Test-words for Sturmian morphisms

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Abstract

J. Berstel and P. Séébold have proved that an acyclic morphism f is Sturmian iff the word $f(baabaababaabab)$ is balanced. More precisely, they have given a set Ω of test-words for Sturmian morphisms. Here, we characterize all such test-words. In particular, we show the optimality of the previous result: there is no test-word of length less or equal to 13, and any test-word has a subword in Ω.

To do this, we describe an efficient algorithm to determine if a finite word is balanced, and we give a short proof of the fact that any finite balanced word is a prefix of an infinite Sturmian word. Finally, we show that the test-words for Sturmian morphisms are exactly the test-words for morphisms preserving finite balanced words.

1 Preliminaries

In all this paper, $A = \{a, b\}$ is a fixed alphabet. The free monoid A^* is the set of finite words over A. We denote by ε the empty word, by $|u|$ the length of a word u and by $|u|_x$ the number of occurrences of the letter x in u. The *mirror image* $\widetilde{u}$ of a word u is the word defined by $\widetilde{\varepsilon} = \varepsilon$, and $\widetilde{ux} = x\widetilde{u}$ with x a letter. A word u is a *subword* of v, if there exist some words x, y such that $v = xuy$. If $x = \varepsilon$, then u is a *prefix* of v. If $y = \varepsilon$, then u is a *suffix* of v.

A word w (finite or infinite) over A is *balanced*, if for all subwords u, v of w, $|u| = |v|$ implies $||u|_a - |v|_a| \leq 1$. Observe that any subword of a balanced word is balanced. Moreover:

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Property 1. *Let $k \geq 2$. For any word u, u^2 is balanced iff u^k is balanced.*

Proof : If u^2 is not balanced, then u^k is not balanced. If u^k is not balanced, let t and t' be two subwords of u^k of same minimal length such that $||t|_a - |t'|_a| \geq 2$. If $|t| \geq |u|$, then $t = xy$, $t' = x'y'$ with $|x| = |x'| = |u|$. The words x and x' are conjugates of u and thus $|x|_a = |x'|_a = |u|_a$. It follows $||y|_a - |y'|_a| \geq 2$: a contradiction with the minimality of $|t|$. Consequently, $|t| < |u|$, and so t and t' are subwords of u^2. ∎

The following proposition will also be useful:

Proposition 2. *[4] A word w over A is not balanced if and only if there exists a word t such that ata and btb are subwords of w.*

A *Sturmian word* is a non ultimately periodic infinite balanced word (an infinite word $\underline{\underline{x}}$ is ultimately periodic if there exist two words u and v such that $\underline{\underline{x}} = \lim_{n\to\infty} uv^n$).

We will consider *morphisms* on A i.e. mappings h from A^* to itself such that for any words u, v, $h(uv) = h(u)h(v)$. The mirror morphism $\widetilde{f}$ of a morphism f is defined by $\widetilde{f}(a) = \widetilde{f(a)}$, $\widetilde{f}(b) = \widetilde{f(b)}$. Observe that for a word w, $\widetilde{f}(w) = \widetilde{f(\widetilde{w})}$. Consequently:

Property 3. *If f and g are 2 morphisms on A, $\widetilde{f \circ g} = \widetilde{f} \circ \widetilde{g}$.*

Proof : For all $u \in \{a,b\}^*$, $\widetilde{f \circ g}(u) = \widetilde{f \circ g(\widetilde{u})} = \widetilde{f}(\widetilde{g(\widetilde{u})}) = \widetilde{f}(\widetilde{g}(u)) = \widetilde{f} \circ \widetilde{g}(u)$. ∎

A *morphism* h is *Sturmian* iff $h(\underline{\underline{w}})$ is Sturmian, whenever $\underline{\underline{w}}$ is Sturmian.

For surveys on Sturmian words and morphisms, see [3, 5, 9].

