The Cauchy integral formulas on the octonions

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Abstract

As the last one of the finite, alternative, division algebra, the Cayley-Graves algebra or the octonion algebra **O**, is a non-commutative, non-associative division algebra, in which the analysis problems that would be a direct generalization of the complex analysis and the quaternion analysis, have been studied systematically. Taking the associator as a measure, the Cauchy integral formulas, the Cauchy theorems and the inverse theorems of the Cauchy integral formulas are obtained on the octonions. Some applications are also given.

1 Introduction

It is well-known that [J], the only finite dimensional alternative division algebras over **R** are

a) Real algebra **R**;
b) Complex algebra **C**;
c) Quaternion algebra **H**;
d) Octonion algebra **O**;
with the embedding relations: $\mathbf{R} \subset \mathbf{C} \subset \mathbf{H} \subset \mathbf{O}$.

R and **C** are commutative and associative, **H** is associative but not commutative, while **O** is neither commutative nor associative.

Quaternions were invented by the Irish mathematician W. R. Hamilton in 1843 after a lengthy struggle to extend the theory of complex numbers to three dimensions. Rejecting the commutative law he got the quaternions. Quaternions have

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been widely used in many fields, especially in physics. One of the known results is that the Maxwell's equations can be expressed quite simply using a quaternion form.

Much earlier the great Swiss mathematician R. Fueter (a student of Hilbert) and his followers developed quaternion analysis up to fifties [F1-F3], which was a great achievement in the development of higher-dimensional analogue of complex analysis.

As a common generalization of Grassmann's exterior algebra and Hamilton's quaternions, Clifford algebra $\mathbf{A_n}$ was constructed by W. K. Clifford in 1878 [C]. It has been intensively studied since then. An important fact is that

$$\mathbf{A_0} = \mathbf{R}, \mathbf{A_1} = \mathbf{C}, \mathbf{A_2} = \mathbf{H}$$

but $\mathbf{A_3} \neq \mathbf{O}$. The Clifford algebra $\mathbf{A_n}$ is associative and is not a division algebra $(n \geq 3)$, while the octonions form a division algebra but not an associative one.

The octonion algebra $\mathbf{O}$ was discovered independently by J. J. Graves in 1843 and A. Cayley in 1845. It is important in both Mathematics and Physics. Recently, octonions are used in antisymmetric tensor gauge fields ([DGT]). It is known that complex analysis, quaternion analysis and Clifford analysis (see [BDS], [GM], [DSS] and [D]) are nearly completed and they play a very important role in many fields. A natural question is: What about the octonion analysis?

In 1976, Habetha ([Hab]) showed that if one wishes to generalize classical function theory by considering algebra-valued functions in such a way that a "simple" Cauchy formula still holds, then one has to restrict to algebra of complex numbers, algebra of quaternion or a Clifford algebra [see [BDS] p.139]. This gives an impression that it seems to be impossible to get the Cauchy formula on $\mathbf{O}$. But when one looks carefully in Habetha's paper, one sees that the algebra he considered was an associative one.

Fortunately, we have obtained the Cauchy integral formulas on the octonions. Our results are closely related with the associative methods, and the associator becomes an indicator of non-associativity. Also, our formulas are still "simple", just a little more complicated than the usual ones.

Let M be an 8-dimensional, compact, oriented C^∞–manifold with boundary ∂M contained in some open connected subset Ω of $\mathbf{R}^8$. For $j : 0 \leq j \leq 7$, let

$$\begin{aligned} d\widehat{x_j} &= dx_0 \wedge dx_1 \wedge \cdots dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_7, \\ d\sigma(x) &= \sum_0^7 (-1)^j e_j d\widehat{x_j}. \end{aligned}$$

Thus to each $f = \sum_{j=0}^7 f_j(x)e_j$ in $C^\infty(\Omega, \mathbf{O})$ there correspond the $\mathbf{O}$–valued 7–forms

$$\begin{aligned} \omega &= d\sigma(x)f(x) = \sum_0^7 (-1)^j e_j d\widehat{x_j} f(x) \\ \nu &= f(x)d\sigma(x) = f(x)\sum_0^7 (-1)^j e_j d\widehat{x_j} \end{aligned}$$

having exterior derivative

$$d\omega = \sum_0^7 (-1)^j e_j \frac{\partial f}{\partial x_j} dx_j \wedge d\widehat{x_j} = Df(x)dV(x)$$
$$d\nu = \sum_0^7 (-1)^j \frac{\partial f}{\partial x_j} e_j dx_j \wedge d\widehat{x_j} = f(x)DdV(x)$$

on Ω, where $dV(x) = dx_0 \wedge \cdots \wedge dx_7$ is the volume element on Ω, and $e_0, e_1, \cdots, e_7$ form a basis of $\mathbf{O}$, $D = \sum_0^7 e_k \frac{\partial}{\partial x_k}$. The operator D is called the Cauchy-Riemann operator on $\mathbf{O}$. We define the left analytic function on $\mathbf{O}$ by $Df = 0$. And the right analytic function on $\mathbf{O}$ by $fD = 0$.

For each $x \in \partial M$, let $n(x) = \sum_0^7 n_j e_j$ be the outer unit normal to ∂M at x. Then $(-1)^j d\widehat{x_j} = n_j(x)dS(x)$, where $dS(x)$ is the scalar element of surface area on ∂M. Consequently on ∂M

$$\begin{aligned} d\sigma &= ndS, \\ \omega &= n(x)f(x)dS(x), \\ \nu &= f(x)n(x)dS(x). \end{aligned}$$

Let

$$\Phi(x-z) = \frac{\overline{x} - \overline{z}}{\omega_8 |x-z|^8} =: \sum_0^7 \Phi_s e_s,$$

where ω_8 is the surface area of the unit sphere in $\mathbf{R}^8$, it is the Cauchy kernel on $\mathbf{O}$.

