Finite partially $\{0, 2\}$-semiaffine linear spaces

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Abstract

In this paper we study finite partially $\{0, 2\}$-semiaffine linear spaces of order n. For the case of different point degrees a complete classification will be given. If the point degree is a constant then we present a classification for those admitting the property that $v \geq n^2$.

1 Introduction

Let us first recall some definitions and results. For more details see [1]. Let $(\mathcal{P}, \mathcal{L})$ be a pair, where $\mathcal{P}$ is a (finite) set of **points** and $\mathcal{L}$ is a family of proper subsets of $\mathcal{P}$. The elements of $\mathcal{L}$ are called **lines** and we assume that each line admits at least two points. The pair $(\mathcal{P}, \mathcal{L})$ is called a **(finite) linear space** if any two distinct points are on a unique line.
We denote by v the number of points and by b the number of lines.
For any point p of $\mathcal{P}$, the **degree** of p is the number $[p]$ of lines through p and for any line ℓ of $\mathcal{L}$, the **length** of ℓ is its cardinality. If $n+1 = \max_{p \in \mathcal{P}} [p]$, the integer n is called the **order** of $(\mathcal{P}, \mathcal{L})$.
Two lines ℓ and m are called **parallel** if $\ell = m$ or $\ell \cap m = \emptyset$.
For any point-line pair (p, ℓ) with $p \notin \ell$, $\pi(p, \ell)$ denotes the number of lines through p, which are parallel to ℓ.

The **near pencil** on v points is the finite linear space on v points with a line of length $v - 1$.

An (h, k)-cross, $h, k \geq 3$ is a linear space whose points are on two incident lines, one of length h and the other one of length k.

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A **projective plane** is a linear space such that any two distinct lines have a point in common and every line has length at least three.

The **Nwpanka-Shrikhande** plane is a linear space on 12 points and 19 lines with constant point degree 5, each point being on one line of length 4 and four lines of length 3.

Let H be a finite set of non-negative integers. The linear space $(\mathcal{P}, \mathcal{L})$ is **H-semiaffine** if $\pi(p, \ell) \in H$ for any point-line pair (p, ℓ) with $p \notin \ell$. The research on H-semiaffine linear spaces has been developed by Dembowski [8] for the case $H = \{0, 1\}$ and has been carried on by many other authors (see [4, 3, 11, 13, 16, 15]).

Furthermore $(\mathcal{P}, \mathcal{L})$ is called **weakly H-semiaffine** if the line set is partitioned into two sets $\mathcal{V}$ and $\mathcal{I}$, the **visible** and **invisible** lines respectively, such that

(a) *A line ℓ is* **visible** *if and only if* $\pi(p, \ell) \in H$ for any $p \notin \ell$.

(b) *There is a* **good** *point of degree $n+1$, that is a point of degree $n+1$ such that all the lines through it are visible.*

(c) *Every visible line has at least two good points.*

(d) *For any point-line pair (p, ℓ) with $p \notin \ell$ we have that $\pi(p, \ell) \geq \min H$.*

Weakly H-semiaffine linear spaces were introduced by A. Beutelspacher and I. Schestag in [5]. They have been studied later also by other authors (see [6, 10, 17]).

The second and third property of the previous definition make sure that the family of visible lines is greater than the family of invisible lines.

In this paper we study finite linear spaces such that the first previous property is verified whereas the other three axioms are replaced by a property assuring that there are "only a few" invisible lines with respect to visible ones.

We begin with the following definition.

Definition

Let H be a finite set of non negative integers. A finite linear space $(\mathcal{P}, \mathcal{L})$ is **2-partially H-semiaffine** if the line set is partitioned into two sets $\mathcal{V}$ and $\mathcal{I}$, the **visible** and **invisible** lines respectively, such that

(1.1) A line ℓ is **visible** if and only if $\pi(p, \ell) \in H$ for any $p \notin \ell$.

(1.2) Through any point there are at most two invisible lines.

From now on a 2-partially H-semiaffine linear space is called a partially H-semiaffine linear space. A point p of $\mathcal{P}$ is called **good** if every line through it is visible. Otherwise it is called a **non-good** point. For every point p of $\mathcal{P}$ the number of invisible lines through p is denoted by μ_p.

From property (1.2) it follows that there are two kinds of non-good points: those of **type I** if $\mu_p = 1$ and those of **type II** if $\mu_p = 2$.

If $\mathcal{V} = \emptyset$ then from property (1.2) it follows that for every point p we have $[p] = 2$, hence $(\mathcal{P}, \mathcal{L})$ is the near-pencil on three points.
From now on we may assume that $\mathcal{V} \neq \emptyset$.

We remark that if $\mathcal{I} = \emptyset$ then $(\mathcal{P}, \mathcal{L})$ is an H-semiaffine linear space.

A partially H-semiaffine linear space is said to be **proper** if for any $h \in H$ there is a point-line pair (p, ℓ) with $p \notin \ell$ such that $\pi(p, \ell) = h$.

In this paper finite partially $\{0,2\}$-semiaffine linear spaces with $\mathcal{I} \neq \emptyset$ are studied. Recall that finite partially $\{0,1\}$-semiaffine linear spaces have been studied in [12].

1.1 Examples of partially $\{0,2\}$-semiaffine linear spaces with $\mathcal{I} \neq \emptyset$ and different point degrees

A1) The $(3,3)$-cross is a partially $\{0\}$-semiaffine linear space with one good point (the intersection point of the two lines of length three) and the remaining points are type II non-good points.

A2) Let $(\mathcal{P}, \mathcal{L})$ be a projective plane of order $n \geq 4$ and let L and L' be two of its lines. We set $q_0 = L \cap L'$, $\Delta = L \cup L' - \{q_0\}$. The linear space $(\mathcal{P}', \mathcal{L}')$ obtained from $(\mathcal{P}, \mathcal{L})$ by deleting the points of Δ and an extra point q different from q_0 is a proper partially $\{0,2\}$-semiaffine linear space. Its points are non-good points of type I.

A3) Let $(\mathcal{P}, \mathcal{L})$ be a projective plane of order $n \geq 4$ and let L and L' be two of its lines. Set $q_0 = L \cap L'$ and $\Delta = L \cup L' - \{q_0\}$. Let $(\mathcal{P}', \mathcal{L}')$ be the linear space obtained from $(\mathcal{P},\mathcal{L})$ by deleting the points of Δ and two other points q and q' different from q_0, such that q, q' and q_0 are collinear. Then we have a proper partially $\{0,2\}$-semiaffine linear space with $n-1$ good points. The remaining points are type II non-good points.