Any Sturmian morphism is *acyclic* i.e. $f(ab) \neq f(ba)$: otherwise, if f is not acyclic, there exist a word u and integers m, n such that $f(a) = u^m$, $f(b) = u^n$ and for any infinite word $\underline{\underline{w}}$, $f(\underline{\underline{w}})$ is periodic.

Let E be the exchange morphism ($E(a) = b$, $E(b) = a$) and φ the Fibonacci morphism ($\varphi(a) = ab$, $\varphi(b) = a$). We have the following characterization:

Theorem 4. *[8] A morphism f is Sturmian iff $f \in \{E, \varphi, \widetilde{\varphi}\}^*$.*

Observe that $E = \widetilde{E}$. The following property is an immediate consequence of Theorem 4 and Property 3:

Property 5. *A morphism is Sturmian iff its mirror morphism is Sturmian.*

In [1, 2], J. Berstel and P. Séébold define for all integers $m, r \geq 1$ the words:

$$w_{m,r} = (ba^{m+1})^{r+1} ba^m (ba^{m+1})^r ba^m b,$$

$$w'_{m,r} = a(ba^m)^{r+1} ba^{m+1} (ba^m)^r ba^{m+1} b.$$

And they prove in [2]:

Theorem 6. *Let m, r be integers greater than or equal to 1 and let $w = w_{m,r}$ or $w = w'_{m,r}$.*

An acyclic morphism f is Sturmian iff $f(w)$ is balanced.

The shortest words of all the words $w_{m,r}$ and $w'_{m,r}$ are $w_{1,1}$ and $w'_{1,1}$: they have 14 letters. Then arises the question of the existence of a word w of length 13 or less such that an acyclic morphism f is Sturmian iff $f(w)$ is balanced (in this note, we prove that such a word cannot exist).

2 Results

In order to answer the previous question, we characterize all the test-words for Sturmian morphisms i.e. each word w such that an acyclic morphism f is Sturmian iff $f(w)$ is balanced. Observe that w is necessary balanced, since the identity morphism is Sturmian.

Theorem 7. *Let w be a balanced word over A, the following two assertions are equivalent:*

1. *for any acyclic morphism f, f is Sturmian iff $f(w)$ is balanced.*
2. *there exist some integers $m \geq 1$ and $r \geq 1$, such that $w_{m,r}$, $w'_{m,r}$, $E(w_{m,r})$, $E(w'_{m,r})$, $\widetilde{w}_{m,r}$, $\widetilde{w}'_{m,r}$, $E(\widetilde{w}_{m,r})$* **or** *$E(\widetilde{w}'_{m,r})$ is a subword of w.*

As a consequence, there is no test-word for Sturmian morphisms of length 13 or less and there are 8 test-words of length 14.

Section 5 is entirely devoted to the proof of Theorem 7.

Before, in order to be self-contained, we recall in the next section a recursive structural property of finite balanced words (see [6]). This property first leads to an efficient algorithm to determine whether a finite word is balanced. This algorithm will be useful to understand details in the proof of Theorem 7. Moreover, it leads to a short proof (see Section 4) of the following result first stated in [6] (see also [7]), used in the proof of Theorem 7:

Proposition 8. *Any finite balanced word over A is a prefix of a Sturmian word.*

Let us call *balanced morphisms* those morphisms which keep balanced all the finite balanced words. Observe also that there exist some balanced morphisms which are non Sturmian (they are not acyclic): for instance, the morphism defined by $f(a) = f(b) = a$.

Corollary 9. *A morphism is Sturmian iff it is balanced and acyclic.*

Proof : We have already mention that a Sturmian morphism is acyclic. As a consequence of Proposition 8, a Sturmian morphism is also balanced. Conversely, let f be a balanced and acyclic morphism. In particular, $f(w_{1,1})$ is balanced. Thus from Theorem 6, f is Sturmian. ∎

Note that this corollary is an analogous for the Sturmian morphisms of the equivalence $2 \Leftrightarrow 5$ of Theorem 2.8 in [5].

