Nonexistence of global solutions to a class of nonlinear wave equations with dynamic boundary conditions

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Abstract

We consider the problem $u_{tt}+\Delta^2 u+\delta u_t-\varphi(\int_\Omega |\nabla u|^2 dx)\Delta u = f(u)$, posed in $\Omega\times(0,T)$, with dynamical boundary conditions. Here $\Omega \subset \mathbb{R}^N$ is a an open smooth bounded domain. We prove, in certain conditions on f and φ that there is absence of global solutions. The method of proof relies on an argument of concavity.

1 Introduction and main result

The aim of the present note is to discuss some nonexistence result of global solutions to the problem

$$\begin{cases} u_{tt} + \Delta^2 u + \delta u_t - \varphi\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(u), & \text{on } \Omega \times (0,T), \\ u = 0, \quad \Delta u + p(\sigma)\frac{\partial u_t}{\partial \nu} = 0, & \text{in } \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.2}$$

for any $x \in \Omega$, where $\Omega \subset \mathbb{R}^N$ is an open smooth bounded domain, $\delta > 0$, $\frac{\partial}{\partial \nu}$ is the normal derivative on $\partial\Omega, p \geq 0$ is a smooth function defined on $\partial\Omega, f, \varphi, u_0$ and u_1 are given functions.

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When Equation $(1.1)_1$ does not includes the term $\Delta^2 u$ and $\delta = 0$, the problem describes, in one dimension, the non-linear vibrations of elastic string. The naturel generalization is given by

$$u_{tt} - \varphi\left(\int_{\mathbb{R}} |\nabla u|^2 dx\right) \Delta u = 0. \tag{1.3}$$

This model was studied by Pohozaev [9] in the case where φ is a real C^1 function defined for nonnegative real satisfying $\varphi \geq a_0 > 0$. The author obtained existence and global solutions for analytic initial data. Later Lions [7] formulated the Pohozaev's result in abstract context.

Returning to our problem, Vasconcellos and Teixera [10] proved the existence and uniqueness of global solutions to

$$\begin{cases} u_{tt} + \Delta^2 u - \varphi\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u + g(u_t) = 0, & \text{on } \Omega \times (0,T), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{in } \partial\Omega \times (0,T), \end{cases} \tag{1.4}$$

where $N \leq 3, \varphi \geq 0$ is a continuous differentiable real and g is a continuous nondecreasing real function.

Our intention here is to prove the non global character of solutions to Problem (1.1) with dynamical boundary condition. Among recent result in this direction we mention the paper by Kirane and Tatar [4]. The authors studied the blow–up phenomena for the problem

$$u_{tt} + \Delta^2 u = f(u) \tag{1.5}$$

with dynamic boundary conditions. They proved, under certain condition on f, that the Problem has no global solutions. The work of Ono [8] deals with

$$u_{tt} + \varphi(\|A^{1/2}u\|_2)Au + \delta u = |u|^{\alpha} u,$$

where $A = -\Delta, \varphi(s) = a + bs^{\gamma}, a \geq 0, b \geq 0, a + b > 0, \delta \geq 0$ and $\gamma > 0$. He proved that for $\alpha > 2\gamma$ the local solution is not global.

In our present work we impose on $\varphi \in C^0(0, +\infty)$ the condition

$$c \int_0^t \varphi(s) ds \geq t\varphi(t), \quad c \geq 1, \tag{1.6}$$

like

$$\varphi(s) = a + bs^{\beta}, \quad \forall\ s > 0,$$

where $-1 < \beta \leq c - 1$. Using the concavity method, we prove that the blow –up occurs for certain initial data. The technique used is to prove that the function Φ, defined by,

$$\Phi(t) = \frac{1}{2}\int_{\Omega} u^2(x,t) dx + \frac{1}{2}\int_0^t \int_{\partial\Omega} p(\sigma) \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma + \frac{1}{2}\Lambda(t+\tau)^2 + M$$

satisfies the concavity argument [6]. The function p is positive and continuous.

Concerning the function f we assume

$$f \in C^0(\mathbb{R}, \mathbb{R}_+), \quad rf(r) \geq 10(1+\gamma)F(r), \quad \gamma > 0, \tag{1.7}$$

for all $r \in \mathbb{R}$, where

$$F(r) := \int_0^r f(s)ds,$$

Without loss of generality we may assume that $\delta = 1$. Let λ_1 be the first eigenvalue of the operator Δ^2 in $H^2(\Omega) \cap H_0^1(\Omega)$; that is

$$\int_\Omega (\Delta v)^2 dx \geq \lambda_1 \int_\Omega v^2 dx,$$

for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Put

$$\mathcal{F}(u_0, u_1) = \frac{1}{2}\int_\Omega u_1^2 dx + \frac{1}{2}\int_\Omega |\Delta u_0|^2 dx + \frac{1}{2}\tilde{\varphi}(\int_\Omega |\nabla u_0|^2 dx) - \int_\Omega F(u_0)dx, \tag{1.8}$$

where

$$\tilde{\varphi}(s) := \int_0^s \varphi(r)dr.$$

The main result is the following.