If $Df = 0$, then for each $z \in M^0$, i.e. z is the interior point of M,

$$\begin{aligned} f(z) &= \int_{\partial M} \Phi(x-z)(d\sigma(x)f(x)) + \int_M \sum_0^7 [e_s, D\Phi_s, f]dV \\ &= \int_{\partial M} (\Phi(x-z)d\sigma(x))f(x) - \int_M \sum_{t=0}^7 [\Phi, Df_t, e_t]dV, \end{aligned}$$

where $\overline{x} = x_0 e_0 - x_1 e_0 - \cdots - x_7 e_7$ if $x = x_0 e_0 + x_1 e_1 + \cdots + x_7 e_7$, and $[x, y, z] =: (xy)z - x(yz)$ is called associator of x, y, z.

Remark The appearance of the big "tails" consisting of the associators is a special phenomenon in octonions. It would be very difficult to calculate

$$\begin{aligned} \sum_0^7 [e_s, D\phi_s, f] &= \sum_0^7 [e_s, \sum_0^7 e_k \frac{\partial \phi_s}{\partial x_k}, f] \\ &= \sum_{s=0}^7 \sum_{k=0}^7 [e_s, e_k \frac{\partial \phi_s}{\partial x_k}, f] \end{aligned}$$

and

$$\begin{aligned} \sum_0^7 [\phi, Df_t, e_t] &= \sum_0^7 [\phi, \sum_0^7 e_k \frac{\partial f_t}{\partial x_k}, e_t] \\ &= \sum_{t=0}^7 \sum_{k=0}^7 [\phi, e_k \frac{\partial f_t}{\partial x_k}, e_t] \end{aligned}$$

for general $\phi =: \sum_0^7 \phi_s e_s$ and f.

A terrific fact is that for the very $\phi = \Phi$ we need, one kind of the "tails" disappeared just as what we have expected after a skillful calculation by using a new method. And the same method is also used to prove the inverse theorem of the Cauchy integral formula.

Our main results are as follows:

Theorem 1 (Cauchy integral formula) *M, Ω are as above, $Df = 0$, $x \in \Omega$. Then*

$$\int_{\partial M} \Phi(x-z)(d\sigma(x)f(x)) = \begin{cases} f(z), & \text{if} \quad z \in M^0, \\ 0, & \text{if} \quad z \in \Omega \backslash M. \end{cases}$$

Theorem 2 *M, Ω are as above, $Df = 0$, $x \in \Omega$. Then*

$$\int_{\partial M} (\Phi(x-z)d\sigma(x))f(x) = \begin{cases} f(z) + \int_M \sum_t [\Phi, Df_t, e_t] dV, & \text{if} \quad z \in M^0, \\ \int_M \sum_t [\Phi, Df_t, e_t] dV, & \text{if} \quad z \in \Omega \backslash M. \end{cases}$$

Theorem 3 (Inverse theorem of the Cauchy integral formula) *Let M be an 8-dimensional, compact, oriented C^∞–manifold with boundary ∂M contained in some open connected subset Ω of $\mathbf{R}^8$,and the function $f : \partial M \to \mathbf{O}$ is continuous. If for each $x \in M^0$*

$$\begin{aligned} f(x) &= \int_{\partial M} \Phi(y-x)(d\sigma(y)f(y)) \\ &= \int_{\partial M} \Phi(y-x)(n(y)f(y))dS(y). \end{aligned}$$

Then f is left $\mathbf{O}$–analytic in M.

And similar results hold for the right $\mathbf{O}$–analytic functions.

In §2 we give some preliminaries on the octonions. In §3 we define the $\mathbf{O}$-analytic functions, and discuss their properties. In §4 we prove our main results and give some further results. Finally in §5 we give some applications of our main results.

2 Preliminaries on the octonions

Recall that [J] a non-associative algebra A over a field F is a vector space equipped with a binary product $(x, y) \longmapsto x \cdot y$ which is bilinear in the sense that

$$\begin{aligned} (x_1 + x_2)y &= x_1 y + x_2 y \\ x(y_1 + y_2) &= xy_1 + xy_2 \\ a(xy) &= (ax)y = x(ay) \end{aligned}$$

where $x, x_i, y, y_i \in A, a \in F$.

An algebra is called alternative if $[x, x, y] = 0 = [y, x, x], \quad \forall x, y \in A$.

The octonion algebra $\mathbf{O}$ is an alternative, non-associative division algebra with the basic octonionic units:

$$e_0, e_1, \cdots, e_6, e_7$$

where e_0 is the unit element in $\mathbf{O}$, satisfying that

$$\begin{aligned} e_0^2 &= e_0, \\ e_\alpha e_0 &= e_0 e_\alpha, \quad (\alpha = 0, 1, 2, \cdots, 7) \\ e_\alpha e_\beta &= -\delta_{\alpha\beta} + \psi_{\alpha\beta\gamma} e_\gamma, \quad (\alpha, \beta, \gamma = 1, 2, \cdots, 7) \end{aligned}$$

where

$$\delta_{\alpha\beta} = \begin{cases} 1 & if \quad \alpha = \beta \\ 0 & if \quad \alpha \neq \beta \end{cases}$$

and the constants $\psi_{\alpha\beta\gamma}$ are totally antisymmetric in (α, β, γ), non-zero and equal to unit for the seven combinations

$$(1,2,3), (1,4,5), (2,4,6), (3,4,7), (2,5,7), (6,1,7), (5,3,6)$$

Clearly, the commutator $[e_\alpha, e_\beta] = 2\psi_{\alpha\beta\gamma} e_\gamma, \quad (\alpha, \beta, \gamma = 1, 2, \cdots, 7)$. For the multiplication table, see [J], [PY] and [MD].

