On dense embeddings of discrete Abelian groups into locally compact groups*

Sergey L. Gefter   Konstantin M. Kulagin

Abstract

Let Γ be an infinite Abelian group with the following property: the image of every embedding of Γ into a locally compact group is either discrete or precompact. We show that either $\Gamma \cong \mathbb{Z} \times F$ or $\Gamma \cong \mathbb{C}_{p^\infty} \times F$, where F is a finite group and $\mathbb{C}_{p^\infty}$ is the group of roots of unity, whose degrees are powers of a prime p.

1 Introduction

In 1940 A.Weil established an interesting property of the group of integers which was used to produce the structural theory of locally compact Abelian groups: the image of every embedding of $\mathbb{Z}$ into a locally compact group is either discrete or precompact (see § 26, Lemma 2 in [7] or Ch II , § 2, Lemma 1 in [1]). This property lead us to provide the following definition.

Definition 1.1. We say that a discrete group Γ has Z-property if the image of every embedding of Γ into a locally compact group is either discrete or precompact.

Equivalently, Γ has Z-property if it cannot be embedded densely into a continuous (i.e. non-discrete) locally compact non-compact group (here, all topological groups are assumed to be Hausdorff).

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Remark 1.2. A similar definition can be given for locally compact groups. A locally compact group H has Z-property if the image of every *continuous* embedding of H into a locally compact group is either *closed* or precompact. In the case of a discrete group H we clearly get the previous definition. For example, $\mathbb{R}$ is a continuous locally compact group with Z-property (see Ch II, § 2, Lemma 1 in [1]).

It is evident that every finite group F and $\mathbb{Z} \times F$ have Z-property. A. Ol'šhanskii group constructed in Theorem 31.5 [6] is a non-trivial example of a countable group with Z-property. There exists no continuous (i.e. non-discrete) topology on this group compatible with the group structure. Hence this group cannot be densely embedded into a continuous topological (and locally compact, in particular) group at all. Let us consider another example of a group with Z-property. For a prime p denote by $\mathbb{C}_{p^\infty}$ the quasicyclic group $\{z \in \mathbb{T} | \; z^{p^n} = 1 \textit{ for some } n\}$.

Proposition 1.3. *$\mathbb{C}_{p^\infty}$ has Z-property.*

Proof. Let $\phi \colon \mathbb{C}_{p^\infty} \to G$ be a dense embedding of $\mathbb{C}_{p^\infty}$ into a locally compact group. Then G is Abelian. Hence there is an open subgroup $G_1 \subset G$ which is topologically isomorphic to $\mathbb{R}^n \times K$, where K is a compact group (see Theorem 24.30 in [4]). Let $\Gamma_1 = \phi^{-1}(G_1)$. Then $\phi(\Gamma_1)$ is a dense subgroup of G_1. If Γ_1 is finite then G_1 is finite and open. Therefore G is discrete. Assume now that Γ_1 is infinite. Then $\Gamma_1 = \mathbb{C}_{p^\infty}$ (see §7 in [5]). Since $\overline{\phi(\mathbb{C}_{p^\infty})} = G$ and G_1 is open, $G_1 = G$. Hence ϕ is a dense embedding of $\mathbb{C}_{p^\infty}$ into $\mathbb{R}^n \times K$. Since the elements of $\mathbb{C}_{p^\infty}$ have finite order, n=0, i.e. $\phi(\mathbb{C}_{p^\infty})$ is precompact. ∎

*Remark 1.4.*It would be interesting to prove this proposition without using the structural theory of locally compact Abelian groups.

2 Main Theorem

The following theorem is the main result of the paper. It provides a complete description of the Abelian groups with Z-property.