A4) Let $(\mathcal{P}, \mathcal{L})$ be a projective plane of order $n \geq 5$ and let L and L' be two of its lines. Set $q_0 = L \cap L'$ and $\Delta = L \cup L' - \{q_0\}$. Consider the linear space $(\mathcal{P}', \mathcal{L}')$ obtained from $(\mathcal{P}, \mathcal{L})$ by deleting the points of Δ and two other points q and q' different from q_0 such that q, q' and q_0 are not collinear. Then we have a proper partially $\{0,2\}$-semiaffine linear space which contains $n-3$ type I non-good points and the remaining points are type II non-good points.

In this note we prove the following theorem.

Theorem I *Let $(\mathcal{P}, \mathcal{L})$ be a partially $\{0,2\}$-semiaffine linear space of order n, with non-constant point degree. Then $(\mathcal{P}, \mathcal{L})$ is either one of the linear spaces described in* A1, A2, A4 *or the pseudo-complement of the union of the set of points of two lines in a projective plane of order n $(n \geq 4)$, except for the point of intersection q_0, and of two points collinear with q_0.*

1.2 Examples of partially $\{0\}$-semiaffine linear spaces with $\mathcal{I} \neq \emptyset$

B1) A punctured projective plane of order n is a partially $\{0\}$-semiaffine linear space. All its points are type I non-good points.

B2) A doubly punctured projective plane of order n is a partially $\{0\}$-semiaffine linear space. If $n \geq 3$ then its points are non-good of both types.

If [12] the following theorem has been proved.

Theorem II *Let $(\mathcal{P}, \mathcal{L})$ be a partially $\{0\}$-semiaffine linear space of order n, with $\mathcal{I} \neq \emptyset$. Then $(\mathcal{P}, \mathcal{L})$ is one of the linear spaces described in* B1 *or* B2.

1.3 Examples of partially $\{2\}$-semiaffine linear spaces with $\mathcal{I} \neq \emptyset$

C1) Let $(\mathcal{P}, \mathcal{L})$ be a projective plane of order $n \geq 4$ and let L and L' be two of its lines and p a point not on $L \cup L'$. Put $\Delta = L \cup L' \cup \{p\}$. The linear space $(\mathcal{P}', \mathcal{L}')$ obtained from $(\mathcal{P}, \mathcal{L})$ by deleting the points of Δ is a partially $\{2\}$-semiaffine linear space with constant point degree $n + 1$. In this linear space there are no lines of length $n+1$ and every non-good point is of type II.

C2) The linear space obtained from a projective plane of square order n by deleting the points of two disjoined Baer subplanes is a partially $\{2\}$-semiaffine linear space whose points are all type II non-good points.

C3) The linear space obtained from a projective plane of square order n, by deleting the points of a Baer subplane and one of its tangent line is a partially $\{2\}$-semiaffine linear space whose points are non-good. There exist both type I and type II non-good points.

C4) The complement of two lines in a projective plane of order n ($n \geq 3$) is a partially $\{2\}$-semiaffine linear space such that all of its points are type I non-good points.

C5) The punctured Nwpanka-Shrikhande plane is a partially $\{2\}$-semiaffine linear space of order 4, with constant point degree 5. It has no lines of length 5. Furthermore, there are three good points and the remaining points are type II non-good ones.

C6) The linear space on six points with two parallel lines of length three is a partially $\{2\}$-semiaffine linear space. All its points are type I non-good points.

C7) The linear space obtained from the projective plane of order 4 by deleting the points of its three concurrent lines and the two pseudo-complements of a triangle in the projective plane of order 4 (see [4]) are three examples of partially $\{2\}$-semiaffine linear spaces. In these examples all points are type II non-good points.

C8) **(a)** Let α_4 be the affine plane of order 4, and let p be one of its points. The linear space obtained from α_4 by deleting p and by breaking up every line through p, in three lines of length two is a partially $\{2\}$-semiaffine linear space. Its points are all type II non-good points.

(b) Let π_5 be the projective plane of order 5 and let L_1, L_2 and L_3 be three lines passing through a fixed point p of π_5. Consider a line L not passing through p and let $(\mathcal{P},\mathcal{L})$ be the linear space obtained from π_5 by deleting the lines L_1, L_2 , L_3 together with their points except the points $p_i = L_i \cap L$, $i = 1,2,3$ and by deleting also the points of L different from p_i, $i = 1,2,3$. The linear space $(\mathcal{P}',\mathcal{L}')$ obtained from $(\mathcal{P},\mathcal{L})$ by breaking up the line $\{p_1,p_2,p_3\}$ in three lines of length 2, is a partially $\{2\}$-semiaffine linear space.

Notice that the linear spaces described in (a) and (b) have the same parameters but they are not isomorphic.

C9) The complement of a hyperoval in the projective plane of order 4 is a partially $\{2\}$-semiaffine linear space with constant point degree 5. All its points are type II non-good points.

In this note we prove the following theorem.

Theorem III *Let $(\mathcal{P},\mathcal{L})$ be a partially $\{2\}$-semiaffine linear space with constant point degree $n+1$, then we have the following properties.*

(a) If there exists a good point then $(\mathcal{P},\mathcal{L})$ is the linear space described in C1 or C5.

(b) If there are no good points and if there exists a line of length n, then $(\mathcal{P},\mathcal{L})$ is the linear space described in C4 or C6 or there exists an integer s ($s \geq 5$) such that:

(1) $(\mathcal{P},\mathcal{L})$ is the pseudo-complement of a Baer subplane together with a tangent line in a projective plane of order $n = (s-2)^2$.

(2) $b = n^2+n+1$, $n = \dfrac{s^2-3s+2}{2}$.

(3) $b = n^2+n+1+z$, with $0 < z < \dfrac{s^2-5s+2}{2}$ and $n \leq \dfrac{s^2-2s}{3}$.

(c) If there are no good points and all the invisible lines have length $n+1-s$, ($s \geq 3$) then $(\mathcal{P},\mathcal{L})$ is the linear space described in C7 or C9 or one of the following cases is true

(4) $(\mathcal{P},\mathcal{L})$ is a hypothetic linear space with $s = 4, n = 7$ and $b = n^2+n-1 = 55$.

(5) $b = n^2+n$, $n = 2s^2-9s+9$, and $(\mathcal{P},\mathcal{L})$ is a hypothetic linear space if $s \geq 5$. If $s = 4$, then $n = 5, b = 30, v = 15$ and two non-isomorphic examples are described in C8.

(6) $b = n^2 + n + 1$ *and* $(\mathcal{P}, \mathcal{L})$ *is the pseudo-complement of two disjoint Baer subplanes in a projective plane of order* $n = (s-2)^2$. *Moreover if* $s \geq 9$, *then* $(\mathcal{P}, \mathcal{L})$ *is embeddable in a projective plane of order* n.

(7) $b = n^2 + n + 1 + z$, *with* $0 < z < s^2 - 5s + 3$, $n \leq \dfrac{2(s^2 - 5s + 3)}{5} + s - 1$, *and* $s \geq 5$.

