Homology of the completion of instanton moduli spaces

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Abstract

Let $M(k,G)$ be the moduli space of based gauge equivalence classes of G-instantons on S^4 with instanton number k. $M(k,G)$ has the Uhlenbeck completion $\overline{M}(k,G)=\bigcup_{q=0}^{k}\mathrm{SP}^q(\mathbf{R}^4)\times M(k-q,G)$, where $\mathrm{SP}^q(\mathbf{R}^4)$ denotes the q-fold symmetric product of $\mathbf{R}^4$. Let $X(k,G)$ be the first two strata of the completion: $X(k,G)=M(k,G)\cup\mathbf{R}^4\times M(k-1,G)$. In this paper we study the homology of $X(k,G)$ for $G=SU(n)$ or $Sp(n)$, and relate this to the homology of a certain homotopy theoretic fibre.

1 Introduction

Let V be a connected complex manifold. For simplicity we assume $\pi_1(V)=0$ and $\pi_2(V)\cong\mathbf{Z}$. Let $\mathrm{Rat}_k(V)$ denote the space of based holomorphic maps of degree k from S^2 to V, and let $i_k:\mathrm{Rat}_k(V)\to\Omega_k^2V$ be the inclusion. Suppose that the following stability principle is satisfied: the inclusion i_k becomes a homotopy equivalence through a range of dimensions which increases to infinity with k. In particular, we have a homotopy equivalence $\mathrm{Rat}_\infty(V)\simeq\Omega_0^2V$, where $\mathrm{Rat}_\infty(V)$ is the direct limit $\lim\limits_{k\to\infty}\mathrm{Rat}_k(V)$.

Let $\mathrm{ad}(i_1):\Sigma\mathrm{Rat}_1(V)\to\Omega V$ be the adjoint map of i_1. We lift $\mathrm{ad}(i_1)$ to a map $\widetilde{\mathrm{ad}}(i_1):\Sigma\mathrm{Rat}_1(V)\to\widetilde{\Omega}V$, where $\widetilde{\Omega}V$ is the universal cover of ΩV. Let $W(V)$ be the homotopy theoretic fibre of $\widetilde{\mathrm{ad}}(i_1)$. Then we have the following sequence of fibrations:

$$\Omega_0^2V\longrightarrow W(V)\longrightarrow\Sigma\mathrm{Rat}_1(V)\xrightarrow{\widetilde{\mathrm{ad}}(i_1)}\widetilde{\Omega}V. \tag{1.1}$$

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We consider the following problem: how to construct a space $X_k(V)$, which is a natural generalization of $\mathrm{Rat}_k(V)$, such that $X_\infty(V)$ approximates $W(V)$.
The problem was solved for $V = \mathbf{C}P^n$ in [7]. We summarize the results. For $f \in \mathrm{Rat}_k(\mathbf{C}P^n)$, we assume the basepoint condition $f(\infty) = [1, \ldots, 1]$. Such holomorphic maps are given by rational functions:

$$\begin{aligned}\mathrm{Rat}_k(\mathbf{C}P^n) = \{(p_0(z), \ldots, p_n(z)) :\ & \text{each } p_i(z) \text{ is a monic, degree-}k \text{ polynomial} \\ & \text{and such that there are no roots common to all } p_i(z)\}.\end{aligned}$$

The stability principle for $i_k : \mathrm{Rat}_k(\mathbf{C}P^n) \to \Omega_k^2 \mathbf{C}P^n$ was proved in [12]: i_k is a homotopy equivalence up to dimension $k(2n-1)$. Later the stable homotopy type of $\mathrm{Rat}_k(\mathbf{C}P^n)$ was described in [4]: $\mathrm{Rat}_k(\mathbf{C}P^n) \simeq_s \bigvee_{q=1}^k D_q(S^{2n-1})$, where $D_q(S^{2n-1})$ is a stable summand of the Snaith's stable splitting $\Omega^2 S^{2n+1} \simeq_s \bigvee_{q\geq 1} D_q(S^{2n-1})$.
We define $X_k^l(\mathbf{C}P^n)$ by

$$\begin{aligned}X_k^l(\mathbf{C}P^n) = \{(p_0(z), \ldots, p_n(z)) :\ & \text{each } p_i(z) \text{ is a monic, degree-}k \text{ polynomial} \\ & \text{and such that there are at most } l \text{ roots common to all } p_i(z)\}.\end{aligned}$$