We end this paper with a corollary of Theorem 7 (Corollary 13): the test-words for balanced morphisms are exactly the same as those for Sturmian morphisms

3 An algorithm for balanced words

For any balanced word over A, there exists an integer n such that between 2 successive occurrences of b, there are exactly n or $n+1$ letters a. Moreover, such balanced word w starts and ends with at most $n+1$ letters a, and thus, if $|w|_b \geq 1$, w can be decomposed in $w = SA_1A_2 \dots A_kP$, with $k \geq 0$, $S \in \bigcup_{l=0}^{n+1} a^l$, $A_i \in \{ba^n, ba^{n+1}\}$ $(1 \leq i \leq k)$, $P \in \bigcup_{l=0}^{n+1} ba^l$. If a word w (balanced or not) has such a decomposition, we will say that it is *n-reducible* or, omitting n, is *reducible*: observe that a word is not reducible, if for all integer n, w is not n-reducible. We can construct from a n-reducible word w a word $reduced_n(w) = sa_1a_2 \dots a_kp$ such that:

$$\begin{cases} s = \varepsilon \text{ if } S \neq a^{n+1}, \text{ else } s = b, \\ a_i = a \text{ if } A_i = ba^n, \\ a_i = b \text{ if } A_i = ba^{n+1} \ (1 \leq i \leq k), \\ p = \varepsilon \text{ if } P \neq ba^{n+1}, \text{ else } p = b. \end{cases}$$

Observe that the same kind of reduction is given in [6]: the difference introduced here is essentially in order to have an efficient algorithm to determine if a word is balanced. If $n \geq 1$, the reduction is important: indeed $|reduced_n(w)| \leq \frac{|w|}{2}$ (since $|s| \leq \frac{|S|}{2}$, $|p| \leq \frac{|P|}{2}$, and $|a_i| \leq \frac{|A_i|}{2}$).

The proposed algorithm is a consequence of:

Lemma 10. *Let w be a word over A containing at least one b. The following are equivalent:*

1. *w is balanced*
2. *w is n-reducible (for an integer $n \geq 0$) and $reduced_n(w)$ is balanced.*

Let h_n be the Sturmian morphism $(\tilde{\varphi} \circ E)^n \circ E \circ \varphi \circ E$ i.e. $h_n(a) = ba^n$ and $h_n(b) = ba^{n+1}$.

Proof of Lemma 10 : Since any balanced word is reducible, we have just to prove that a n-reducible word w is not balanced iff $reduced_n(w)$ is not balanced.

First assume w is not balanced. From Proposition 2, there exists a word t with ata and btb subwords of w. If $|t|_b = 0$, from btb subword of w comes $t = a^n$ or a^{n+1}. In this case, a^{n+2} is a subword of ata and then w is not n-reducible: a contradiction. Thus there exists an integer $l \geq 0$ with a^lb prefix of t. From btb subword of w comes $l = n$ or $l = n+1$. Since a^{n+2} is not a subword of w and aa^l is a subword of at, we have $l = n$. On the same way, ba^n is a suffix of t. So $t = a^nh_n(u)ba^n$ with $u \in \{a, b\}^*$. Since $ba^nh_n(u)ba^nb$ is a subword of w, aua is a subword of $reduced_n(w)$.

Since $a^{n+1}h_n(u)ba^{n+1}$ is a subword of w, bub is a subword of $reduced_n(w)$. Then, $reduced_n(w)$ is not balanced.

Now assume $reduced_n(w)$ is not balanced. From Proposition 2, there exists a word t with ata and btb subwords of $reduced_n(w)$. Since btb is a subword of $reduced_n(w)$, $a^{n+1}h_n(t)ba^{n+1}$ is a subword of w. Now observe that if ata is a suffix of $reduced_n(w)$, then there exists an integer l, $0 \leq l \leq n$ with $h_n(ata)ba^l$ suffix of w. Thus $ba^n h_n(t)ba^n b$ is a subword of w, and, w is not balanced.