Theorem 2.1. *Let Γ be an infinite Abelian group, then: Γ has the Z-property if and only if Γ is isomorphic to either $\mathbb{Z} \times F$ or $\mathbb{C}_{p^\infty} \times F$, where F is a finite Abelian group.*

Let us first mention the following well known lemma:

Lemma 2.2. *Every infinite Abelian group can be densely embedded into a continuous compact group.*

Proof. See Section 26.11 in [4]. ∎

In order to prove the main theorem, we will need the following easy facts:

Lemma 2.3. *Let $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_1, Γ_2 are infinite Abelian groups. Then Γ does not have Z-property.*

Proof. According to Lemma 2.2, there are compact group K and a dense embedding $\phi\colon \Gamma_1 \to K$. We define a map $\psi\colon \Gamma \to K \times \Gamma_2$ by setting $\psi(\gamma_1, \gamma_2) = (\phi(\gamma_1), \gamma_2)$. Then ψ is a dense embedding of Γ into $K \times \Gamma_2$ which image is continuous and non-compact. ■

Lemma 2.4. *Let F be a finite group. Then Γ has Z - property iff $\Gamma \times F$ has Z-property.*

Proof. Clear from the definition. ■

Lemma 2.5. *Let K be a compact group, H a topological group, $G = K \times H$, and $\Gamma \subset G$. Let us denote by π_H the canonical projection onto H.*

i) If H is continuous and $\pi_H(\Gamma)$ is dense in H then Γ is non-discrete.

ii) If H is infinite and $\pi_H(\Gamma) = H$ then the closure of Γ is non-compact. Moreover if $(\ker \pi_H) \cap \Gamma$ is an infinite group then the closure of Γ is continuous.

Proof. i) Notice that if Γ is closed, then $\pi_H(\Gamma)$ is closed.

ii) Clear. ■

Proof of the main theorem. The "only if" part is clear. Now, for the "if" part, consider the minimal divisible extension Λ of Γ (see §23 in [5]). We first consider the case when Γ is non-periodic. Then Λ can be expanded as $\Lambda \cong (\bigoplus_{i\in I} \mathbb{Q}_i) \oplus P$, where $\mathbb{Q}_i = \mathbb{Q}$, $I \neq \emptyset$ (because Γ is non-periodic) and P is the direct sum of a family of copies of $\mathbb{C}_{p^\infty}$ for some primes p (see §23 in [5]). Since $\mathbb{Q}$ and $\mathbb{C}_{p^\infty}$ are subgroups in $\mathbb{R}$ and $\mathbb{T}$ respectively, Λ is embedded into a group $A \cong (\bigoplus_{i\in I} \mathbb{R}_i) \times K$, where $\mathbb{R}_i = \mathbb{R}$ and $K = \prod_{s\in S} \mathbb{T}_s$,$\mathbb{T}_s = \mathbb{T}$. Let $\pi_j\colon A \to \mathbb{R}$ be the projection onto the j-th component of $\bigoplus_{i\in I} \mathbb{R}_i$. Since every subgroup of $\mathbb{R}$ is either dense or discrete, there are two possibilities:

1) $\pi_{j_0}(\Gamma)$ is dense in $\mathbb{R}$ for some $j_0 \in I$;

2) $\pi_j(\Gamma)$ is discrete in $\mathbb{R}$ for every $j \in I$.

Let us consider the case 1). We embed Λ into

$$G = (\prod_{i\in I\setminus\{j_0\}} \mathbb{T}_i) \times \mathbb{R} \times K,$$

by considering homomorphisms $q_i\colon \mathbb{R}_i \to \mathbb{T}$ such that $q_i|_{\mathbb{Q}_i}$ is injective, $\forall i \neq j_0$. Since K is compact, G is a locally compact non-compact group. Let B stand for the closure of the image of Γ in G. Then B is non-compact. By Lemma 2.5 part i), B is continuous. Hence Γ does not have Z-property.

In the case 2) we fix some index $j_0 \in I$. Since Λ is a minimal divisible extension of Γ (which implies that $\pi_{j_0}(\Gamma) \neq \{e\}$), and $\pi_{j_0}(\Gamma)$ is a discrete subgroup in $\mathbb{R}$, $\pi_{j_0}(\Gamma) \cong \mathbb{Z}$. Hence $\Gamma \cong \mathbb{Z} \times (\ker \pi_{j_0} \cap \Gamma)$. Since Γ has Z-property, we deduce from Lemma 2.3 that $(\ker \pi_{j_0}) \cap \Gamma$ is finite.