Thus as sets we have

$$X_k^l(\mathbf{C}P^n) = \coprod_{q=0}^{l} \mathbf{C}^q \times \mathrm{Rat}_{k-q}(\mathbf{C}P^n), \tag{1.2}$$

where $\mathbf{C}^q \times \mathrm{Rat}_{k-q}(\mathbf{C}P^n)$ corresponds to the subspace of $X_k^l(\mathbf{C}P^n)$ consisting of elements $(p_0(z), \ldots, p_n(z))$ such that there are exactly l roots common to all $p_i(z)$.
Let $J^l(S^{2n})$ denote the l-th stage of the James construction which builds ΩS^{2n+1}, and let $W^l(S^{2n})$ be the homotopy theoretic fibre of the inclusion $J^l(S^{2n}) \hookrightarrow J(S^{2n}) \simeq \Omega S^{2n+1}$. In [7] we proved a stable homotopy equivalence $X_k^l(\mathbf{C}P^n) \simeq_s \bigvee_{q=1}^k D_q \xi^l(S^{2n})$, where $D_q \xi^l(S^{2n})$ is a stable summand of the stable splitting $W^l(S^{2n}) \simeq_s \bigvee_{q\geq 1} D_q \xi^l(S^{2n})$.
We consider the case $l = 1$. Since $J^1(S^{2n}) \simeq S^{2n}$, $W^1(S^{2n})$ is the homotopy theoretic fibre of the Freudenthal suspension $E : S^{2n} \to \Omega S^{2n+1}$. Since $\mathrm{Rat}_1(\mathbf{C}P^n) \simeq S^{2n-1}$, $\widetilde{\mathrm{ad}}(i_1) : \Sigma \mathrm{Rat}_1(\mathbf{C}P^n) \to \widetilde{\Omega} \mathbf{C}P^n \simeq \Omega S^{2n+1}$ in (1.1) is also the Freudenthal suspension. Hence $W(\mathbf{C}P^n) \simeq W^1(S^{2n})$ and $X_k^1(\mathbf{C}P^n)$ gives an answer to the problem for $V = \mathbf{C}P^n$.

The purpose of this paper is to study the problem for $V = \Omega G$, where $G = SU(n)$ or $Sp(n)$. In this case $\mathrm{Rat}_k(V)$ is identified with $M(k, G)$, the moduli space of based gauge equivalence classes of G-instantons on S^4 with instanton number k. The moduli space $M(k, G)$ is a smooth connected non-compact complex manifold of complex dimension $2kn$ if $G = SU(n)$ and $2k(n+1)$ if $G = Sp(n)$. Let $i_k : M(k, G) \to \Omega_k^3 G$ be the inclusion. The stability principle for i_k is called the Atiyah-Jones conjecture, which is now solved (see Section 3). Let $C = C_G(SU(2))$ be the centralizer of $SU(2)$ in G. Define a map $J : G/C \to \Omega_0^3 G$ by $J(gC)(x) = gxg^{-1}x^{-1}$, where $x \in SU(2)$. Particular examples are: $SU(n)/C$ is diffeomorphic to the unit tangent bundle of $\mathbf{C}P^{n-1}$ and $Sp(n)/C$ is diffeomorphic to $\mathbf{R}P^{4n-1}$ (see [2], [11]). $M(1, G)$ is diffeomorphic to $\mathbf{R}^5 \times G/C$ such that the following homotopy commutative diagram holds (see [2]):

$$\begin{array}{ccc} M(1,G) & \xrightarrow{i_1} & \Omega^3_1 G \\ \Big\downarrow \simeq & & \Big\downarrow \simeq \\ G/C & \xrightarrow{J} & \Omega^3_0 G \end{array} \tag{1.3}$$