Theorem 1.1. *Assume that* $u_1 \not\equiv 0$ *and*

$$\mathcal{F}(u_0, u_1) \leq 0. \tag{1.9}$$

Let f *and* φ *satisfy conditions* (1.6),(1.7) *where*

$$\gamma \geq \max\left\{\frac{1}{20\lambda_1} - \frac{4}{5}, \frac{c}{5} - 1\right\}. \tag{1.10}$$

Then Problem (1.1) *has no global solution in* $C^2((0,+\infty), H^2(\Omega) \cap H_0^1(\Omega))$.

Remark 1.1. It is noticing that condition (1.9) may be not satisfied by a small initial data. For example consider $\varphi(t) = t^\beta, -1 < \beta < c-1, f(u) = |u|^q u, q > 0$. We have, for any $\sigma \in \mathbb{R}_+$,

$$\mathcal{F}(\sigma u_0, \sigma u_1) = \sigma^2(A + \sigma^{2\beta}B - \sigma^q C) := \sigma^2 L(\sigma),$$

where

$$A = \frac{1}{2}\left(\int_\Omega u_1^2 dx + \int_\Omega (\Delta u_0)^2 dx\right), \qquad B = \frac{1}{a+1}\left(\int_\Omega |\nabla u_0|^2\right)^{\beta+1},$$

and

$$C = \frac{1}{q+2}\int_\Omega |u_0|^{q+2} dx.$$

Therefore $\lim_{\sigma \to 0} L(\sigma) = +\infty$. Hence condition (1.9) is violated if σ is small enough.

2 Proof

We assume that the problem has a global solution $u \in C^2(\mathbb{R}_+, H_0^1(\Omega) \cap H^2(\Omega))$. The proof will be done by applying the concavity argument [6] as follows.

Lemma 2.1. *Let $\Phi \in C^2$ be a nonnegative function for which there exists a constant $\gamma > 0$ such that*

$$\Phi''\Phi \geq (1+\gamma)(\Phi')^2. \tag{2.1}$$

Assume

$$\Phi(0) > 0, \quad \Phi'(0) > 0, \tag{2.2}$$

then there exists $t_0 \leq \dfrac{\Phi(0)}{\gamma\Phi'(0)}$, such that

$$\lim_{t \to t_0^-} \Phi(t) = +\infty. \tag{2.3}$$

The proof of this Lemma is very simple. We deduce from (2.1) that $(\Phi^{-\gamma})'' \leq 0$ as long as $\Phi > 0$. Hence

$$\Phi^{-\gamma}(t) \leq \Phi^{-\gamma}(0) + (\Phi^{-\gamma})'(0)t.$$

This implies that $\Phi^{-\gamma}$ must cross the t axis, thus we see that Φ blows-up in finite time. First we prove the following.

Lemma 2.2. *Assume hypothesis* (1.6)–(1.7) *are satisfied and $u_1 \not\equiv 0$. Then*

$$u(.,t) \not\equiv 0, \tag{2.4}$$

for any $t \geq 0$.

Proof. First we have $u_0 \not\equiv 0$. Otherwise we deduce from (1.9) that $u_1 \equiv 0$. This is impossible. Assume, on the contrary that there exists $t_0 > 0$ such that

$$u(x, t_0) = 0, \text{ for any } x \in \Omega.$$

We multiply Equation $(1.1)_1$ by u_t and integrate over $\Omega \times (0, t_0)$, one sees

$$\begin{aligned} \mathcal{F}(u_0, u_1) &= \frac{1}{2}\int_\Omega u_t^2(x, t_0)dx + \int_0^{t_0}\int_\Omega u_t^2(s)dxds + \frac{1}{2}\int_\Omega |\Delta u|^2(x, t_0)dx \\ &+ \int_0^{t_0}\int_{\partial\Omega} p(\sigma)\left(\frac{\partial u_t}{\partial \nu}\right)^2 d\sigma + \frac{1}{2}\tilde{\varphi}\left(\int_\Omega |\nabla u|^2(x, t_0)dx\right) - \int_\Omega F(u)(x, t_0)dx, \end{aligned} \tag{2.5}$$