1.4 Some examples of proper partially $\{0, 2\}$-semiaffine linear spaces with $\mathcal{I} \neq \emptyset$ and constant point degree

D1) Let $(\mathcal{P}, \mathcal{L})$ be the linear space obtained from a projective plane of order n by deleting two points. $(\mathcal{P}, \mathcal{L})$ is a proper partially $\{0, 2\}$-semiaffine linear space with $n-1$ good points whereas every other point is a type II non-good point.

D2) Let $(\mathcal{P}, \mathcal{L})$ be a projective plane of order $n \geq 3$ and n odd, and let Γ be an $(n+1)$-arc. The linear space $(\mathcal{P}', \mathcal{L}')$ obtained from $(\mathcal{P}, \mathcal{L})$ by deleting the points of Γ is a proper partially $\{0, 2\}$-semiaffine linear space with constant point degree $n+1$. The internal points with respect to Γ are good points. The remaining points are type II non-good points.

In this note we prove the following statement.

Theorem IV *Let* $(\mathcal{P}, \mathcal{L})$ *be a* $\{0, 2\}$*-partially semiaffine linear space of order* n *with constant point degree. If* $v \geq n^2$ *then* $(\mathcal{P}, \mathcal{L})$ *is the complement of an* $(n+1)$*-arc in a projective plane of order* n *or it is a doubly punctured projective plane of order* n, *with* $n \geq 3$.

2 First properties of a finite partially $\{0, 2\}$-semiaffine linear space

From now on $(\mathcal{P}, \mathcal{L})$ is a partially $\{0, 2\}$-semiaffine linear space of order n, with $\mathcal{I} \neq \emptyset$.

Proposition 2.1. *If there exist two lines* ℓ *and* m *such that* $\mathcal{P} = \ell \cup m$ *then* $(\mathcal{P}, \mathcal{L})$ *is one of the linear spaces described in A1 and C6.*

Proof. We distinguish two cases: (i) $\ell \cap m = \emptyset$ and (ii) $\ell \cap m \neq \emptyset$.

Case (i). Since ℓ and m have both exactly one parallel line, they are invisible lines. From $\mathcal{V} \neq \emptyset$ it follows that there exists a visible line t of length 2, so $|\ell| = |m| = 3$ and $(\mathcal{P}, \mathcal{L})$ is the linear space on six points with two lines of length 3.

Case (ii). In this case ℓ and m are visible lines and their common point is a good point. If ℓ (or m) has length 2 then $(\mathcal{P}, \mathcal{L})$ is the near pencil on $|m| + 1$ ($|\ell| + 1$) points and this is not possible since $\mathcal{I} \neq \emptyset$. Hence both ℓ and m have at least three points. If $|\ell| = |m| = 3$ then $(\mathcal{P}, \mathcal{L})$ is the $(3, 3)$-cross. Hence we may suppose that ℓ (or m) has at least four points. We prove that

$|\ell| = 4$. Assume on the contrary that $|\ell| \geq 5$ and put $m = \{O, a_1, a_2, ...\}$ and $\ell = \{O, b_1, b_2, b_3, b_4, ...\}$. The line $t = a_1 b_1$ is invisible, because $\pi(a_2, t) \geq 3$. If L is a line passing through a_2 and parallel to t, then L is invisible, because $\pi(a_1, L) \geq 3$. Hence at least three invisible lines pass through a_2 and this is a contradiction! In a similar way, we may prove that $|m| = 4$. Thus $(\mathcal{P}, \mathcal{L})$ is the $(4,4)$-cross, and this is false, since $\mathcal{I} \neq \emptyset$. ∎

Hence from now on we may suppose that $(\mathcal{P}, \mathcal{L})$ is not the union of two lines and so that every point has degree at least three.

Proposition 2.2. *Let $(\mathcal{P}, \mathcal{L})$ be a proper partially $\{0,2\}$-semiaffine linear space of order n, then for each point p, $[p] \in \{n-1, n+1\}$ and for each visible line ℓ, $|\ell| \in \{n-3, n-1, n+1\}$.*

Proof. First we prove that for every point p there exists a visible line not passing through p. Assume on the contrary that there exists a point p such that all the visible lines pass through it. Then every point different from p is a type II non-good point of degree 3 and p is a good point. Hence the lines have length 2 or 3. Thus if L is a visible line and x is a point not on L, then at most one parallel line to L passes through x, contradicting the assumption that $(\mathcal{P}, \mathcal{L})$ is a proper partially $\{0,2\}$-semiaffine linear space.
Let p_0 be a point of degree $n+1$ and let ℓ be a visible line not passing through p_0. Then $|\ell| \in \{n-1, n+1\}$. Hence each point not on ℓ has degree $n-1$ or $n+1$ and each visible line not passing through p_0 has length $n-1$ or $n+1$. If L is a visible line passing through p_0 then $|L| \in \{n-3, n-1, n+1\}$, because every point not in $\ell \cup L$ has degree $n-1$ or $n+1$. Therefore the points of $(\mathcal{P}, \mathcal{L})$ have degree $n-3$, $n-1$ or $n+1$ and the visible lines have length $n-3$, $n-1$ or $n+1$.
Thus we have to show that there are no points of degree $n-3$. Assume on the contrary that there exists a point x of degree $n-3$. Then every visible line not passing through x has length $n-3$ and passes through p_0. Since $[x] = n-3 \geq 3$, it follows that $n \geq 6$, hence at least two visible lines of length $n-3$ pass through p_0. Then the points different from p_0 not lying on ℓ have degree $n-1$, so $|\ell| = n-1$. Now consider a line t parallel to ℓ and passing through p_0 and let $y \in t \setminus \{p_0\}$. Then $[y] = n+1$, a contradiction. Hence each point has degree $n-1$ or $n+1$, so the assertion follows. ∎

Proposition 2.3. *If ℓ is a visible line, then $|\ell| \in \{n-1, n+1\}$.*

Proof. From proposition 2.2 it follows that if ℓ is visible line then $|\ell| \in \{n-3, n-1, n+1\}$. We prove that there are no visible lines of length $n-3$. Assume on the contrary that there exists a visible line ℓ of length $n-3$. Then every $x \notin \ell$ has degree $n-1$ and if p is a point of degree $n+1$, then $p \in \ell$. Let p_0 be a point of degree $n+1$ and let p be a point of ℓ different from p_0. We prove that ℓ is the only visible line passing through p.
Indeed if L is a visible line through p different from ℓ, then $|L| = n-1$, because $p_0 \in \ell$. Let L' be a parallel line to L and let y be a point of $L' \setminus \{p_0\}$ then $[p] \geq n$, a contradiction. Therefore $[p] = n-1 = 3$, contradicting $|\ell| = n-3 \geq 2$. ∎