Hence $W(\Omega G)$ in (1.1) is identified with the homotopy theoretic fibre of $\widetilde{\mathrm{ad}}(J) : \Sigma G/C \to \widetilde{\Omega}^2 G$.
On the other hand, $M(k,G)$ has the Uhlenbeck completion (see [6]), which is similar to (1.2):

$$\overline{M}(k,G) = \bigcup_{q=0}^{k} \mathrm{SP}^q(\mathbf{R}^4) \times M(k-q,G),$$

where $\mathrm{SP}^q(\mathbf{R}^4)$ denotes the q-fold symmetric product of $\mathbf{R}^4$. Let $X(k,G)$ be the first two strata of the completion:

$$X(k,G) = M(k,G) \cup \mathbf{R}^4 \times M(k-1,G). \tag{1.4}$$

For $X_k(\Omega G)$ in the problem, we consider $X(k,G)$.
Now our results are as follows.

Theorem A. *Let $n \geq 3$ and p a prime with $p = 2$ or $p \geq 2n+1$. Then we have the following isomorphism for $q \leq 2k$:*

$$H_q(X(k,SU(n)); \mathbf{Z}/p) \cong H_q(W(\Omega SU(n)); \mathbf{Z}/p).$$

Theorem B. *Let $n \geq 1$ and p a prime with $p = 2$ or $p \geq 2n+3$. Then we have the following isomorphism for $q \leq \left[\frac{k}{2}\right] - 1$:*

$$H_q(X(k,Sp(n)); \mathbf{Z}/p) \cong H_q(W(\Omega Sp(n)); \mathbf{Z}/p).$$

For $G = SU(2) = Sp(1)$, we have

Theorem C. *For all primes p, we have the following isomorphism for $q \leq k-1$:*

$$H_q(X(k,SU(2)); \mathbf{Z}/p) \cong H_q(W(\Omega SU(2)); \mathbf{Z}/p).$$

If we form $k \to \infty$ in Theorems A and B, we have the following:

Corollary D. *Put $\nu(G) = 2n+1$ if $G = SU(n)$ $(n \geq 2)$ and $\nu(G) = 2n+3$ if $G = Sp(n)$ $(n \geq 1)$. Then for p a prime with $p = 2$ or $p \geq \nu(G)$, we have the following isomorphism:*

$$H_*(X(\infty,G); \mathbf{Z}/p) \cong H_*(W(\Omega G); \mathbf{Z}/p).$$

The structure of $H_*(W(\Omega G); \mathbf{Z}/p)$ is determined in Section 2. (See Propositions 2.7, 2.9 and Remark 2.10 for $G = SU(n)$, and Propositions 2.12 and 2.13 for $G = Sp(n)$.)
Finally we remark about the situation where $n = \infty$. By [9] we have a homotopy equivalence $M(k,SU) \simeq BU(k)$ hence $W(\Omega SU)$ is the homotopy theoretic fibre

of the natural map $\Sigma BU(1) \to SU$. Similarly, we have a homotopy equivalence $M(k, Sp) \simeq BO(k)$ hence $W(\Omega Sp)$ is the homotopy theoretic fibre of the natural map $\Sigma BO(1) \to SU/SO$. In particular, localized away from 2, $W(\Omega Sp)$ is homotopy equivalent to $\Omega SU/SO$.
This paper is organized as follows. In Section 2 we determine $H_*(W(\Omega G); \mathbf{Z}/p)$ and in Section 3 we prove Theorems A, B and C.