The basic elements of $\mathbf{O}$ can be written as

$$1 = e_0, e_1, e_2, e_1e_2, e_4; e_1e_4, e_2e_4, (e_1e_2)e_4.$$

And any real octonion $x \in \mathbf{O}$, which labels say a point in $\mathbf{R}^8$, the eight-dimensional Euclidean space-time, is of the form

$$\begin{aligned} x = \sum_0^7 x_k e_k =& (x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3) \\ &+ (x_4e_0 + x_5e_1 + x_6e_2 + x_7e_3)e_4, \end{aligned}$$

$x_j \in \mathbf{R}, \quad (j = 0, 1, \cdots, 7)$. Its conjugate $\overline{x} = \sum_0^7 x_k \overline{e_k}$ where $\overline{e_0} = e_0, \quad \overline{e_j} = -e_j, \quad (j = 1, 2, \cdots 7)$. Then

$$\overline{e_i e_j} = \overline{e_j}\ \overline{e_i}, \quad \forall i, j = 1, 2, \cdots, 7.$$

$x\overline{x} = \overline{x}x = \sum_0^7 x_i^2 =: \ |x|^2$. So if $\mathbf{O} \ni x \neq 0$, $x^{-1} = \frac{\overline{x}}{|x|^2}$, i.e. $\mathbf{O}$ is a division algebra.

Let $a = e_0a_0 + e_1a_1 + e_2a_2 + e_3a_3 + e_4a_4 + e_5a_5 + e_6a_6 + e_7a_7$, $b = e_0b_0 + e_1b_1 + e_2b_2 + e_3b_3 + e_4b_4 + e_5b_5 + e_6b_6 + e_7b_7$. We consider the product ab (see [PY]): Denote an associated matrix to a by $A_8(a)$

$$A_8(a) = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{pmatrix},$$

then the product can be written in the form of matrix:

$$ab = (e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7) A_8(a) b'$$

where b' is the transfer matrix of $b = (b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7)$.

The matrix $A_8(a)$ gives a matrix "representation" of $\mathbf{O}$. But it is not really a representation, we have

$$A_8(a)A_8(b) \neq A_8(ab).$$

The multiplication of matrix is associative, the multiplication of octonion $\mathbf{O}$ is non-associative, so there is no matrix representation for $\mathbf{O}$.

Denote

$$C_7(a_I) = \begin{pmatrix} 0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_3 & 0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ -a_2 & a_1 & 0 & -a_7 & a_6 & -a_5 & a_4 \\ a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{pmatrix}.$$

It turns out that the matrix $C_7(a_I)$ gives the matrix form of the cross product in $\mathbf{R}^7$:

$$a_I \times b_I = C_7(a_I)b_I',$$

where $a_I = (a_1, a_2, a_3, a_4, a_5, a_6, a_7), b_I = (b_1, b_2, b_3, b_4, b_5, b_6, b_7)$. Now the expression $A_8(a)b$ becomes

$$A_8(a)b = \begin{pmatrix} 0 & -a_I \cdot b_I \\ b_0 a_I' & a_I \times b_I \end{pmatrix} + a_0 b'. \tag{1-22}$$

So the matrix $A_8(a)$ gives the concepts of the scalar product $(\alpha a_I')$, the inner product $(a_I \cdot b_I)$ and the cross product $(a_I \times b_I)$ in $\mathbf{R}^7$. Denote a, b by

$$a = a_0 + \vec{A}, \quad b = b_0 + \vec{B},$$

Then

$$ab = a_0 b_0 + a_0 \vec{B} + b_0 \vec{A} - \vec{A} \cdot \vec{B} + \vec{A} \times \vec{B},$$

And we can show that [cf. PY]

$$\begin{aligned} &(\vec{A} \times \vec{B}) \cdot \vec{A} = 0, \\ &(\vec{A} \times \vec{B}) \cdot \vec{B} = 0, \\ &\vec{A} \parallel \vec{B} \Longleftrightarrow \vec{A} \times \vec{B} = 0, \\ &\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}. \end{aligned}$$

Although the octonions do not satisfy the associative law, we still have

$$[\overline{x}, x, y] = 0 = [y, x, \overline{x}],$$

$$[x, x, y] = 0 = [y, x, x],$$

and

$$[x,y,z]=[y,z,x]=[z,x,y],$$
$$[x,z,y]=-[z,x,y],$$
$$[y,x,z]=-[y,z,x],$$
$$[y,x,z]=-[z,x,y],$$

for all $x,y,z\in\mathbf{O}$.

Also, the octonions obey some weakened associative laws, such as the so-called R. Moufang identities: [J] and [Sc].

$$(uvu)x=u(v(ux))$$
$$x(uvu)=((xu)v)u$$
$$u(xy)u=(ux)(yu).$$

3 O–analytic functions

Let Ω be an open connected set in $\mathbf{R}^8$, and f be the function

$$f:\Omega\longrightarrow\mathbf{O},$$

$$f(x)=\sum_0^7 e_k f_k(x).$$

The Dirac D-operator and its adjoint $\overline{D}$ are the first-order systems of differential operators on $C^\infty(\Omega,\mathbf{O})$ defined by

$$D=\sum_0^7 e_k\frac{\partial}{\partial x_k},\quad \overline{D}=\sum_0^7\overline{e_k}\frac{\partial}{\partial x_k}.$$

Definition 1 *A function $f\in C^\infty(\Omega,\mathbf{O})$ is said to be left (right)* **O**–*analytic on Ω when*

$$Df=\sum_0^7 e_k\frac{\partial f}{\partial x_k}=0\quad (fD=\sum_0^7\frac{\partial f}{\partial x_k}e_k=0).$$

Since

$$D\overline{D}=\overline{D}D=\triangle_8=\sum_0^7\frac{\partial^2}{\partial x_k^2}$$

the real-valued components of any left (right) **O**–analytic function are always harmonic. And there are big differences between the Clifford analytic functions and the **O**–analytic functions [L].

Examples

1) Each complex–valued analytic function is both left and right **O**–analytic function;

2) Each left (right) **H**–analytic function is left (right) **O**–analytic function;

3) If ϕ is any real-valued harmonic function on Ω, then $f=\overline{D}\phi$ is both left and right **O**–analytic function. Especially, for

$$\Gamma_8(x) = \frac{-1}{6\omega_8|x|^6},$$

$$\Phi(x) = \overline{D}\Gamma_8(x) = \frac{\overline{x}}{\omega_8|x|^8}$$

is both left and right **O**–analytic function everywhere away from the origin.

4) $q_k^n(x) = (x_k e_0 - x_0 e_k)^n$, $n \in N$, $(k = 1, 2, \cdots, 7)$ are left and right **O**–analytic functions.