From propositions 2.2 and 2.3 follows

Proposition 2.4. *A line ℓ is visible if and only if $|\ell| \in \{n-1, n+1\}$.*

Proposition 2.5. *There exists at most one point of degree $n-1$.*

Proof Assume on the contrary that there are two different points q and q' of degree $n-1$. Let L be the line qq'. We prove that each point of L has degree $n-1$. In fact if there exists a point $x \in L$ of degree $n+1$, then counting v from x and q resp., we have

$$|L| + 2 + (n-2)^2 \leq v \leq |L| + (n-2)^2, \tag{1}$$

which is not possible. Let p_0 be a point of degree $n+1$, then $p_0 \notin L$. Counting v from p_0 and q respectively, we have

$$v \geq 3 + (n-1)(n-2), \tag{2}$$

$$v \leq |L| + (n-2)^2. \tag{3}$$

From equations (2) and (3) it follows that $|L| = n+1$ and $v = n+1+(n-2)^2 = n^2 - 3n + 5$.
Then all the points of L are good points. Thus each point of $(\mathcal{P}, \mathcal{L})$ is a good point. Hence there are no invisible lines, contradicting the assumption that $\mathcal{I} \neq \emptyset$. ■

3 Partially $\{0,2\}$-semiaffine linear spaces of order n with a good point of degree $n-1$

In the previous section we have seen that there exists at most one point q_0 of degree $n-1$. In this section we assume that q_0 is a good point.

Theorem 3.1. *Let $(\mathcal{P}, \mathcal{L})$ be a partially $\{0,2\}$-semiaffine linear space of order n with a good point of degree $n-1$. Then $(\mathcal{P}, \mathcal{L})$ is the pseudo-complement of the union of two lines L and L', except their common point q_0, and of two different points collinear with q_0 in a projective plane of order n.*

Proof. Let q_0 be the good point of degree $n-1$. We prove that there exists at least a line ℓ of length $n-1$ passing through q_0. Assume on the contrary that each line through q_0 is a line of length $n+1$, then $v = n^2 - n + 1$. Thus every line not through q_0 has length $n-1$, then all the lines are visible lines which is a contradiction.
Let L be a line of length $n-1$ through q_0. Denote by λ the number of lines of length $n-1$ through q_0, then $\lambda \geq 1$. Counting v from q_0 we have

$$v = 1 + \lambda(n-2) + (n-1-\lambda)n = n^2 - n + 1 - 2\lambda. \tag{4}$$

Computing v from a point of degree $n+1$ we have

$$v \geq 2 + 1 + (n-1)(n-2) = n^2 - 3n + 5. \tag{5}$$

From equations (4) and (5) it follows that $\lambda \leq n-2$. Therefore there exists a line of length $n+1$ passing through q_0. Each point of $L \setminus \{q_0\}$ is a non-good point, otherwise

$v = n^2 - n + 1$ and from equation (4) it follows that $\lambda = 0$, a contradiction.
Now let t be an invisible line through x and t' the possible other invisible line through x. Counting v from x we have

$$v = n + 1 + t + t' - 2 + (n-2)^2 = n^2 - 3n + 3 + t + t'. \tag{6}$$

Since the $n-1-\lambda$ lines of length $n+1$ through q_0 meet t and t' we have $|t|, |t'| \geq n-1-\lambda$, hence from equation (6) it follows that $v \geq n^2 - 3n + 3 + 2(n-1) - 2\lambda$. Then from equation (4) it follows that $|t| = |t'| = n - 1 - \lambda$.
Thus if L is a line through q_0 meeting t and t' then L has length $n+1$. It follows that if ℓ' is a line of length $n-1$ through q_0 and $y \in \ell' \setminus \{q_0\}$ then y is a good point. Computing v from y we have $v = n^2 - n - 1$. Thus from equation (4) it follows that $\lambda = 1$.
Hence through q_0 there passes a unique line ℓ of length $n-1$ and the remaining lines are all of length $n+1$. All the points of ℓ are good points and the invisible lines (which are parallel to ℓ) have length $n-2$, because they meet the $n-2$ lines of lenght $n+1$ passing through q_0.
Therefore the lines have length $n-2$, $n-1$ and $n+1$ and all the points not on ℓ are points of type II. If L is a line of length $n+1$ it meets all the lines, then $b = n^2 + n - 1$. From $v = n^2 - n - 1$ the assertion follows. ∎

4 Partially $\{0,2\}$-semiaffine linear spaces of order n with a non-good point of degree $n-1$

In this section we study the case when the point q_0 of degree $n-1$ is a non-good point. Denote by λ the number of lines of length $n-1$ passing through q_0.

We distinguish two cases

Case I *q_0 is a point of type* I.

Lemma 4.1. *There are no good points.*

Proof. Assume on the contrary that there exists a good point x. Then computing v from x and q_0, respectively, we obtain

$$|xq_0| + n^2 - 2n = v \leq |xq_0| + |t| - 1 + (n-3)n, \tag{7}$$

where t is the invisible line passing through q_0.
It follows that $|t| \geq n+1$, contradicting $|t| \leq n$ as t is an invisible line. ∎

Theorem 4.1. *Let $(\mathcal{P}, \mathcal{L})$ be a partially $\{0,2\}$-semiaffine linear space of order n, with a non-good point of degree $n-1$ of type* I. *Then $(\mathcal{P}, \mathcal{L})$ is the linear space obtained from a projective plane of order n by deleting two lines L and L' except their common point together with a point not on $L \cup L'$.*

Proof. Let t be the invisible line through q_0. Counting v from q_0, we have

$$v = |t| + \lambda(n-2) + (n-2-\lambda)n = n^2 - 2n + |t| - 2\lambda. \tag{8}$$

First we prove that all the points of $t \setminus \{q_0\}$ are of type I. Indeed if x is a point of $t \setminus \{q_0\}$ of type II, then counting v from x

$$v = |t| + |t'| - 1 + (n-1)(n-2) = n^2 - 3n + 1 + |t| + |t'|. \tag{9}$$

From equation (8) it follows that $|t'| = n - 1 - 2\lambda$.
Since the $n - 2 - \lambda$ lines of length $n + 1$ passing through q_0 meet t', then

$$n - 1 - \lambda \leq |t'| = n - 1 - 2\lambda, \tag{10}$$

from which it follows that $\lambda = 0$. Thus t' is a visible line, which is a contradiction.
Now let x be a point of $t \setminus \{q_0\}$, since x is of type I it follows that $v = |t| + n(n-2)$. Comparing with equation (8) we have $\lambda = 0$, so through q_0 there pass $n - 2$ lines of length $n + 1$ and the line t.
Let L be a line of length $n+1$ and let $y \in L \setminus \{q_0\}$. Counting v from q_0 and y resp., we have

$$|t| + n^2 - 2n = v = |s| + |s'| - 1 + n + (n-2)^2 = n^2 - 3n + 3 + |s| + |s'|. \tag{11}$$