2 Homology of $W(\Omega G)$

First we determine $H_*(W(\Omega SU(n)); \mathbf{Z}/2)$. Since $SU(n)/C$ is diffeomorphic to the unit tangent bundle of $\mathbf{C}P^{n-1}$, $H^*(SU(n)/C; \mathbf{Z}/2)$ is given as follows.
(1) For n even,

$$H^*(SU(n)/C; \mathbf{Z}/2) \cong H^*(\mathbf{C}P^{n-1}; \mathbf{Z}/2) \otimes H^*(S^{2n-3}; \mathbf{Z}/2).$$

(2) For n odd,

$$H^*(SU(n)/C; \mathbf{Z}/2) \cong H^*(\mathbf{C}P^{n-2}; \mathbf{Z}/2) \otimes H^*(S^{2n-1}; \mathbf{Z}/2). \tag{2.1}$$

We write the generators of $H_*(SU(n)/C; \mathbf{Z}/2)$ as follows.
(1) For n even,

$$\begin{cases} \alpha_{2i} & 1 \le i \le n-1 \\ \beta_{2i+1} & n-2 \le i \le 2n-3. \end{cases}$$

(2) For n odd,

$$\begin{cases} \alpha_{2i} & 1 \le i \le n-2 \\ \beta_{2i+1} & n-1 \le i \le 2n-3. \end{cases}$$

Theorem 2.2 ([2]). *For $n \ge 3$, there are choices of elements*
(1) x_{2i} $\quad i = 1$ *or* $2 \le i \le n-2, i \equiv 0 \pmod 2$, (2) y_{4i+1} $\quad \left[\frac{n-1}{2}\right] \le i \le n-2, i \equiv 1 \pmod 2$, *such that $H_*(\Omega_0^3 SU(n); \mathbf{Z}/2)$ is isomorphic to the following algebra:*

$$\begin{aligned} \mathbf{Z}/2&\left[Q_2^a(x_{2i}) : a \ge 0, i = 1 \text{ or } 2 \le i \le \left[\frac{n-3}{2}\right], i \equiv 0 \pmod 2\right] \\ &\otimes \mathbf{Z}/2\left[Q_1^a Q_2^b(x_{2i}) : a, b \ge 0, \left[\frac{n-1}{2}\right] \le i \le n-2, i \equiv 0 \pmod 2\right] \\ &\otimes \mathbf{Z}/2\left[Q_1^a Q_3^b(y_{4i+1}) : a, b \ge 0, \left[\frac{n-1}{2}\right] \le i \le n-2, i \equiv 1 \pmod 2\right]. \end{aligned}$$

We generalize the elements x_{2i} and y_{4i+1} in Theorem 2.2 to the following ones:
(1) For n even,

$$\begin{cases} x_{2i} & 1 \le i \le n-1 \\ y_{4i+1} & \frac{n}{2} - 1 \le i \le n-2. \end{cases} \tag{2.3}$$

(2) For n odd,

$$\begin{cases} x_{2i} & 1 \le i \le n-2 \\ y_{4i+1} & \frac{n-1}{2} \le i \le n-2. \end{cases} \tag{2.4}$$

The definitions are as follows.
(i) If $i = 1$ or $i \equiv 0 \pmod 2$, then we define x_{2i} to be the one in Theorem 2.2 (1).
(ii) If $i \equiv 1 \pmod 2$, then there exist a and j uniquely such that $\deg Q_2^a(x_{4j}) = 2i$ (where we put $x_0 = [1]$). Then we define x_{2i} by $x_{2i} = Q_2^a(x_{4j})$.
(iii) If $i \equiv 1 \pmod 2$, then we define y_{4i+1} to be the one in Theorem 2.2 (2).
(iv) If $i \equiv 0 \pmod 2$, then we define y_{4i+1} by $y_{4i+1} = Q_1(x_{2i})$. Recall that we defined a map $J : SU(n)/C \to \Omega_0^3 SU(n)$ in Section 1.