Proposition 1 *Suppose* $F(x) = \sum_0^7 f_k e_k \in L^1(\mathbf{R}^8)$, $\frac{\partial F}{\partial x_k} \in L^1(\mathbf{R}^8)$, $(k = 1, 2, \cdots, 7)$, *then*

$$DF = 0 \Longleftrightarrow F = 0.$$

The equalities hold for almost all $x \in \mathbf{R}^8$.

Proof

$$F(x) \in L^1(\mathbf{R}^8) \Longleftrightarrow f_k(x) \in L^1(\mathbf{R}^8), \quad (k = 0, 1, \cdots, 7)$$

The matrix representation of $DF(x) = 0$ is

$$A_8(\partial)F = 0$$

where $\partial = (\frac{\partial}{\partial x_0}, \cdots, \frac{\partial}{\partial x_7})$ is the gradient operator. Taking the Fourier transform, we have

$$A_8(\xi)\begin{pmatrix} \widehat{f}_0 \\ \widehat{f}_1 \\ \vdots \\ \widehat{f}_7 \end{pmatrix} = 0.$$

From [PY]

$$detA_8(\xi) = (2\pi)^4(\xi_0^2 + \xi_1^2 + \cdots + \xi_7^2)^4.$$

So, $detA_8(\xi) \neq 0$ for $\xi \neq 0$. Hence, $\widehat{f}_k = 0$, i.e. $f_k(x) = 0$ $(k = 0, 1, \cdots, 7)$. This means that $F = 0$, a. e. $\mathbf{R}^8$.

The opposite is obvious. And the proof is completed.

Given a real-valued rational function $u(x, y)$, we can construct a **O**–analytic function whose real part is just $u(x, y)$ [LP1].

In order to develop the classical Hardy space theory in higher-dimensional space, E. M. Stein and G. Weiss introduced the following important concept:

Definition A ([SW1][SW2]) *A vector-valued function* $F = (u_1, u_2, \cdots, u_n)$ *on a domain* Ω *of* $\mathbf{R}^n$ *is called a S–W conjugate harmonic system if there exists a real-valued harmonic function* U *such that* $F = grad\, U$ *on* Ω.

Equivalently, $F = (u_1, u_2, \cdots, u_n)$ is called a S–W conjugate harmonic function, if it satisfies the so-called generalized Cauchy-Riemann equations

$$\sum_1^n \frac{\partial u_j}{\partial x_j} = 0$$

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \quad (i, j = 1, 2, \cdots, n)$$

Theorem A ([S] [SW1]) *If $F = (u_1, u_2, \cdots, u_n)$ is a S–W conjugate harmonic system, then $|F|^p$ is subharmonic if $p \geq \frac{n-2}{n-1}$, and $\frac{n-2}{n-1}$ is the best constant. Where $|F| = (\sum_1^n |u_j|^2)^{1/2}$ denotes the norm of F.*

The index $\frac{n-2}{n-1}$ is very important in the Hardy space theory (see [SW1] and [SW2]). For the **H**–analytic function, we have obtained the following results:

Proposition 2 [LP2] *Let*

$$D = \frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3},$$
$$f = f_0 + e_1 f_1 + e_2 f_2 + e_3 f_3.$$

Then $fD = Df = 0 \iff \overline{f} = (f_0, -f_1, -f_2, -f_3)$ is S-W conjugate harmonic system.

Corollary *Let Ω be any open set in R^4. Then whenever $p \geq 2/3$ and f is a left and right* **H**–*analytic function on Ω, $|f|^p$ is subharmonic on Ω, and $2/3$ is the best constant.*

For the **O**–analytic functions we only have the partial result.

Proposition 3 [LP2] *If $(f_0, f_1, \cdots, f_7)$ is a S-W conjugate harmonic system, then $\overline{f} = (f_0, -f_1, -f_2, \cdots, -f_7)$ is both left and right* **O**–*analytic function.*

Open problem: According [SW1], if $Df = 0$, there must be a p_0: $0 < p_0 < 1$, such that $|f|^p$ is a subharmonic whenever $p > p_0$. Our question is $p_0 =?$ in the octonion algebra **O**.

4 Proofs of the main results

Lemma 1 [GM] *Let ϕ, f be smooth scalar-valued functions on Ω.* Then $\forall j : 0 \leq j \leq 7$

$$\int_M (\phi\frac{\partial f}{\partial x_j} + \frac{\partial \phi}{\partial x_j} f)dV = \int_{\partial M} \phi f n_j dS.$$

Lemma 2 *Let ϕ, f be smooth* **O**– *valued functions*: $\phi = \sum_0^7 \phi_s e_s, f = \sum_0^7 f_s e_s$. *Then*

1) $\int_M (\phi(Df) - \sum_0^7 [e_s, D\phi_s, f] + (\phi D)f)dV = \int_{\partial M} \phi(nf)dS,$

2) $\int_M \Big(\phi(Df) + \sum_t [\phi, Df_t, e_t] + (\phi D)f\Big)dV = \int_{\partial M} (\phi n)f dS,$

3) $\int_M (fD)\phi + \sum_s [f, D\phi_s, e_s] + f(D\phi)]dV = \int_{\partial M} (fd\sigma)\phi,$

4) $\int_M ((fD)\phi - \sum_t [e_t, Df_t, \phi] - f(D\phi))dV = \int_{\partial M} f(d\sigma\phi).$

Proof For any s, t and j, applying Lemma 1, we have

$$\int_M (\phi_s \frac{\partial f_t}{\partial x_j} + \frac{\partial \phi_s}{\partial x_j} f_t)dV = \int_{\partial M} \phi_s f_t n_j dS.$$

Multiplying e_j on both side and taking summation for j, we have

$$\int_M [(\phi_s (\sum_j e_j \frac{\partial f_t}{\partial x_j}) + (\sum_j e_j \frac{\partial \phi_s}{\partial x_j}) f_t] dV = \int_{\partial M} \phi_s f_t n dS.$$

