well-known formula $\Sigma(A \times B) = \Sigma A \vee \Sigma B \vee \Sigma(A \wedge B)$, we get :

$$\Sigma\Omega X \sim_{(p)} \Sigma(\prod_{i\in J} \Omega^{n_i} \times E) = \Sigma E \vee \left[\Sigma(E \wedge \prod_{i\in J} \Omega^{n_i})\right] \vee \Sigma(\prod_{i\in J} \Omega^{n_i}).$$

Denote by $E' = \Sigma E \vee [\Sigma(E \wedge \prod_{i\in J} \Omega^{n_i})]$, the space E' is of course $(m+1)$-connected. The formula $\Sigma\left(\prod_{i\in J} \Omega^{n_i}\right) = \bigvee_{i\in\Lambda} S^{n_i}$ implies

$$\Sigma\Omega X \sim_{(p)} \bigvee_{i\in\Lambda} S^{n_i} \vee E'.$$

Replacing $\Sigma\Omega X_{(p)}$ by this in the cofibration sequence $(*)$ we obtain :

$$\bigvee_{i\in I} S^{n_i} \vee E_1 \xrightarrow{j} \bigvee_{k\in K} S^{n_k} \vee E_2 \xrightarrow{\rho} X \vee \bigvee_{l\in L} S^{n_l} \vee E_3$$

where E_1, E_2 and E_3 are at least $(m+1)$-connected. By the cellular approximation theorem, we may assume that there exists a cofibration sequence:

$$\bigvee_{i\in I} S^{n_i} \xrightarrow{j} \bigvee_{k\in K} S^{n_k} \xrightarrow{\rho} X \vee \bigvee_{l\in L} S^{n_l}$$

(We keep the notation j, ρ for all restrictions of the previous maps.) Since all spaces are of finite type, Theorem 2 is a consequence of Theorem 1.

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Proof of Theorem 3:

The proof is analogous to those of Theorem 1 and 2. In the rational setting, we have a homotopy cofibration:

$$\bigvee_{i\in I} S^{n_i} \xrightarrow{j} \bigvee_{k\in K} S^{n_k} \xrightarrow{\rho} X_0 \vee \bigvee_{l\in L} S^{n_l} \quad (*)$$

The homology long exact sequence of $(*)$ is now an exact sequence of rational vector spaces :

$$\ldots \longrightarrow H_n(\bigvee_{i\in I} S^{n_i}) \xrightarrow{H_n(j)} H_n(\bigvee_{k\in K} S^{n_k}) \xrightarrow{H_n(\rho)} H_n(X_0) \oplus H_n(\bigvee_{l\in L} S^{n_l}) \longrightarrow \ldots \quad (**)$$

We can split $H_n(\bigvee_{l\in L} S^{n_l}) = H_n(\bigvee_{l\in L_1} S^{n_l}) \oplus H_n(\bigvee_{l\in L_2} S^{n_l})$ in such a way that $\bigvee_{l\in L_1} S^{n_l}$ comes from a subwedge of $\bigvee_{k\in K} S^{n_k}$ via ρ and $\bigvee_{l\in L_2} S^{n_l}$ goes to a subwedge of $\bigvee_{i\in I} S^{n_i+1}$ via the connecting homomorphism. Using the same diagrams as in case 1 and case 2.1 for Theorem 2, we obtain $\mathrm{Cat} X_0 \leq 2$.

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Remark: For the p-local case (Theorem 1) the cancellation phenomenon is applied to one sphere each time. In the rational setting (Theorem 3), since this exact sequence splits, we can cancel any family of spheres.

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