Lemma 2.5. *We have the following relations:*
(1) (i) *When* $i = 1$ *or* $i \equiv 0 \pmod 2$, $J_*(\alpha_{2i}) = x_{2i}$.
(ii) *When* $i \equiv 1 \pmod 2$, $J_*(\alpha_{2i})$ *contains the term* x_{2i}.
(2) (i) *When* $i \equiv 1 \pmod 2$, $J_*(\beta_{4i+1}) = y_{4i+1}$.
(ii) *When* $i \equiv 0 \pmod 2$, $J_*(\beta_{4i+1})$ *contains the term* y_{4i+1}.
(iii) $J_*(\beta_{4i+3})$ *is decomposable.*

Proof. (1) follows from the following homotopy commutative diagram:

$$\begin{array}{ccc} SU(n)/C_{SU(n)}(SU(2)) & \xrightarrow{J} & \Omega_0^3 SU(n) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ SU/C_{SU}(SU(2)) & \xrightarrow{J} & \Omega_0^3 SU \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ BU(1) & \xrightarrow{k} & BU \end{array}$$

where i, j and k are the inclusions and $\Omega_0^3 SU \simeq BU$ is the Bott periodicity. (The homotopy commutativity of the bottom square follows from the third diagram of [11, p. 4054] for $k = 1$ and $l = \infty$.)
(2) is proved in [8]. In particular, (iii) is a consequence of dimensional reasons. This completes the proof of Lemma 2.5. ∎

We define an increasing sequence of ideals of $H_*(\Omega_0^3 SU(n); \mathbf{Z}/2)$, I_ν, as follows: Consider the elements x_{2i} and y_{4i+1} in (2.3) or (2.4). Put

$$I_\nu = \text{the ideal generated by } x_{2i} \text{ and } y_{4i+1} \text{ whose degree is less than } \nu. \tag{2.6}$$

For $\gamma \in H_*(SU(n)/C; \mathbf{Z}/2)$, let $s\gamma \in H_{*+1}(\Sigma SU(n)/C; \mathbf{Z}/2)$ be the suspension.

Proposition 2.7. *For* $n \geq 3$, $H_*(W(\Omega SU(n)); \mathbf{Z}/2)$ *is isomorphic to the following module:*

$$\bigoplus_{i=2}^{a} s\alpha_{2i} \otimes I_{2i} \oplus \bigoplus_{i=b}^{n-2} s\beta_{4i+1} \otimes I_{4i+1} \oplus \bigoplus_{i=b}^{n-2} s\beta_{4i+3} \otimes H_*(\Omega_0^3 SU(n); \mathbf{Z}/2) \oplus \frac{H_*(\Omega_0^3 SU(n); \mathbf{Z}/2)}{I_{4n-3}},$$

where

$$a = \begin{cases} n-1 & n : \text{even} \\ n-2 & n : \text{odd} \end{cases} \quad \text{and} \quad b = \begin{cases} \frac{n}{2} - 1 & n : \text{even} \\ \frac{n-1}{2} & n : \text{odd.} \end{cases}$$

Proof. Consider the homology Serre spectral sequence of the fibration $\Omega_0^3 SU(n) \to W(\Omega SU(n)) \to \Sigma SU(n)/C$. By Lemma 2.5 (1) (i), we have $d^3(s\alpha_2) = x_2$ hence $E^\infty_{3,*} = 0$. Next, we have $d^5(s\alpha_4) = x_4$. Note that $d^5(s\alpha_4 \otimes z) = 0$ if $z \in I_4 = (x_2)$. Hence $E^\infty_{5,*} \cong s\alpha_4 \otimes I_4$. Following the same procedures, we obtain Proposition 2.7. ■

Let $n \geq 2$ and p a prime with $p \geq 2n+1$. Then $H^*(SU(n)/C; \mathbf{Z}/p)$ is of the form (2.1), and we define α_{2i} and β_{2i+1} similarly. On the other hand, localized at p, $SU(n)$ is homotopy equivalent to $\prod_{i=1}^{n-1} S^{2i+1}$ (see [10]) and $H_*(\Omega^3 S^{2i+1}; \mathbf{Z}/p)$ is given in [5]. For $1 \leq i \leq n-2$, let $x_{2i} \in H_{2i}(\Omega^3 S^{2i+3}; \mathbf{Z}/p)$ be the generator. Form $x_2, \dots, x_{2n-4}$, we define an increasing sequence of ideals of $H_*(\Omega_0^3 SU(n); \mathbf{Z}/p)$, I_ν, in the same way as in (2.6).