■

Let w be a balanced word: $E(w)$ is also balanced. If $|w|_b \geq 2$ (i.e. $w \notin a^*ba^* \cup a^*$), then we can compute the greater integer n such that w is n-reducible (if $w \in a^*ba^*$, for any integer $k \geq |w|_a$, w is k-reducible): observe that ba^nb is a subword of w. On the same way, if $|w|_a \geq 2$, we can compute the greater integer m such that $E(w)$ is m-reducible. Moreover, $n \geq 1$ or $m \geq 1$: indeed, if $n = 0$ and $m = 0$, then aa and bb are subwords of the balanced word w: contradiction.

Finally, we give the *recursive scheme* of the algorithm to test if a finite word w is balanced:

If $|w|_a \leq 1$ or $|w|_b \leq 1$, **Then** w is balanced
Else
  If w or $E(w)$ is not reducible,
  Then w is not balanced
  Else
    Let n be the greater integer such that w is n-reducible
    Let m be the greater integer such that $E(w)$ is m-reducible
    If $n \geq 1$ **Then** let $w' = reduced_n(w)$
      Else let $w' = reduced_m(E(w))$
    w is balanced iff w' is balanced.

The previous algorithm has a *time complexity* in $O(|w|)$. Indeed in $O(|w|)$, we can simultaneously determine if $|w|_a \leq 1$, $|w|_b \leq 1$, determine if w is reducible, and compute the integers n and m. So one recursive step can be achieved in time bounded by $\lambda|w|$ where λ is a strictly positive constant. Moreover, we have seen that $|w'| \leq \frac{|w|}{2}$. Thus, if $2^{k-1} < |w| \leq 2^k$, then the time complexity of the algorithm is bounded by $\sum_{i=0}^{k} \lambda 2^i = \lambda(2^{k+1} - 1) < 4\lambda|w|$, and then is in $O(|w|)$.

Observe that using the definition of balanced words (and counting the number of a in subwords of same length of w), or using Proposition 2, we obtain algorithms whose time complexity is in $O(|w|^2)$.

4 Extension of a finite balanced word

In this section, we prove that any finite balanced word can be extended on a longer balanced word. As a consequence, any finite balanced word is a prefix of some infinite balanced words. Moreover, at least one of these infinite words is not ultimately periodic (Proposition 8) which means that it is Sturmian.

Lemma 11. *For any balanced word w over A, there exist (at least) two letters x and y in A such that wx and yw are balanced.*

Proof : Assume that wa and wb are not balanced but w is balanced. There exist some words t and t' such that ata is a suffix of wa, $bt'b$ is a suffix of wb, and btb and $at'a$ are subwords of w. We cannot have $|t| = |t'|$. If $|t| < |t'|$, then $t' = uat$ for a word u. Thus ata and btb are subwords of w: contradiction. On the same way, if $|t| > |t'|$, $bt'b$ and $at'a$ are subwords of w.

Now, if w is balanced, so is $\widetilde{w}$, thus there exists $y \in \{a, b\}$ with $\widetilde{w}y$ balanced: yw is balanced. ∎

Proof of Proposition 8 : Let w be a finite balanced word. Since any suffix of a Sturmian word is Sturmian, it is sufficient to prove that there exists a Sturmian morphism f such that w is a subword of $f(a)$.

We prove this by induction on $|w|$.

First observe that for all integers n and p, $a^nba^p = (\varphi \text{o} E)^n \text{o} (\widetilde{\varphi} \text{o} E)^p \text{o} E(a)$. Moreover, the property is true for w iff it is true for $E(w)$.