Since s and s' have at least $n-2$ points (since they meet all the lines of length $n+1$ passing through q_0) it follows that

$$|t| = -n + 3 + |s| + |s'| \geq 2n - 4 - n + 3 = n - 1, \tag{12}$$

hence $|t| = n$, because t is an invisible line.
Therefore equation (12) becomes $2n - 3 = |s| + |s'|$, from which it follows that one of these lines has length $n - 1$ and the other one has length $n - 2$.
Then all the points are of type I, the lines have length $n - 2, n - 1, n$ and $n + 1$, $v = n^2 - n$ and $b = n^2 + n - 1$. Hence $(\mathcal{P}, \mathcal{L})$ is the pseudo-complement of two lines L and L' except their common point together with a point not on $L \cup L'$ in a projective plane of order n. Since $|t| = n$ and $q_0 \in t$ the number of lines parallel to t is $n + 1$ and they form a partition Π of $\mathcal{P}$. Adding a "new" point to the lines of Π we get a linear space $(\mathcal{P}_0, \mathcal{L}_0)$ whose lines have length $n - 1$ and $n + 1$ and the points different from q_0 have degree $n + 1$. From results in [16] it follows that $(\mathcal{P}_0, \mathcal{L}_0)$ is the complement of two lines L and L' except their common point, hence the assertion follows. ■

Case II *The point q_0 is of type* II.

Lemma 4.2. *There are no good points.*

Proof. Assume on the contrary that there exists a good point x. Then counting v from x and q_0 resp., we have

$$v = |xq_0| + n^2 - 2n \leq v \leq |xq_0| + |t| + |t'| - 2, \tag{13}$$

where t and t' are the two invisible lines through q_0. Then $|t| + |t'| \geq 2(n+1)$, which is a contradiction because t and t' are invisible lines, and so they have at most n points. ■

Theorem 4.2. *Let $(\mathcal{P},\mathcal{L})$ be a partially $\{0,2\}$-semiaffine linear space of order n, with a type* II *non-good point of degree $n-1$. Then $(\mathcal{P},\mathcal{L})$ is the linear space obtained from a projective plane of order n by deleting two lines L and L', except their common point q_0, together with two points that are collinear with q_0.*

Proof. Let q_0 be the point of degree $n-1$, t and t' the two invisible lines through q_0. As usual let λ be the number of lines of length $n-1$ passing through q_0. Counting v from q_0 we have:

$$v = |t| + |t'| - 1 + \lambda(n-2) + (n-3-\lambda)n = n^2 - 3n - 2\lambda + |t| + |t'| - 1. \quad (14)$$

Let p be a point of t. If p is of type I then $v = |t| + n(n-2)$ and comparing with equation (14) we have $|t| \geq n+1$, which is a contradiction because t is an invisible line.

Thus p is of type II. Let t'' be the second invisible line through p, counting v from p we have

$$v = |t| + |t''| - 1 + (n-1)(n-2) = n^2 - 3n + 1 + |t| + |t''|. \quad (15)$$

Comparing with equation (14) we have

$$|t'| - 2 - 2\lambda = |t''|. \quad (16)$$

Since the $n-3-\lambda$ lines of length $n+1$ pass through q_0, it follows that $|t''| \geq n-2-\lambda$. Hence from equation (16) it follows that $|t'|-2-2\lambda \geq n-2-\lambda$ that is $|t'| \geq n+\lambda$. Thus $|t'| = n$ and $\lambda = 0$. Interchanging t and t' we have also $|t| = n$. It follows that $v = n^2 - n - 1$ and $b = n^2 + n - 1$.

Let $x \notin t \cup t'$, if x is a non-good point of type I, denote by s the invisible line through x. Then $v = |s| + n + (n-1)(n-2)$. Comparing this expression with $v = n^2 - n - 1$ we have $|s| = n-3$ and so s is parallel to t and t'. If x is a non-good point of type II then let s and s' be the two invisible lines through x. Counting v from x we have

$$v = |s| + |s'| - 1 + n + (n-2)^2. \quad (17)$$

Comparing with $v = n^2 - n - 1$ it follows that $|s| + |s'| = 2(n-2)$. Thus $|s| = |s'| = n-2$.

Since $|t| = |t'| = n$ and $b = n^2 + n - 1$, there exists a unique line m parallel to t and t'. Since $q_0 \notin m$ it follows that $|m| \leq n-3$. Thus $|m| = n-3$ and the points of $(\mathcal{P},\mathcal{L})$ are all of type I. Moreover m is the unique line of length $n-3$, because every line different from m meets t and t'.

Let $\Pi = \{\ell : \ell = t \text{ or } \ell \| t\}$, then Π is a partition of $\mathcal{P}$ and $|\Pi| = n+1$. If $\ell \in \Pi \setminus \{t, m\}$, then $|\ell| = n-2$, because the lines of $\Pi \setminus \{t, m\}$ partition the set of points of Π not on $t \cup m$. Similarly the set $\Pi' = \{\ell : \ell = t' \text{ or } \ell \| t'\}$ is a set of $n+1$ lines partitioning $\mathcal{P}$ and every line of $\Pi' \setminus \{t', m\}$ has length $n-2$. Since there exists a line L of length $n+1$ and each line intersects L, it follows that $|\mathcal{I}| = 2n+1$, hence $\mathcal{I} = \Pi \cup \Pi'$. Consider the pair $(\mathcal{P}_0, \mathcal{L}_0)$, where

$$\mathcal{P}_0 = \mathcal{P} \cup \{\omega, \omega'\}, \quad \mathcal{L}_0 = \{\ell \in \mathcal{L} : \ell \notin \Pi \cup \Pi'\} \cup \{\ell \cup \omega : \ell \in \Pi\} \cup \{\ell \cup \omega' : \ell \in \Pi'\}.$$

This is a linear space with constant point degree $n+1$ and whose lines have length either $n-1$ or $n+1$. Then from the results in [16] it follows that $(\mathcal{P}_0, \mathcal{L}_0)$ is the complement of two lines except their common point, in a projective plane. Thus, the assertion follows. ∎

5 Partially $\{2\}$-semiaffine linear spaces

In this section we study partially $\{2\}$-semiaffine linear spaces of order n and we prove III of section 1.
Hence suppose that $(\mathcal{P}, \mathcal{L})$ is a partially $\{2\}$-semiaffine linear space of order n. If $\mathcal{P}$ is the union of two lines, then arguing as in proposition 2.1 one obtains that $(\mathcal{P}, \mathcal{L})$ is the linear space C6 or C4 with $n = 3$. Thus from now on we may suppose that each point x has degree at least 3.