Now let Γ be periodic. It follows from Theorem A3 in [4] and Lemma 2.3 that without loss of generality we may assume Γ to be p-primary. Let Λ be a minimal divisible extension of Γ. Then $\Lambda \cong \bigoplus\limits_{i \in I} P_i$, where $P_i = \mathbb{C}_{p^\infty}, i \in I$ (see §23 in [5]). Now let us denote by π_j the projection onto the j-th component of Λ. There exists two possibilities again:

i) $\pi_{j_0}(\Gamma)$ is an infinite group for some $j_0 \in I$;

ii) $\pi_j(\Gamma)$ is a finite group for every $j \in I$.

Let us consider the case i). We deduce that $\pi_{j_0}(\Gamma) = \mathbb{C}_{p^\infty}$ from the properties of $\mathbb{C}_{p^\infty}$. If $(\ker \pi_{j_0}) \cap \Gamma$ is infinite then $|I| \geq 2$ and Λ can be densely embedded into $G = (\prod\limits_{i \in I \setminus \{j_0\}} \mathbb{T}_i) \times \mathbb{C}_{p^\infty}$ by considering the identity embeddings from P_i into $\mathbb{T}$, $\forall i \neq j_0$. However this contradicts to Lemma 2.5, part ii). Thus $(\ker\ \pi_{j_0}) \cap \Gamma$ is finite. Denote it by F. Then $\Gamma \,/\, F \;\cong\; \mathbb{C}_{p^\infty}$. If Γ does not contain nontrivial direct summands, then $\Gamma \;\cong\; \mathbb{C}_{p^\infty}$ (see Corollary 27.4 and Theorem 3.1 in [2]). Let $\Gamma_1 \subset \Gamma$ be a nontrivial infinite direct summand. Then $\pi_{j_0}(\Gamma_1) = \pi_{j_0}(\Gamma) = \mathbb{C}_{p^\infty}$ and $\Gamma = \Gamma_1 \bigoplus L_1$, where L_1 is finite (see Lemma 2.3) and Γ_1 has Z-property (see Lemma 2.4). We set $F_1 = \Gamma_1 \cap F$. Then $\Gamma_1/F_1 \cong \mathbb{C}_{p^\infty}$ and $|F_1| < |F|$. If Γ_1 does not contain nontrivial direct summands, then $\Gamma \cong \mathbb{C}_{p^\infty} \times L_1$. In the opposite case we proceed with the above constructions. Finally we conclude that either $\Gamma \cong \mathbb{C}_{p^\infty} \times L_k \times \ldots \times L_1$ after some steps or $F_n = \Gamma_n \cap F = \{e\}$ for some $n \leq |F|$, i.e. $\Gamma_n \cong \mathbb{C}_{p^\infty}$ and $\Gamma \cong \Gamma_n \times F$.

Let us consider the case ii). Then Γ is a subgroup of a direct sum of cyclic groups, hence it can be expanded into a direct sum of finite cyclic groups (see §24 in [5]). As Γ is infinite, this contradicts to Lemma 2.3. Now the theorem is completely proved. ∎

Corollary 2.7. *A discrete Abelian group with Z-property is finite or countable.*

Remark 2.8. There exist uncountable noncommutative discrete groups with Z-property. The permutation group of a countable set $S(\mathbb{N})$ is an example of such a group. This group cannot be densely embedded into a continuous locally compact group (see [3]).

Corollary 2.9. *Let K be an infinite compact Abelian group. Suppose that K does not admit continuous embedding of any continuous non-compact locally compact group. Then K is topologically isomorphic either to $\mathbb{T} \times F$ or to $\mathbb{Z}_p \times F$, where F is a finite Abelian group, $\mathbb{Z}_p$ is the group of integer p-adic numbers.*

Proof. Since $\mathbb{T}$ and $\mathbb{Z}_p$ are the duals to $\mathbb{Z}$ and $\mathbb{C}_{p^\infty}$ respectively, we use Pontryagin duality functor to deduce the statement of this corollary from Ch II, §1, Corollary 6 in [1]. ∎

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Sergey L. Gefter.
Department of Mechanics and Mathematics
Kharkov National University,
Freedom Sq. 4,
61077 Kharkov,
Ukraine
email : gefter@univer.kharkov.ua

Konstantin M. Kulagin.
Institute for Low Temperature Physics and Engineering,
Lenin Ave 47,
61103 Kharkov,
Ukraine
email : K_Kulagin@ukr.net