If $|w| \leq 4$, the property is true since w or $E(w)$ is a subword of a^4ba^3 or of $abbabab = E \text{o} \widetilde{\varphi} \text{o} \varphi \text{o} (E \text{o} \widetilde{\varphi})^2(a)$.

If $|w| \geq 5$, the property is true if $|w|_a \leq 1$ or $|w|_b \leq 1$ i.e. if $w \in a^*ba^* \cup b^*ab^* \cup a^* \cup b^*$. In the other cases, since w is balanced, w is reducible: let w' be the word computed in the algorithm of the previous section. We have $|w'| \leq \frac{|w|}{2}$ and from Lemma 10, w' is balanced. From Lemma 11, there exist two letters x and y such that $xw'y$ is balanced. The word w is a subword of $h_n(xw'y)$ or of $E \text{o} h_n(xw'y)$. Since $|xw'y| \leq \frac{|w|}{2} + 2 < |w|$ ($|w| \geq 5$), by induction hypothesis, there exists a Sturmian morphism g such that $xw'y$ is a subword of $g(a)$. Finally, w is a subword of $f(a)$ where $f = h_n \text{o} g$ or $f = E \text{o} h_n \text{o} g$ (in the two cases, f is Sturmian). ∎

5 Proof of Theorem 7

Let w be a balanced word.

First assume that there exist integers $m \geq 1$ and $r \geq 1$ such that $w_{m,r}$, $w'_{m,r}$, $E(w_{m,r})$, $E(w'_{m,r})$, $\widetilde{w}_{m,r}$, $\widetilde{w}'_{m,r}$, $E(\widetilde{w}_{m,r})$, or $E(\widetilde{w}'_{m,r})$ is a subword of w.

From Proposition 8, there exists a Sturmian word $\underline{\underline{x}}$ such that w is a prefix of $\underline{\underline{x}}$. Thus for any Sturmian morphism f, $f(w)$ is balanced. Conversely, if $f(w)$ is balanced then $f(w_{m,r})$, $f(\widetilde{w}_{m,r}) = \widetilde{f}(w_{m,r})$, $f(E(w_{m,r})) = f \text{o} E(w_{m,r})$, $f(E(\widetilde{w}_{m,r})) = \widetilde{f \text{o} E}(w_{m,r})$, $f(w'_{m,r})$, $f(\widetilde{w}'_{m,r}) = \widetilde{f}(w'_{m,r})$, $f(E(w'_{m,r})) = f \text{o} E(w'_{m,r})$ or $f(E(\widetilde{w}'_{m,r})) = \widetilde{f \text{o} E}(w'_{m,r})$ is a balanced word. From Theorem 6, if f is an acyclic morphism, f, $\widetilde{f}$, $f \text{o} E$ or $\widetilde{f \text{o} E}$ is Sturmian. Thus, from $f = f \text{o} E \text{o} E$, Property 3 and Property 5, f is Sturmian.

To prove Part 1 $\Rightarrow$ 2 of Theorem 7, we consider the morphisms f_n, g_n $(n \geq 0)$ defined by:

$$\begin{cases} f_n(a) = a, \\ f_n(b) = aba^nba, \end{cases} \qquad \begin{cases} g_n(a) = bab, \\ g_n(b) = (abb)^na. \end{cases}$$

Now, let us suppose that for any acyclic morphism f, f is Sturmian iff $f(w)$ is balanced. In particular, since for all $n \geq 0$, f_n, $f_n \text{o} E$, g_n and $g_n \text{o} E$ are not Sturmian, $f_n(w)$, $f_n(E(w))$, $g_n(w)$ and $g_n(E(w))$ are not balanced.

First observe that $w \notin a^* \cup b^*$, since for all $k \geq 1$, $f_0(a^k) = g_0(b^k) = a^k$ is balanced. Moreover $w \notin a^*ba^*$ since for all $u \in a^*ba^*$, $f_{|u|}(u)$ is a subword of $a^{|u|}ba^{|u|}ba^{|u|}$ and so is balanced. For the same reason (consider $E(w)$), $w \notin b^*ab^*$.