Proposition 5.1. *If $(\mathcal{P}, \mathcal{L})$ is a partially $\{2\}$-semiaffine linear space of order n, then every point has degree $n+1$ and a line is visible if and only if it has length $n-1$.*

Proof. Let p_0 be a point of degree $n+1$. We prove that there exists a visible line not passing through p_0. Assume on the contrary that all visible lines pass through p_0. Since at most one visible line passes through a point different from p_0, it follows that every point different from p_0 has degree 3. Thus if L is a visible line and x is a point not on L then $|L| \leq 1$, a contradiction.
Let L be a visible line not passing through p_0, then $|L| = n-1$ so $n \geq 3$ and every point not lying on L has degree $n+1$. Let M be a visible line passing through p_0, if L is parallel to M then each point has degree $n+1$ and each visible line has length $n-1$. If L intersects M in a point y, then each point different from y has degree $n+1$ and each visible line has degree $n-1$. Moreover, since there are at least two visible lines through p_0 it follows that there exists a visible line not passing through y, hence $[y] = n+1$. ∎

Proposition 5.2. *Let $(\mathcal{P}, \mathcal{L})$ be a partially $\{2\}$-semiaffine linear space of order n with a good point, then $n \geq 4$ and $(\mathcal{P}, \mathcal{L})$ is either the linear space obtained from a projective plane of order n, by deleting the points of two lines L and L' and an extra point outside L and L', or the punctured Nwpanka - Shrikhande plane.*

Proof. Counting v from a good point, we have $v = n^2 - n - 1$. We prove that there are no lines of length $n+1$. Assume on the contrary that an invisible line L of length n exists. Let L' be an invisible line different from L and put $p = L \cap L'$. Counting v from p we have

$$v = |L'| + n^2 - 2n + 2. \tag{18}$$

Comparing with the previous value of v we obtain $|L'| = n-3$. Hence every line different from L has length at most $n-1$. Moreover if x is a point of L then x is a type II non-good point and the invisible lines through it have length $n+1$ and $n-3$ respectively. Consider an invisible line t of length $n-3$, let y be a point on t different from $L \cap t$, then counting v from y we have

$$v \leq n - 3 + n - 1 + (n-1)(n-2) = n^2 - n - 2, \tag{19}$$

a contradiction.
Hence every invisible line has at most n points. Now we prove that there are no type I points. Let p be a non-good point, then counting v from p we have

$$v = |t| + |t'| - 1 + (n-1)(n-2), \tag{20}$$

where at least one of the lines t and t' is an invisible line. Comparing with the previous value of v, it follows that $|t|+|t'|=2n-2$. Since p is a non-good point, t and t' are both invisible lines. Thus *if p is a non-good point, then p is of type* II.
Let p be a non-good point and let t and t' be the two invisible lines through p. Then $|t|+|t'|=2n-2$, so through p there pass a line of length $n-2$, a line of length n and $n-1$ lines of length $n-1$. Hence if ℓ is a line, then $|\ell|\in\{n-2,n-1,n\}$. By the above argument it follows that the lines of length n are mutually parallel and similarly those of length $n-2$. Since a line of length $n-2$ exists, it follows that $n\geq 4$.
Consider a line L of length n, for each point x on L denote by ℓ_x the other invisible line through x. The lines ℓ_x, when $x\in L$, are mutually parallel and cover a set X of $n(n-2)$ points of $(\mathcal{P},\mathcal{L})$. Let y be a point of L and let ℓ_y be the line of length $n-2$ passing through y. For each point z of ℓ_y denote by L_z the line of length n through z. The lines L and L_z, with $z\in\ell_y$ are mutually parallel. Moreover the line L_z meets all the lines ℓ_y, because it is a line of length n parallel to L.
Therefore L and the lines L_z cover the same set X of points. Let $p\notin\cup_{x\in L}\ell_x$ and let ℓ be the parallel line to L. The line ℓ is parallel to all the lines L_z, so it passes through $n-1$ points outside the set X. Hence ℓ is parallel to the lines ℓ_x. Then ℓ and the lines ℓ_x partition $\mathcal{P}$. Let Π be this set of parallel lines, then $|\Pi|=n+1$. It follows that the lines ℓ_x are all the lines of length $n-2$ in $(\mathcal{P},\mathcal{L})$. Let $(\mathcal{P}^*,\mathcal{L}^*)$ the linear space defined as follows:

$$\mathcal{P}^*=\mathcal{P}\cup\{\infty\},\mathcal{L}^*=\{\ell\in\mathcal{L}:\ell\notin\diamond\cup\diamond'\}\cup\{\ell\cup\infty:\ell\in\diamond\}.$$

It is a linear space with constant point degree $n+1$ and two line length: $n-1$ and n. Hence $(\mathcal{P}^*,\mathcal{L}^*)$ is a $\{1,2\}$-semiaffine linear space and from the results in [15], the assertion follows. ∎

From now on we may suppose that there are no good points.

Proposition 5.3. *If $(\mathcal{P},\mathcal{L})$ is a partially $\{2\}$-semiaffine linear space with constant point degree $n+1$ and with all points of type* I*, then $(\mathcal{P},\mathcal{L})$ is either the complement of two lines in a projective plane of order n or the Nwpanka-Shrikhande plane.*

Proof. Since every point is of type I, it follows that the invisible lines have constant length and are mutually parallel. Let $n+1-s$ be the length of the invisible lines. Counting v from a point p we have $v=n^2-n+1-s$. Counting the pairs (p,L) with $p\in L$ and $|L|=n-1$ we have $nv=(n-1)b_{n-1}$. Hence $n-1|vn=n^2(n-1)+1-s$. It follows that $n-1|s-1$. Since $n+1-s\geq 2$ and since the invisible lines have length different from $n-1$, it follows that $s=1$. Thus $(\mathcal{P},\mathcal{L})$ is a $\{1,2\}$-semiaffine linear space on $v=n^2-n$ points and from the results in [15] the assertion follows. ∎

By proposition 5.3 it follows that from now on we may suppose that at least a type II point exists.
Let p be a non-good point of type II, and let t and t' be the two invisible lines passing through p. Counting v from p we obtain

$$v=|t|+|t'|-1+(n-1)(n-2)=n^2-3n+1+|t|+|t'|. \tag{21}$$

It follows that

$$n^2 - 3n + 5 \leq v \leq n^2 - n + 3. \tag{22}$$

If $v = n^2 - n + 3$ then through every point p pass two lines of length $n + 1$. Since a line of length $n + 1$ meets all the other lines, the number of the invisible lines is the number of the lines of length $n + 1$ meeting a line of length $n - 1$. Hence $|\mathcal{I}| = 2(n - 1)$. On the other hand the number of invisible lines is the number of lines of length $n+1$, that is $|\mathcal{I}| = n+2$. Thus $n = 4$ and $(\mathcal{P}, \mathcal{L})$ is a $\{0, 2\}$-semiaffine linear space with all the points of degree 5. From the results in [13, 16] it follows that $(\mathcal{P}, \mathcal{L})$ is the complement of a hyperoval in the projective plane of order 4.