Lemma 2.8. *For $p \geq 2n+1$, we have the following relations:*
(1) For $1 \leq i \leq n-2$, $J_*(\alpha_{2i}) = x_{2i}$. (2) For $n-1 \leq i \leq 2n-3$, $J_*(\beta_{2i+1}) = 0$.

Proof. We can prove (1) in the same way as in Lemma 2.5 (1). For (2), a minimal odd dimensional element of $H_*(\Omega_0^3 S^3; \mathbf{Z}/p)$ is $\beta Q_2[1]$, whose degree is $2p-3$. On the other hand, $\deg \beta_{2i+1} \leq 4n-5$. Since $p \geq 2n+1$, dimensional reasons show $J_*(\beta_{2i+1}) = 0$. This completes the proof of Lemma 2.8. ■

Proposition 2.9. *For $n \geq 2$ and $p \geq 2n+1$, $H_*(W(\Omega SU(n)); \mathbf{Z}/p)$ is isomorphic to the following module:*

$$\bigoplus_{i=2}^{n-2} s\alpha_{2i} \otimes I_{2i} \oplus \bigoplus_{i=n-1}^{2n-3} s\beta_{2i+1} \otimes H_*(\Omega_0^3 SU(n); \mathbf{Z}/p) \oplus \frac{H_*(\Omega_0^3 SU(n); \mathbf{Z}/p)}{I_{2n-2}}.$$

Proof. We can prove the proposition in the same way as in Proposition 2.7. ■

Remark 2.10. In order to determine $H_*(W(\Omega SU(2)); \mathbf{Z}/p)$ for all primes p, we need to consider the cases $p = 2$ and 3. For $p = 2$, see Proposition 2.12. For $p = 3$, we have $J_*(\beta_3) = \beta Q_2[1]$ (where β in the right-hand side is the Bockstein homomorphism). Hence

$$H_*(W(\Omega SU(2)); \mathbf{Z}/3) \cong s\beta_3 \otimes (\beta Q_2[1]) \oplus \frac{H_*(\Omega_0^3 S^3; \mathbf{Z}/3)}{(\beta Q_2[1])}.$$

Next we study $H_*(W(\Omega Sp(n)); \mathbf{Z}/p)$. Recall that $Sp(n)/C$ is diffeomorphic to $\mathbf{R}P^{4n-1}$. Let α_i $(1 \leq i \leq 4n-1)$ be the generator of $H_i(Sp(n)/C; \mathbf{Z}/2)$.

Theorem 2.11 ([3]). *For $n \geq 1$, $H_*(\Omega_0^3 Sp(n); \mathbf{Z}/2)$ is isomorphic to the following algebra:*

$$\mathbf{Z}/2\left[Q_1^a Q_2^b[1] * [-2^{a+b}] : a, b \geq 0\right] \otimes \mathbf{Z}/2\left[Q_1^a Q_2^b(x_{4j}) : a, b \geq 0, 1 \leq j \leq n-1\right].$$

We generalize x_{4j} $(1 \leq j \leq n-1)$ in Theorem 2.11 to x_i $(1 \leq i \leq 4n-1)$ in the same way as in (2.3) or (2.4). That is,
(i) If $i \equiv 0 \pmod 4$, then we define x_i to be the one in Theorem 2.11.
(ii) If $i \not\equiv 0 \pmod 4$, then there exist a, b and j uniquely such that $\deg Q_1^a Q_2^b(x_{4j}) = i$ (where we put $x_0 = [1]$). Then we define x_i by $x_i = Q_1^a Q_2^b(x_{4j})$. From the elements $x_1, \dots, x_{4n-1}$, we define an increasing sequence of ideals of $H_*(\Omega_0^3 Sp(n); \mathbf{Z}/2)$, I_ν, in the same way as in (2.6).