Thus $|w|_b \geq 2$ and $|w|_a \geq 2$.

Let m be the minimal number of a between two consecutive b in w. One can suppose (else consider $E(w)$) that $m \geq 1$. Since w is balanced, all subwords ba^pb of w are such that $p = m$ or $p = m + 1$.

If w does not contain any subword $ba^{m+1}b$ then w is a subword of an element of the set $a^{m+1}(ba^m)^+ba^{m+1}$, and $f_{m+2}(w)$ is a subword of an element of $a^{m+2}(ba^{m+2})^+ba^{m+2}$ and then is balanced: a contradiction.

So, ba^mb and $ba^{m+1}b$ are subwords of w.

Now let us denote $\{n, n'\} = \{m, m + 1\}$ (this means $n = m$ and $n' = m + 1$, or $n = m + 1$ and $n' = m$).

If ba^nb occurs only once as a subword of w, (since w is balanced, a^{m+2} is not a subword of w) w is a subword of an element of $a^{m+1}(ba^{n'})^*ba^n(ba^{n'})^*ba^{m+1}$. But, since a^{m+2} is a subword of $a^{n'+2}$, $f_{n'+2}(w)$ is a subword of an element of $(ba^{n'+2})^*ba^{n+2}(ba^{n'+2})^*$ and then it is balanced (since $\{n+2, n'+2\} = \{m+2, m+3\}$): a contradiction.

Thus each of the two words ba^nb and $ba^{n'}b$ has at least two occurrences as a subword of w. Moreover, since w is balanced, ba^nba^nb and $ba^{n'}ba^{n'}b$ are not simultaneously subwords of w. However, at least one of these words is a subword of w. Indeed, otherwise w is a subword of an element of $(a^{m+1} \cup a^{m+1}ba^{n'})(ba^nba^{n'})^*(ba^{m+1} \cup ba^nba^{m+1})$. Thus, since a^{m+2} is a subword of $a^{n'+2}$ and of a^{n+2}, $f_{n+2}(w)$ is a subword of an element of $(ba^{n+2}ba^{n'+2})^*$ and then it is balanced (since $\{n + 2, n' + 2\} = \{m + 2, m + 3\}$): a contradiction.

Now let us suppose that ba^nba^nb is a subword of w and $ba^{n'}ba^{n'}b$ is not.

Since $ba^{n'}b$ has at least 2 occurrences in w, and since w is balanced, there exists an integer $r \geq 1$ such that $ba^{n'}(ba^n)^rba^{n'}b$ is a subword of w: let us choose r minimal for this property. For each other subword $ba^{n'}(ba^n)^sba^{n'}b$ of w, since w is balanced and because of the minimality of r, $s = r$ or $s = r + 1$.

If $(ba^n)^{r+1}b$ is not a subword of w, then w is a subword of an element of

$$[a^{m+1}(ba^n)^{r-1} \cup a^{m+1}(ba^n)^r][ba^{n'}(ba^n)^r]^*ba^{n'}b[(a^nb)^{r-1}a^{m+1} \cup (a^nb)^ra^{m+1}].$$

Since $f_{n+2}(w)$ is a subword of an element of $[ba^{n'+2}(ba^{n+2})^{2r+1}]^*$ (let us recall that a^{m+2} is a subword of both a^{n+2} and $a^{n'+2}$), $f_{n+2}(w)$ is balanced: a contradiction.

Thus $(ba^n)^{r+1}b$ is a subword of w, and then, one of the two words v or $\widetilde{v}$, where $v = (ba^n)^{r+1}ba^{n'}(ba^n)^rba^{n'}b$, is a subword of w.

If $n = m + 1$ then $n' = m$ and thus $v = w_{m,r}$ and $\widetilde{v} = \widetilde{w}_{m,r}$.