If $v = n^2 - n + 2$ then through every point p pass a line of length n and a line of length $n + 1$. Hence every point is a type II point. Let L be a line of length $n + 1$ and let p be a point not on L. The line L' of length $n + 1$ passing through p meets L in a point q through which two lines of length $n + 1$ pass, a contradiction.

If $v = n^2 - n + 1$ then $|t| = |t'| = n$, every point is a point of type II and all the invisible lines have length n. Hence $(\mathcal{P}, \mathcal{L})$ is a $\{1, 2\}$-semiaffine linear space, a contradiction because there are no $\{1, 2\}$-semiaffine linear spaces of order n on $v = n^2 - n + 1$ points.

If $v = n^2 - n$ then $|t| + |t'| = 2n - 1$, contradicting the assumption that t and t' are invisible lines.
Hence if $(\mathcal{P}, \mathcal{L})$ is a $\{2\}$-semiaffine linear space with constant point degree $n+1$ and with a type II point, that is not the complement of a hyperoval in the projective plane of order 4, then

$$n^2 - 3n + 5 \leq v \leq n^2 - n - 1. \tag{23}$$

Proposition 5.4. *Let $(\mathcal{P}, \mathcal{L})$ be a partially $\{2\}$-semiaffine linear space with constant point degree $n + 1$, without good points. If there exists a line of length n, then there exists an integer $s \geq 5$ such that one of the following assertions is true:*

(i) $(\mathcal{P}, \mathcal{L})$ is the pseudo-complement of a Baer subplane and a tangent line, in a projective plane of order $n = (s - 2)^2$.

(ii) $b = n^2 + n + 1$, $n = \dfrac{s^2 - 3s + 2}{2}$ and $(\mathcal{P}, \mathcal{L})$ is a hypothetic linear space.

(iii) $b = n^2 + n + 1 + z$, with $0 < z \leq \dfrac{s^2 - 5s + 2}{2}$ and $n \leq \dfrac{s^2 - 2s}{3}$.

Proof. Let t_1 be a line of length n, from equation (23) it follows that every point on t_1 is a point of type II. Denote by r_i, $i = 1, .., n$ the invisible lines meeting t_1, then these lines have the same length and are mutually parallel. Let $n + 1 - s$ be the length of a line r_i. For every point of r_1 there is a line of length n parallel to t_1; let $t_2, \ldots, t_{n+1-s}$ be these lines. The lines t_i are mutually parallel and every line r_i meets all the lines t_j. Counting v from a point on t_1 we obtain

$$v = (n - 1)(n - 2) + n - s + n = n^2 - n + 2 - s. \tag{24}$$

If $\mathcal{P} = \cup_i t_i$ then $v = n(n + 1 - s)n$. Comparing with (24) it follows that $s = 2$ or $n = 1$, a contradiction. It follows that there exists at least a point p outside the lines t_i. If p is a type II point, then the two invisible lines passing through it are parallel to t_1, that is a contradiction because t_1 is a line of length n. Hence p and every other point outside the lines t_i are type I points. Let t' be the invisible line passing through p, then $|t'| = n+2-s$ and t' is parallel to every line t_i. It follows that the points not on the lines t_i lie on the invisible lines of length $n+2-s$ that are mutually parallel. Then

$$v = (n+1-s)n + b_{n+2-s}(n+2-s), \tag{25}$$

comparing with (24) we have

$$b_n + 2 - s = s - 2 + \frac{s^2-5s+2}{n+2-s}. \tag{26}$$

If $\dfrac{s^2-5s+2}{n+2-s} = 1$, then $n = (s-2)^2$, $b_{n+2-s} = s-1 = \sqrt{n}+1$.
Moreover $|L| \in \{n-1-\sqrt{n}, n-\sqrt{n}, n-1, n\}$ for every line L, $b = n^2+n+1-s+b_{n+2-s} = n^2+n$ and $v = n^2-n-\sqrt{n}$. Hence $(\mathcal{P}, \mathcal{L}\ )$ is the pseudo-complement described in (i).
If $\dfrac{s^2-5s+2}{n+2-s} = 2$, then $n = \dfrac{s^2-3s+2}{2}$, $b = n^2+n+1$ and $(\mathcal{P},\mathcal{L})$ is a hypothetic $\{1,2,s-1,s\}$-semiaffine linear space.
If $\dfrac{s^2-5s+2}{n+2-s} \geq 3$, then $n \leq \dfrac{s^2-2s}{3}$ and $b = n^2+n+1+z$, with $z > 0$. Since $b_n{+}2{-}s \leq \dfrac{s^2-3s+2}{2}$, from $b = n^2{+}n{+}1 + b_n{+}2{-}s$ it follows that $z \leq \dfrac{s^2-5s+2}{2}$. ■

Finally when all the invisible lines have constant length, we have the following result.

Proposition 5.5. *Let $(\mathcal{P},\mathcal{L})$ be a partially $\{2\}$-semiaffine linear space of order n, without good points and with all the invisible lines of length $n+1-s$, then one of the following assertions is true:*

(1) $(\mathcal{P},\mathcal{L})$ is the complement of a hyperoval in the projective plane of order 4.

(2) $(\mathcal{P},\mathcal{L})$ is one of the pseudo-complements of a triangle in the projective plane of order 4 *$(s=3)$.*

(3) $(\mathcal{P},\mathcal{L})$ is a hypothetical linear space with $s=4, n=7,$ and $b = n^2+n-1 = 55$.

(4) $b = n^2+n, n = 2s^2-9s+9$ and $(\mathcal{P},\mathcal{L})$ is a hypothetic linear space if $s \geq 5$. If $s=4$, then $n=5, b=30, v=15$ and in C8 two non-isomorphic examples are described.

(5) $b = n^2+n+1$ and $(\mathcal{P},\mathcal{L})$ is the pseudo-complement of two disjoint Baer subplanes in a projective plane of order $n=(s-2)^2$, $(s \geq 5)$. Moreover, if $s \geq 9$, then $(\mathcal{P},\mathcal{L})$ is embeddable in a projective plane of order n[1].

[1]Notice that de Resmini [7] has shown that if a linear space as that described in (5) is embeddable in a projective plane of order n, then it is possibly not a complement. Moreover, if $s > 14$, Batten and Sane [2] get a condition such that a pseudo-complement of two disjoint Baer subplanes is a complement.