Proposition 2.12. *For $n \geq 1$, $H_*(W(\Omega Sp(n)); \mathbf{Z}/2)$ is isomorphic to the following module:*

$$\bigoplus_{i=2}^{4n-1} s\alpha_i \otimes I_i \oplus \frac{H_*(\Omega_0^3 Sp(n); \mathbf{Z}/2)}{I_{4n}}.$$

Proof. In the same way as in the proof of Lemma 2.5 (1), we see that $J_*(\alpha_i)$ contains the term x_i. Then from the fibration $\Omega_0^3 Sp(n) \to W(\Omega Sp(n)) \to \Sigma \mathbf{R}P^{4n-1}$, we can prove the proposition in the same way as in Proposition 2.7. ∎

Let p be a prime with $p \geq 2n+3$ and α_{4n-1} be the generator of $H_{4n-1}(\mathbf{R}P^{4n-1}; \mathbf{Z}/p)$. Localized at p, $Sp(n)$ is homotopy equivalent to $\prod_{i=1}^{n} S^{4i-1}$.

Proposition 2.13. *For $n \geq 1$ and $p \geq 2n+3$, $H_*(W(\Omega Sp(n)); \mathbf{Z}/p)$ is isomorphic to the following module:*

$$\begin{aligned} s\alpha_{4n-1} \otimes H_*(\Omega_0^3 Sp(n); \mathbf{Z}/p) \oplus H_*(\Omega_0^3 Sp(n); \mathbf{Z}/p) \\ \cong H_*(S^{4n}; \mathbf{Z}/p) \otimes H_*(\Omega_0^3 Sp(n); \mathbf{Z}/p). \end{aligned}$$

Proof. In the same way as in the proof of Lemma 2.8 (2), we have $J_*(\alpha_{4n-1}) = 0$ for $p \geq 2n+3$. Then the proposition follows easily. ∎

3 Proofs of Theorems A, B and C

The Atiyah-Jones conjecture suggests the stability principle for $i_k : M(k,G) \to \Omega_k^3 G$. The conjecture was first proved in [1] for $G = SU(2)$. Later the conjecture was confirmed for general G and the range of homotopy equivalence for i_k was improved. At present, the following solutions are known:

Theorem 3.1 ([9], [13], [14]). *The map $i_k : M(k,G) \to \Omega_k^3 G$ induces homomorphisms in homotopy groups, which are*
(i) *if $G = SU(n)$ $(n \geq 3)$, isomorphisms in dimensions less than $2k+1$, and an epimorphism in dimension $2k+1$;*
(ii) *if $G = Sp(n)$ $(n \geq 2)$, isomorphisms in dimensions less than $\left[\frac{k}{2}\right]$, and an epimorphism in dimension $\left[\frac{k}{2}\right]$;*
(iii) *if $G = SU(2) = Sp(1)$, isomorphisms in dimensions less than k, and an epimorphism in dimension k.*

By [2] we have a C_4-structure in $\coprod_{k \geq 1} M(k,G)$, in particular we have a loop sum $* : M(k,G) \times M(k',G) \to M(k+k',G)$. Similarly, we have a map $* : X(k,G) \times M(k',G) \to X(k+k',G)$, which is an extension of the above loop sum. Passing to homology, $\oplus_{k \geq 1} H_*(X(k,G); \mathbf{Z}/p)$ is a module over $\oplus_{k \geq 1} H_*(M(k,G); \mathbf{Z}/p)$.
We prove the following:

Lemma 3.2. *For all primes p, we have the following long exact sequence:*

$$\begin{aligned} \cdots \to H_q(M(k,G); \mathbf{Z}/p) \to H_q(X(k,G); \mathbf{Z}/p) \\ \to \bigoplus_{i+j=q} \widetilde{H}_i(\Sigma G/C; \mathbf{Z}/p) \otimes H_j(M(k-1,G); \mathbf{Z}/p) \\ \xrightarrow{\phi_q} H_{q-1}(M(k,G); \mathbf{Z}/p) \to \cdots. \end{aligned}$$

Moreover, ϕ_q is given by

$$\phi_q(s\gamma \otimes z) = \gamma * z,$$

where $\gamma \in \widetilde{H}_i(G/C; \mathbf{Z}/p)$, $s\gamma$ is the suspension of γ and $z \in H_j(M(k-1,G); \mathbf{Z}/p)$. As in Section 1, let $C = C_G(SU(2))$ be the centralizer of $SU(2)$ in G.