Multiplying e_t from the right and taking summation for t, we have

$$\int_M [(\phi_s (\sum_j e_j \frac{\partial f}{\partial x_j}) + (\sum_j e_j \frac{\partial \phi_s}{\partial x_j}) f] dV = \int_{\partial M} \phi_s n f dS,$$

i.e.

$$\int_M [\phi_s(Df) + (\phi_s D)f]dV = \int_{\partial M} \phi_s n f dS.$$

Multiplying e_s from the left and taking summation for s, finally we have

$$\int_M [\phi(Df) + \sum_s e_s((\phi_s D)f)]dV = \int_{\partial M} \phi(nf)dS.$$

Note that

$$e_s((\phi_s D)f) = (e_s(\phi_s D))f - [e_s, \phi_s D, f].$$

Then we obtain

$$\int_M (\phi(Df) + (\phi D)f - \sum_0^7 [e_s, D\phi_s, f])dV = \int_{\partial M} \phi(nf)dS.$$

This finishes the proof of 1). By changing the order of e_t, e_s, and, the order of "left" and "right", we obtain 2), 3) and 4).

Proof of Theorem 1 Suppose $z \in M^0$, i.e. z is an interior point of M, and for all sufficiently small $\epsilon > 0$, let $M_\epsilon = M \backslash B_\epsilon(z)$ where $B_\epsilon(z)$ is the open ball of radius ϵ centered at z. Denote that

$$\Phi(x) = \overline{D}\Gamma(x) = \frac{\overline{x}}{\omega_8 |x|^8},$$

then

$$\Phi(x - z) = \frac{\overline{x} - \overline{z}}{\omega_8 |x - z|^8}$$

is both left and right $\mathbf{O}$–analytic function in M_ϵ. Applying 1) of lemma 2, we thus obtain

$$\begin{aligned}\int_{M_\epsilon} \{\Phi(x - z)(Df) - \sum_0^7 [e_s, D\Phi_s, f]\}dV &= \int_{M_\epsilon} \Phi(x - z)(nf)dS \\ &= (\int_{\partial M} - \int_{\Sigma_{\epsilon(z)}})\Phi(x - z)(nf)dS\end{aligned}$$

where $\Sigma_\epsilon(z)$ is the sphere of radius ϵ centered at z. Noticing that on $\Sigma_\epsilon(z)$, $\Phi(x - z)(nf) = \frac{(\overline{x}-\overline{z})}{\omega_8|x-z|^8}(\frac{x-z}{|x-z|}f(x)) = \frac{1}{\omega_8}(\frac{\overline{x}-\overline{z}}{|x-z|^8}\frac{x-z}{|x-z|})f(x) = \frac{1}{\omega_8|x-z|^7}f(x)$, we have

$$\int_{\Sigma_{\epsilon(z)}} \Phi(x - z)(nf)dS = \frac{1}{\omega_8 \epsilon^7} \int_{\Sigma_{\epsilon(z)}} f(x)dS \longrightarrow f(z)$$

as $\epsilon \longrightarrow 0$. i.e.

$$f(z) = \int_{\partial M} \Phi(x - z)(d\sigma(x)f(x)) + \int_M \sum_0^7 [e_s, D\Phi_s, f]dV.$$

We shall prove that $\sum_0^7[e_s, D\Phi_s, f] = \sum_0^7[e_s, \sum_0^7 e_k\frac{\partial\Phi_s}{\partial x_k}, f] = \sum_{s=0}^7\sum_{k=0}^7[e_s, e_k\frac{\partial\Phi_s}{\partial x_k}, f] = 0$. In fact, let $x = x_0 + e_1x_1 + \cdots + e_7x_7, \quad z = z_0 + e_1z_1 + \cdots + e_7z_7$, then

$$\begin{aligned}
\Phi_0 &= \frac{x_0 - z_0}{\omega_8|x-z|^8},\\
\frac{\partial\Phi_0}{\partial x_0}, &= \frac{|x-z|^2 - 8(x_0-z_0)^2}{\omega_8|x-z|^{10}},\\
\Phi_s &= \frac{-x_s + z_s}{\omega_8|x-z|^8},\\
\frac{\partial\Phi_s}{\partial x_s} &= \frac{-|x-z|^2 + 8(x_s-z_s)^2}{\omega_8|x-z|^{10}},\\
\frac{\partial\Phi_s}{\partial x_j} &= \frac{8(x_s-z_s)(x_j-z_j)}{\omega_8|x-z|^{10}}, \quad j \neq s\\
&(j, s = 1, 2, \cdots, 7)
\end{aligned}$$

While

$$[e_s, \Phi_s D, f] = [e_s, \frac{\partial\Phi_s}{\partial x_o}e_o + \cdots + \frac{\partial\Phi_s}{\partial x_s}e_s + \cdots + \frac{\partial\Phi_s}{\partial x_7}e_7, f].$$

Since $[e_s, \frac{\partial\Phi_s}{\partial x_0}e_0, f] = 0, \quad [e_s, \frac{\partial\Phi_s}{\partial x_s}e_s, f] = 0, \quad (s = 1, 2, \cdots, 7)$, the terms $\frac{\partial\Phi_s}{\partial x_o}e_o$ and $\frac{\partial\Phi_s}{\partial x_s}e_s$ can be omitted.

Thus we have

$$[e_s, \Phi_s D, f] = [e_s, \frac{\partial\Phi_s}{\partial x_1}e_1 + \cdots + \frac{\partial\Phi_s}{\partial x_{s-1}}e_{s-1} + 0 + \frac{\partial\Phi_s}{\partial x_{s+1}}e_{s+1} + \cdots + \frac{\partial\Phi_s}{\partial x_s}e_7, f].$$

Replacing the term "0" by

$$\frac{8(x_s-z_s)(x_s-z_s)}{\omega_8|x-z|^{10}}e_s,$$

we obtain

$$\begin{aligned}
[e_s, \Phi_s D, f] =& \frac{8(x_s-z_s)}{\omega_8|x-z|^{10}}[e_s, \sum_1^7(x_j-z_j)e_j, f]\\
=& \frac{8}{\omega_8|x-z|^{10}}[(x_s-z_s)e_s, \sum_1^7(x_j-z_j)e_j, f].
\end{aligned}$$

By the alternative property of **O** and $[e_0, \Phi_0 D, f] = 0$, we have

$$\sum_0^7[e_s, D\Phi_s, f] = \frac{8}{\omega_8|x-z|^{10}}[\sum_1^7(x_j-z_j)e_j, \sum_1^7(x_j-z_j)e_j, f] = 0,$$

i.e.

$$f(z) = \int_{\partial M}\Phi(x-z)(d\sigma(x)f(x)).$$

For exterior points it is sufficient to repeat the previous proof without bothering to exclude $B_\epsilon(z)$.