If $n = m$, then $n' = m + 1$ and $v = (ba^m)^{r+1}ba^{m+1}(ba^m)^rba^{m+1}b$. One of the two words $w'_{m,r} = av$ and $\widetilde{w}'_{m,r}$ is a subword of w. Indeed, otherwise w is a subword of one word in

$$(ba^m)^{r+1}[ba^{m+1}(ba^m)^r]^+ba^{m+1}(ba^m)^{r+1}b.$$

In this case, $g_{m-1}(w)$ (let recall $m \geq 1$) is balanced because it is a subword of a word in the following set of balanced words

$$[X^{m-1}Y]^{2r+3}[X^mY(X^{m-1}Y)^{2r+1}]^+X^mY(X^{m-1}Y)^{2r+2}X^{m-1}a$$

with $X = abb$ and $Y = ab$: a contradiction.

Consequently, in all the cases, if w is a test-word for the Sturmian morphisms then there exist two strictly positive integers m and r such that w contains as a subword at least one of the eight words: $w_{m,r}$, $\widetilde{w}_{m,r}$, $w'_{m,r}$, $\widetilde{w}'_{m,r}$, $E(w_{m,r})$, $E(\widetilde{w}_{m,r})$, $E(w'_{m,r})$ or $E(\widetilde{w}'_{m,r})$ (the last four words appear in the previous study when $m = 0$ i.e. when we have to consider $E(w)$).

6 Balanced morphisms

Here, we study the balanced morphisms i.e. the morphisms such that the image of any finite balanced word is balanced. First observe that for a non acyclic morphism f ($f(ab) = f(ba)$), there exist a non empty word u and integers k, l such that $f(a) = u^k$ and $f(b) = u^l$. If $k \neq 0$ or $l \neq 0$, we say that f is u-cyclic.

Property 12. *Let $u \in A^+$ and f a u-cyclic morphism. The morphism f is balanced iff the word u^2 is balanced.*

Proof : Since f is u-cyclic, for any word v over A, there exists an integer n such that $f(v) = u^n$. If u^2 is balanced, then from Property 1, f is balanced. Conversely, since u^2 is a subword of $f(abab)$, if f is balanced, then u^2 is balanced. ■

Thus, the balanced morphisms are the Sturmian morphisms, the empty morphism (the image of a and b is ε) and the u-cyclic morphisms with u^2 balanced. Consequently from Theorem 7, the test-words for Sturmian morphisms are exactly the test-words for balanced morphisms. More precisely:

Corollary 13. *Let w be a balanced word over A, the two following assertions are equivalent:*

1. *a morphism f is balanced iff $f(w)$ is balanced;*
2. *there exist some integers $m \geq 1$ and $r \geq 1$, such that $w_{m,r}$, $w'_{m,r}$, $E(w_{m,r})$, $E(w'_{m,r})$, $\widetilde{w}_{m,r}$, $\widetilde{w}'_{m,r}$, $E(\widetilde{w}_{m,r})$* **or** *$E(\widetilde{w}'_{m,r})$ is a subword of w.*

Proof : First, if w verifies Part 1, then in particular for any acyclic morphism, f is balanced iff $f(w)$ is balanced. From Corollary 9 and Theorem 7, Part 2 of the corollary is true.

From now on, let us suppose that w verifies Part 2.

From Corollary 9 and Theorem 7, Part 1 is verified for acyclic morphisms.

Let f be a non acyclic morphisms. If $f(a) = f(b) = \varepsilon$, then f and $f(w)$ are balanced. Else, there exists a non empty word u, such that f is u-cyclic. Since $|w|_a \geq 2$ and $|w|_b \geq 2$, $f(w) = u^n$, for an integer $n \geq 2$. From Property 1, $f(w)$ is balanced iff u^2 is balanced. From Property 12, Part 1 is verified for non acyclic morphisms. ■

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