Proof. By the propositions 5.4 and 5.3 we may suppose that $s \neq 1$ and that there exists at least a point p of type II. Counting v from p we obtain

$$v = (n-1)(n-2) + n - s + n + 1 - s = n^2 - n + 3 - 2s. \qquad (27)$$

It follows that every point is a type II point. If $s = 0$ then $v = n^2 - n + 3$ hence we have that $(\mathcal{P}, \mathcal{L})$ is the complement of a hyperoval in the projective plane of order 4 (see the argumentation after equation (22)). Assume $s \geq 3$ then $(\mathcal{P}, \mathcal{L})$ is a proper $\{2, s\}$-semiaffine linear space with constant point degree n + 1 and through every point p pass exactly two lines of length $n + 1 - s$. From [4, 9, 11] the assertion follows. ∎

6 Partially $\{0, 2\}$-semiaffine proper linear spaces of order n with constant point degree

In this section $(\mathcal{P}, \mathcal{L})$ is a proper partially $\{0, 2\}$-semiaffine proper linear space, that is there exist visible lines of both length $n - 1$ and $n + 1$ and all the points have degree $n + 1$. If L is a line of length $n + 1$, then $b = n^2 + n + 1$ and $|\mathcal{I}| \leq 2(n + 1)$, because every line meets L.
Counting v from a point p, we obtain

$$n^2 - 3n + 7 \leq v \leq n^2 + n - 1. \qquad (28)$$

In this section we prove Theorem IV. Hence we study the case $v \geq n^2$.
Denote by λ_p the number of lines of length $n - 1$ passing through a point p.

Proposition 6.1. *If $v = n^2$ then $(\mathcal{P}, \mathcal{L})$ is the complement of a $(n + 1)$- arc in a projective plane of order n, with n odd.*

Proof. Since $v = n^2$, from [14] it follows that $(\mathcal{P}, \mathcal{L})$ is embeddable in a projective plane of order n. If L is a line of length $n - 1$, then it is in two parallel classes, and each class has $n + 1$ lines. It follows that if Π is a parallel class of L then it contains a line of length n. Hence there exist two intersecting lines of length n and parallel to L. Let t_1 and t_2 be two such lines of length n, and let p be their common point then $\lambda_p = \dfrac{n-1}{2}$, so n is odd. Each line L of length $n - 1$ passing through p determines a pair of two meeting lines of length n parallel to L. In such a way we obtain $n - 1$ lines of length n pairwise incident, because different lines of length $n - 1$ give distinct pairs of parallel lines of length n. Moreover these lines of length n meet t_1 and t_2, thus there exist $n + 1$ lines of length n pairwise incident. Using the same argumentation used as in proposition 5.3 in [10] the assertion follows. ∎

Hence we may suppose $v \geq n^2 + 1$.

Proposition 6.2. *If $v \geq n^2 + 1$ then $(\mathcal{P}, \mathcal{L})$ is the complement of two points in a projective plane of order n, with $n \geq 3$.*

Proof. Put $v = n^2 + h$, $(h \geq 1)$ let L be a line of length $n-1$, in each parallel class of L there are at least $h+1$ lines of length n. Let t_1 and t_2 be two intersecting lines of length n parallel to L, and let p be their common point. Since $\lambda_p = \dfrac{n-h-1}{2}$, using the same argumentation as in the previous proposition one obtains that there are $(n-h-1)(h+1)$ lines of length n parallel to all the lines of length $n-1$ passing through p. At least $h+1$ lines of length n are parallel to t_1 and at least $h+1$ lines of length n are parallel to t_2. Since these lines are different and invisible and since $|\mathcal{I}| \leq 2n+2$, it follows that $(n-h-1)(h+1) + 2(h+1) \leq 2n+2$, from which follows that $n(h-1) \leq h^2 + 1$.
If $h \geq 2$ then $n \leq h+1$, i.e $h \geq n-1$, then $v \geq n^2 + n - 1$, and $(\mathcal{P}, \mathcal{L})$ is a doubly punctured projective plane of order n.
Hence consider $h = 1$. Then there exist at least $2(n-2)+4 = 2n$ lines of length n. Since $h = 1$ it follows that if L is a line of length $n-1$, then in each parallel class of L there are exactly two line of length n. Let s_1 and s_2 be two lines of length n in a parallel class of L. Then the remaining $2n-2$ lines of length n meet both s_1 and s_2. Since through every point there pass at most two invisible lines, it follows that $2n-2 \leq n$, a contradiction because $n-1 \geq 2$. ∎

References

[1] L. M. Batten and A. Beutelspacher. *The theory of finite linear spaces*. Cambridge University Press, 1993.

[2] L. M. Batten and S. Sane. A characterization of the complement of the union of two disjoint Baer subplanes. *Arch. Math.*, 44:569 – 576, 1985.

[3] A. Beutelspacher and A.Kersten. Finite semiaffine planes. *Arch. Math.*, 44:557 – 768, 1984.

[4] A. Beutelspacher and J. Meinhardt. On finite h-semiaffine planes. *European J. Combin.*, 5:113 – 122, 1984.

[5] A. Beutelspacher and I Schestag. On a-semiaffine planes with invisible lines. *Geom. Dedicata*, 24:319 – 327, 1987.

[6] P. Biondi. Piani $\{0, 1, 2,\}$-semiaffini con rette invisibili. *Ricerche di Matematica*, 51:41 – 75, 1992.

[7] M. J. de Resmini. On 2-blocking sets in projective planes. *Ars Combin.*, 20-B:59 – 69, 1985.

[8] P. Dembowski. Semiaffine Ebenen. *Arch. Math.*, 13:120 – 131, 1962.

[9] N. Durante. Sugli spazi lineari $\{2, s\}$-semiaffini, 1997. Manuscript.

[10] N. Durante and V. Napolitano. Weakly $\{0, 2\}$-semiaffine planes. *Ricerche di Matematica*, 55:179 – 192, 1996.

[11] N. Durante and V. Napolitano. Finite $\{2,4\}$-semiaffine planes. *Ricerche di Matematica*, 56:1 – 12, 1997.

[12] N. Durante, V. Napolitano, and D. Olanda. Finite partially $\{0,1\}$-semiaffine linear spaces, 1997. Submitted to Geom. Dedicata.

[13] W. Hauptmann. Endliche $[0,2]$-Ebenen. *Geom. Dedicata*, 9:77 – 86, 1980.

[14] D. McCarthy and S. A. Vanstone. Embedding $(r,1)$-designs in finite projective planes. *Discrete Math.*, 19:67–76, 1977.

[15] M. Oehler. Endliche biaffine Inzidenzebenen. *Geom. Dedicata*, 4:419 – 436, 1975.

[16] P. M. Lo Re and D. Olanda. On $[0,2]$-semiaffine planes. *Simon Stevin*, 60:157 – 182, 1986.

[17] P. M. Lo Re and D. Olanda. Piani debolmente $\{1,3\}$-semiaffini. *Ricerche di Matematica*, 54:249 – 268, 1996.

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