This completes the proof of Theorem 1.

Using 2) of lemma 2) and with the upper part of the above proof repeated, we obtain the proof of Theorem 2.

In order to prove theorem 3, we need the following lemma:

Lemma 3 (Cauchy integral formula outside a ball) *Let f be left $\mathbf{O}$–analytic in $\mathbf{R}^8 \setminus \overline{B}(0,R)$ with $\lim_{x\to\infty} f(x) = \lambda$, Then for each $x \in \mathbf{R}^8 \setminus \overline{B}(0,R)$,*

$$f(x) = \lambda - \int_{\partial B(0,R')} \Phi(y-x)(d\sigma f(x))$$

where R' is suitably chosen such that $R < R' < |x|$.

The proof is similar with [BDS], so it is omitted.

Proof of Theorem 3 Since

$$\begin{aligned}
\frac{\partial f(x)}{\partial x_i} &= \int_{\partial M} \frac{\partial \Phi(y-x)}{\partial x_i}(d\sigma(y)f(y))\\
e_i\frac{\partial f(x)}{\partial x_i} &= \int_{\partial M} e_i(\frac{\partial \Phi(y-x)}{\partial x_i}(d\sigma(y)f(y)))\\
&= \int_{\partial M}(e_i\frac{\partial \Phi(y-x)}{\partial x_i})(d\sigma(y)f(y)) - [e_i, \frac{\partial \Phi(y-x)}{\partial x_i}, d\sigma f]\\
D\Phi(y-x) &= 0,
\end{aligned}$$

so

$$\begin{aligned}
Df &= \int_{\partial M}(D\Phi(y-x))(d\sigma f) - \int_{\partial M}\sum_0^7[e_i, \frac{\partial \Phi(y-x)}{\partial x_i}, d\sigma f]\\
&= -\int_{\partial M}\sum_0^7[e_i, \frac{\partial \Phi(y-x)}{\partial x_i}, d\sigma f].
\end{aligned}$$

Let $\Phi(y-x) = \sum_0^7 e_s\Phi_s$,then

$$\sum_0^7[e_i, \frac{\partial \Phi(y-x)}{\partial x_i}, d\sigma f] = \sum_0^7\sum_0^7[e_i, \sum_0^7 e_s\frac{\partial \Phi_s}{\partial x_i}, d\sigma f].$$

where

$$\begin{aligned}
\Phi_0 &= \frac{y_0 - x_0}{\omega_8|x-y|^8},\\
\Phi_s &= \frac{x_s - y_s}{\omega_8|x-y|^8}, \quad (s = 1, 2, \cdots, 7)\\
\frac{\partial \Phi_s}{\partial x_i} &= \frac{8(y_s - x_s)(y_i - x_i)}{\omega_8|x-y|^{10}}, \quad s \neq i\\
\frac{\partial \Phi_s}{\partial x_s} &= \frac{-|x-y|^2 + 8(x_s - y_s)^2}{\omega_8|x-y|^{10}},\\
&[e_0, \frac{\partial \Phi(y-x)}{\partial x_i}, d\sigma f] = 0,\\
&[e_i, \frac{\partial \Phi_i}{\partial x_i}e_i, d\sigma f] = 0,\\
&(i = 0, 1, 2, \cdots, 7)
\end{aligned}$$

Taking $\frac{8(y_i-x_i)(y_i-x_i)}{\omega_8|x-y|^{10}}$ in place of $\frac{\partial g_i}{\partial x_i}$ then we have

$$\begin{aligned}[e_i, \frac{\partial \Phi(y-x)}{\partial x_i}, d\sigma f] &= [e_i, \sum_{s=0}^{7} \frac{8e_s(y_s-x_s)(y_i-x_i)}{\omega_8|x-y|^{10}}, d\sigma f] \\ &= [(y_i-x_i)e_i, \sum_{s=1}^{7} \frac{8e_s(y_s-x_s)}{\omega_8|x-y|^{10}}, d\sigma f],\end{aligned}$$

hence

$$\begin{aligned}&\sum_0^7 [e_i, \frac{\partial \Phi(y-x)}{\partial x_i}, d\sigma f] \\ &= \frac{8}{\omega_8|x-y|^{10}}[\sum_1^7 (y_i-x_i)e_i, \sum_1^7 (y_s-x_s)e_s, d\sigma f] \\ &= 0,\end{aligned}$$

so, $Df = 0, \forall x \in M^0$, i.e. f is left $\mathbf{O}$–analytic in M^0.

This completes the proof of Theorem 3.

Similarly, we also obtain the corresponding Cauchy integral formulas, Cauchy theorems and the inverse theorems of the Cauchy integral formulas for right $\mathbf{O}$–analytic functions:

Theorem 4 *M, Ω are as above, if $fD = 0$, $x \in \Omega$, then*

$$\frac{1}{\omega_8}\int_{\partial M}(fd\sigma)\frac{\overline{x}-\overline{z}}{|x-z|^8} = \begin{cases} f(z), & \text{if} \quad z \in M^0, \\ 0, & \text{if} \quad z \in \Omega \backslash M. \end{cases}$$

Theorem 5 *M, Ω are as above, if $fD = 0$, $x \in \Omega$, then*

$$\frac{1}{\omega_8}\int_{\partial M} f(x)(d\sigma\frac{\overline{x}-\overline{z}}{|x-z|^8}) = \begin{cases} f(z) - \int_M \sum_t [e_t, Df_t, \phi]dV, & \text{if} \quad z \in M^0, \\ -\int_M \sum_t [e_t, Df_t, \phi]dV, & \text{if} \quad z \in \Omega \backslash M. \end{cases}$$

Furthermore, by taking $\phi = 1$ in the lemma 2, we thus get the Cauchy theorems:

Theorem 6 *Let M be a compact, 8-dimensional, oriented C^∞ manifold in Ω. Then*

$$\int_{\partial M}\omega = \int_{\partial M} n(x)f(x)dS(x) = 0$$

for any function f which is left $\mathbf{O}$–analytic in Ω.