where $\mathcal{F}$ is given by (1.8). Therefore

$$\mathcal{F}(u_0, u_1) = \frac{1}{2}\int_\Omega u_t^2(x, t_0)dx + \int_0^{t_0}\int_\Omega u_t^2(s)dxds + \int_0^{t_0}\int_{\partial\Omega} p(\sigma)\left(\frac{\partial u_t}{\partial \nu}\right)^2 d\sigma \geq 0,$$

and then $\mathcal{F}(u_0, u_1) = 0$ thanks to (1.9). The latter implies that

$$u_t(., t) \equiv 0,$$

for any $t \leq t_0$. Hence $u_1 \equiv 0$. A contradiction. This ends the proof. ∎

Now we are in force to prove the theorem. We apply Lemma 2.1 by choosing the following function

$$\Phi(t) = \frac{1}{2}\int_\Omega u^2(x,t)dx + \frac{1}{2}\int_0^t\int_{\partial\Omega} p(\sigma)\left(\frac{\partial u}{\partial\nu}\right)^2 d\sigma + \frac{1}{2}\Lambda(t+\tau)^2 + M,$$

where τ, Λ and M are nonnegative parameters. Set, for T large

$$N(T) = \inf_{0\le t\le T}\int_\Omega u^2(x,t)dx > 0.$$

We shall seek

$$E(t) = \Phi^{"}\Phi - (1+\gamma)(\Phi^{'})^2 \ge 0, \tag{2.6}$$

for $\gamma > 0$ given by (1.7). Differentiating Φ once and twice, we infer

$$\Phi^{'}(t) = \int_\Omega uu_t dx + \frac{1}{2}\int_{\partial\Omega} p(\sigma)\left(\frac{\partial u}{\partial\nu}\right)^2 d\sigma + \Lambda(t+\tau) \tag{2.7}$$

and

$$\Phi''(t) = \int_\Omega u_t^2 dx + \int_\Omega u_{tt}u dx + \int_{\partial\Omega} p(\sigma)\frac{\partial u_t}{\partial\nu}\frac{\partial u}{\partial\nu}d\sigma + \Lambda. \tag{2.8}$$

Set

$$H(t) = \int_\Omega uu_{tt}dx - b\int_\Omega u_t^2 dx + \int_{\partial\Omega} p(\sigma)\frac{\partial u_t}{\partial\nu}\frac{\partial u}{\partial\nu}d\sigma,$$

it follows from (2.7), (2.8),

$$\begin{aligned}
E(t) \;=\;& \Phi(t)H(t) + \Phi(t)\Lambda + \frac{a}{2}\int_\Omega u_t^2 dx\int_\Omega u^2 dx \\
&+\frac{a}{2}\int_\Omega u_t^2 dx\int_0^t\int_{\partial\Omega} p(\sigma)\left(\frac{\partial u}{\partial\nu}\right)^2 d\sigma + \frac{a}{2}\Lambda(t+\tau)^2\int_\Omega u_t^2 dx \\
&-(1+\gamma)\left(\int_\Omega uu_t dx\right)^2 - 2(1+\gamma)\Lambda(t+\tau)\int_\Omega uu_t dx \\
&-(1+\gamma)\Lambda^2(t+\tau)^2 - \frac{(1+\gamma)}{4}\left[\int_{\partial\Omega} p(\sigma)(\frac{\partial u}{\partial\nu})^2 d\sigma\right]^2 \\
&-(1+\gamma)\int_\Omega uu_t dx\int_{\partial\Omega} p(\sigma)(\frac{\partial u}{\partial\nu})^2 d\sigma - (1+\gamma)\Lambda(t+\tau)\int_{\partial\Omega} p(\sigma)(\frac{\partial u}{\partial\nu})^2 d\sigma,
\end{aligned}$$

where $b = a-1, a = 5(1+\gamma)$.
Using Cauchy–Schwarz inequality we get

$$\int_\Omega u_t^2 dx\int_\Omega u^2 dx - \left(\int_\Omega uu_t dx\right)^2 \ge 0,$$

which implies the following estimate

$$E(t) \ge \Phi(t)H(t) + \Phi(t)\Lambda - \frac{a}{2}\Lambda^2(t+\tau)^2 - \frac{a}{2}\left[\int_{\partial\Omega} p(\sigma)(\frac{\partial u}{\partial\nu})^2 d\sigma\right]^2. \tag{2.9}$$