Theorem 7 *Let M be a compact, 8-dimensional, oriented C^∞ manifold in Ω. Then*

$$\int_{\partial M}\nu = \int_{\partial M} f(x)n(x)dS(x) = 0$$

for any function f which is right $\mathbf{O}$–analytic in Ω.

Theorem 8 *For continuous function $f : \partial M \to \mathbf{O}$, if for each $x \in M^0$*

$$\begin{aligned}f(x) &= \int_{\partial M}(f(y)d\sigma(y))\Phi(y-x) \\ &= \int_{\partial M}(n(y)f(y))\Phi(y-x)dS(y).\end{aligned}$$

Then f is right $\mathbf{O}$*–analytic in* M*.*

At the end of this section, we would like to talk about an interesting fact. If f is the S-W conjugate harmonic system, the calculation of the associators would be simple. For then

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad (i, \quad j = 1, 2, \cdots, 7)$$

We have

$$\begin{aligned}
\sum_{t=0}^{7}[\phi, Df_t, e_t] &= \sum_{t=0}^{7}\sum_{j=0}^{7}[\phi, e_j, e_t]\frac{\partial f_t}{\partial x_j} \\
&= \sum_{t=1}^{7}\sum_{j=1}^{7}[\phi, e_j, e_t]\frac{\partial f_t}{\partial x_j} \\
&= \sum_{t\neq j, t,j=1}^{7} [\phi, e_j, e_t]\frac{\partial f_t}{\partial x_j} \\
&= \sum_{1\leq t<j\leq 7} ([\phi, e_j, e_t]\frac{\partial f_t}{\partial x_j} + [\phi, e_t, e_j]\frac{\partial f_j}{\partial x_t}) \\
&= \sum_{1\leq t<j\leq 7} ([\phi, e_j, e_t] + [\phi, e_t, e_j])\frac{\partial f_j}{\partial x_t} \\
&= 0.
\end{aligned}$$

So, we get our general results: M, Ω and Φ are as above, let z be a interior point of M. Then, if $Df = 0$,

$$f(z) = \int_{\partial M} \Phi(d\sigma f) = \int_{\partial M} (\Phi d\sigma) f - \int_M \sum_{t=0}^{7}[\Phi, Df_t, e_t]dV,$$

if $fD = 0$,

$$f(z) = \int_{\partial M} (f d\sigma)\Phi = \int_{\partial M} f(d\sigma\Phi) + \int_M \sum_{t=0}^{7}[e_t, Df_t, \Phi]dV,$$

and if $\overline{f}$ is a conjugate harmonic system,

$$f(z) = \int_{\partial M} \Phi d\sigma f = \int_{\partial M} f d\sigma\Phi.$$

5 Some applications

Theorem 9 (Mean value theorem) *Suppose that* Ω *is an open connected set in* R^8*,* $B_r(z_0)$ *is the open ball of radius* r *centered at* z_0*,* $B_r(z_0) \subset \Omega$*. If* $Df = 0$*, then*

$$f(z_0) = \frac{1}{|B_r(z_0)|}\int_{B_r(z_0)} f(x)dV(x)$$

where $|B_r(z_0)|$ *is the volume of* $B_r(z_0)$*.*

Proof By the Cauchy integral formula, we have

$$\begin{aligned} f(z_0) &= \int_{\partial B_r(z_0)} \Phi(x-z)(n(x)f(x))dS(x) \\ &= \frac{1}{\omega_8 r^8}\int_{\partial B_r(z_0)} (\overline{x}-\overline{z}_0)(\frac{\overline{x}-\overline{z}_0}{|x-z|}f(x))dS(x). \end{aligned}$$

Then, applying Lemma 2 of the last section, we have

$$f(z) = \frac{1}{\omega_8 r^8}\int_{B_r(z_0)} ((\overline{x}-\overline{z}_0)D)f - \sum_0^7 [e_s, D\Phi_s, f])dV$$

where

$$\begin{aligned} &\phi_s = -x_s - a_s, \quad (s = 1, 2, \cdots, 7) \\ &\phi_0 = x_0 - a_0, \\ &z_0 = \sum_0^7 a_s e_s. \end{aligned}$$

Note that

$$\begin{aligned} (\overline{x}-\overline{z}_0)D &= \sum_0^7 \frac{\partial(\overline{x}-\overline{z}_0)}{\partial x_j} e_j = 8 \\ D\Phi_s &= \sum_0^7 \frac{\partial \Phi_s}{\partial x_j} e_j = -e_s, \end{aligned}$$

so, $\sum_0^7 [e_s, D\Phi_s, f] = 0$. Thus we have

$$\begin{aligned} f(z_0) &= \frac{1}{\omega_8 r^8}\int_{B_r(z_0)} 8f(x)dV(x) \\ &= \frac{1}{|B_r(z_0)|}\int_{B_r(z_0)} f(x)dV(x). \end{aligned}$$

Remark Same result still holds if $fD = 0$.