To show $E(t) \geq 0$, we have to demonstrate

$$\Phi(t)H(t) + \Phi(t)\Lambda - \frac{a}{2}\Lambda^2(t+\tau)^2 - \frac{a}{2}\left[\int_{\partial\Omega} p(\sigma)(\frac{\partial u}{\partial \nu})^2 d\sigma\right]^2 \geq 0 \tag{2.10}$$

for a suitable number $\tau > 0$.
Note that

$$\Phi'(0) = \int_\Omega u_0 u_1 dx + \frac{1}{2}\int_{\partial\Omega} p(\sigma)\left(\frac{\partial u_0}{\partial \nu}\right)^2 d\sigma + \Lambda\tau.$$

In the sequel we demande, for fixed $\Lambda > 0$, that τ verify

$$\int_\Omega u_0 u_1 dx + \frac{1}{2}\int_{\partial\Omega} p(\sigma)\left(\frac{\partial u_0}{\partial \nu}\right)^2 d\sigma + \Lambda\tau > 0, \tag{2.11}$$

and

$$\frac{\Phi(0)}{\gamma\Phi'(0)} < T. \tag{2.12}$$

Let us now verify inequality (2.10). Multiplying Equation $(1.1)_1$ by u and integrating over Ω, we have

$$\begin{aligned}\int_\Omega u u_{tt} dx &= -\int_\Omega u u_t dx - \int_\Omega (\Delta u)^2 dx - \int_{\partial\Omega} p(\sigma)\frac{\partial u_t}{\partial \nu}\frac{\partial u}{\partial \nu} d\sigma \\ &\quad -\varphi\left(\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |\nabla u|^2 dx + \int_\Omega u f(u) dx.\end{aligned}$$

Then we estimate the first term on the right–hand by the following inequality

$$-\int_\Omega u u_t dx \geq -\frac{1}{4}\int_\Omega u^2 dx - \int_\Omega u_t^2 dx,$$

so one has

$$\begin{aligned}H(t) &\geq -\frac{1}{4}\int_\Omega u^2 dx - a\int_\Omega u_t^2 dx - \int_\Omega (\Delta u)^2 dx \\ &\quad -\varphi\left(\int_\Omega |\nabla u|^2 dx\right)\left(\int_\Omega |\nabla u|^2 dx\right) + \int_\Omega u f(u) dx.\end{aligned} \tag{2.13}$$

Using equality (2.5), and hypothesis (1.7), (1.6) we arrive at

$$\begin{aligned}H(t) &\geq \left[-\frac{1}{4} + \lambda_1(a-1)\right]\int_\Omega u^2 dx + a\tilde{\varphi}\left(\int_\Omega |\nabla u|^2 dx\right) \\ &\quad -\varphi\left(\int_\Omega |\nabla u|^2 dx\right)\left(\int_\Omega |\nabla u|^2 dx\right).\end{aligned}$$

Back to E, it follows

$$\begin{aligned}E(t) &\geq \frac{k}{2}\Lambda(t+\tau)^2 N(T) - \frac{5(1+\gamma)}{2}\left(\int_{\partial\Omega} p(\sigma)\left(\frac{\partial u}{\partial \nu}\right)^2 dx\right)^2 \\ &\quad +M\Lambda - \tfrac{5}{2}(1+\gamma)\Lambda^2(t+\tau)^2,\end{aligned} \tag{2.14}$$

where
$$k = -\frac{1}{4} + \lambda_1(a-1),$$
which is nonnegative thanks to (1.10). Consequently, we arrive at a new sufficient condition for the blow–up, namely

$$\frac{k}{2}\Lambda(t+\tau)^2 N(T) - \frac{5(1+\gamma)}{2}\left(\int_{\partial\Omega} p(\sigma)\left(\frac{\partial u}{\partial \nu}\right)^2 dx\right)^2 + M\Lambda - \frac{5}{2}(1+\gamma)\Lambda^2(t+\tau)^2 \geq 0.$$

Now, choosing Λ and M such that
$$0 < \Lambda \leq \frac{kN(T)}{5(1+\gamma)}$$
and
$$M \geq \frac{5(1+\gamma)}{2\Lambda}\sup_{0\leq t\leq T}\left[\int_{\partial\Omega} p(\sigma)\left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma\right]^2,$$
we arrive, for τ large, at the following
$$\Phi(0) > 0, \quad \Phi'(0) > 0,$$
and
$$\Phi''\Phi - (1+\gamma)(\Phi')^2 \geq 0,$$

Hence the blow–up takes place in the interval $(0, T)$ thanks to (2.12). A contradiction. The proof of the theorem is finished. ■

Corollary 2.1. *Assume that conditions* (1.7)–(1.9) *hold. Let* $u \in C^2$ *be a global solution to (1.1). Then* $u_t(.,0) \equiv 0$.

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