Theorem 10 (Maximum modules theorem) *Let f be a left* **O***–analytic function in the open and connected set Ω. If there exists a point $\omega_0 \in \Omega$, such that $|f(w)| \leq |f(w_0)|, \ \ \forall \omega \in \Omega, \ \ \omega_0 \in \Omega$, then f must be a constant function in Ω.*

Corollary *Under the assumption of Theorem 10, and $f \in C(\overline{\Omega})$. Then*

$$\sup_{x\in\Omega} |f(\omega)| = \sup_{x\in\partial\Omega} |f(x)|.$$

Theorem 11 (Weierstrass type theorem) *Let $\{f_j\}_{j\in N}$ be a sequence of left* **O***–analytic functions in Ω. If for each compact set $K \subset \Omega$ and $\forall \epsilon > 0$, there exists a natural number $N(\epsilon, K)$, such that*

$$\sup_{x\in k} |f_i(x) - f_j(x)| < \epsilon$$

whenever $i, j > N(\epsilon, K)$. Then there exists a function f in Ω such that

(i) $Df = 0$;

(ii) *the sequence* $\{\partial^\beta f_j\}_{j\in N}$ *converges uniformly on the compact subsets of* Ω *to* $\partial^\beta f$, *for any multi-index* $\beta \in N^8$.

The proofs of the above theorems are similar to [BDF], so they are omitted.

There are lots of other applications of the Cauchy integral formulas on the octonions **O**. In the coming papers, we will discuss the Taylor series [LP3], the three-line theorems which are closely related with the interpolation theory [JP], the Laurent series, the Mittag-Leffler theorem and the Liouville theorem etc.

Remark After we finished this work, Professor J. Ryan and Professor M. Shapiro informed us in a Beijing conference about the works of Paolo Dentoni and Michele Sce [PM], T. Dray and C. A. Manogue [DM1-DM2] [MD], and K. Nono [N]. We checked and found that some results of this paper have obtained in [PM], [N]. But the methods we use are quite different. Our methods are more elementary and can be used furthermore to develop octonion analysis.

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References

[BDS] F. Brackx, R.Delanghe, and F. Sommen, "Clifford analysis", Pitman Advanced Publishing Program, 1982.

[C] W. K. Clifford, *Application of Grassman's extensive algebra,* Amer. J. of Math., **Vol.1**(1878), 350-358.

[D] R. Delanghe, *On regular-analytic functions with values in Clifford algebra,* Math. Ann., **185**(1970), 91-111.

[De] C. A. Deavours, *The Quaternion calculus,* Amer. Math. Monthly, **80**(1973), 307-30.

[DGT] R. Dündarer, F. Gürsey and C-H, Tze, *Self-duality and octonionic analyticity of* S^7*–valued antisymmetric fields in eight dimensions,* Nuclear Physics, **266**(1986), 440–450.

[DM1] T. Dray, C. A. Manogue, *The octonion eigenvalue problem,* Adv. Appl. Clifford Algebras **8**(1998), 341-364.

[DM2] T. Dray, C. A. Manogue, *Finding octonionic eigenvector using Mathematica,* Compute. Phys. Comm.**115**(1998), 536-547.

[DSS] R. Delanghe, F. Sommen, and V. Soucek, "Clifford algebra and spinor-valued functions, a function theory for the Dirac operator", Kluwer Academic Publishers, London, 1992.

[F1] R. Fueter, *Die Funktionentheorie der Differentialgleichungen* $\Delta u = 0$ *und* $\Delta\Delta u = 0$ *mit vier reallen Variablen,* Comm. Math. Helv., **7**(1934), 307–30.

[F2] R. Fueter, *Zur Theotie der regulären Funktionen einer Quaternionenvariablem,* Monat. fürMath. und Phys. **43**(1935), 69-74.

[F3] R. Fueter, C. R. Congress Int. Math. Olso **1** (1936), 75.

[GM] J. E. Gilbert, and M. A. M. Murray, "Clifford algebras and Dirac operators in harmonic analysis",Cambridge University Press, Cambridge, 1991.

[Hab] K.Habetha, *Eine Memerkung zur Funktionentheorie in Algebra, (A Note to function theory in algebras),* Lecture Notes in Math., **561**(1976), 502-509, Springer-Verlag.

[J] N. Jacobson, "Basic algebras I", W.H. Freeman and Company, New York, 1985, Second Edition.

[JP] Savente Janson and Jaak Peetre, *Harmonic interpolation,* Lecture Notes in Math., **1070**, 92-124, Springer-Verlag, 1984.

[L] Xingmin Li, *On two questions in Clifford analysis and octonion analysis,* Lecture Notes in Pure and Applied mathematics, vol. 214, Finite or infinite dimensional complex analysis, edited by Joji Kajiwara, Zhong li, Kwang Ho Shon, 2000, 293-299, Marcel Dekker.

[LP1] Xingmin Li, Zhao kai and Lizhong Peng, *Characterization of octonionic analytic functions,* to appear.

[LP2] Xingmin Li and Lizhong Peng, *On Stein-Weiss conjugate harmonic function and octonion analytic function,* Approx. Theory and its Appl. **16**(2000), 28-36.

[LP3] Xingmin Li and Lizhong Peng, Taylor series and orthogonality of the octonion analytic functions, Acta of Math. Scientia, **21**(2001), Ser.**B**, 323-330.

[MD] C. A. Manogue, T. Dray, *Octonionic möbius transformations,* Modern Physics Letters A,**14**(1999), 1243-1255.

[N] K. Nono, *On the octonionic linearization of Laplacian and octonionic function theory,* Bull. Fukuoka Univ. Ed., Part III, **37**(1988), 1-15.

[PM] P. Dentoni, M. Sce., *Funzioni regolari nell'algebra di Cayley,* Rend. Sem. Mat. Univ. Padova **50**(1973), 251-267.

[PY] Lizhong Peng and Lei Yang, *The curl in 7-dimensional space and its applications,* Approximation theory and its applications, **15**(1999), 66-80

[S] E.M.Stein, "Singular integrals and differentiability properties of functions", Princeton Univ. Press, 1970.

[Sc] R. D. Schafer, "An introduction to non-associative algibras," Academic Press, New York, 1966.

[Su] A. Sudbery, *Quaterinonic analysis,* Math. Proc. Camb. Phil. Soc., **85**(1979), 199-225.

[SW1] E. M. Stein and G. Weiss, *On the theory of harmonic functions of several variables, 1. The theory of H_p spaces,* Acta Math., **103**(1960), 26-62.

[SW2] E. M. Stein and G. Weiss, "Introduction to Fourier analysis on Euclidean spaces", Princeton Univ. Press, (1971).

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