Proof. We recall how the stratum $\mathbf{R}^4 \times M(k-1,G)$ is attached to $M(k,G)$ and builds $X(k,G)$ in (1.4) (see [6, Section 3.4]). By [2] we have a diffeomorphism $M(1,G) \cong \mathbf{R}^5 \times G/C$. Hence $M(1,G)$ is parametrized by $\mathbf{R}^4 \times (0,\delta) \times G/C$, where $\delta > 0$ is a small number. The point $x \in \mathbf{R}^4$ represents the center of the instanton, while the scale $\lambda \in (0,\delta)$ represents the spread of the curvature density function. The remaining parameter is G/C. For $G = SU(2)$, $SU(2)/C \cong SO(3)$ represents the framing at infinity.
Let the scale $\lambda \in (0,\delta)$ approach 0. At the moment when $\lambda = 0$, all the elements of G/C are identified to a point. Thus, when $\lambda = 0$, the remaining parameter is only $\mathbf{R}^4$ and a tubular neighborhood of this $\mathbf{R}^4$ in $X(1,G)$ is given by $\left(\mathbf{R}^4 \times [0,\delta] \times G/C\right)/\sim$, where we put $(x,0,u) \sim (x,0,u')$.
Similarly, we apply the above construction to $X(k,G)$. We consider $\mathbf{R}^4 \times (0,\delta) \times G/C \times M(k-1,G) \subset M(k,G)$. As above, let the scale $\lambda \in (0,\delta)$ approach 0. When $\lambda = 0$, we obtain a new stratum $\mathbf{R}^4 \times M(k-1,G)$ and a tubular neighborhood of the stratum in $X(k,G)$ is given by

$$\nu = \left(\mathbf{R}^4 \times [0,\delta] \times G/C \times M(k-1,G)\right)/\sim,$$

where we put $(x,0,u,A) \sim (x,0,u',A)$.
Let $\partial\nu$ be the boundary of ν. We consider the homology long exact sequence of the pair $(X(k,G), M(k,G))$. By excision, we have

$$H_*(X(k,G), M(k,G); \mathbf{Z}/p) \cong H_*(\nu, \partial\nu; \mathbf{Z}/p). \tag{3.3}$$

Moreover,

$$\begin{aligned} &H_*(\nu,\partial\nu;\mathbf{Z}/p) \\ \cong &H_*\left(\frac{[0,\delta]\times G/C}{\{0\}\times G/C}, \{\delta\}\times G/C; \mathbf{Z}/p\right) \otimes H_*(M(k-1,G);\mathbf{Z}/p) \\ \cong &\widetilde{H}_*(\Sigma G/C;\mathbf{Z}/p) \otimes H_*(M(k-1,G);\mathbf{Z}/p). \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we obtain the long exact sequence of Lemma 3.2.
By the same argument as in [7], it is easy to see that in the long exact sequence of the pair $(X(k,G), M(k,G))$, the connecting homomorphism $\partial_* : H_q(X(k,G), M(k,G); \mathbf{Z}/p) \to H_{q-1}(M(k,G); \mathbf{Z}/p)$ corresponds to ϕ_q in Lemma 3.2. This completes the proof of Lemma 3.2. ∎

In order to prove Theorem A, we first determine $H_q(X(k,SU(n)); \mathbf{Z}/2)$ for $q \leq 2k$. We consider the long exact sequence of Lemma 3.2 starting from

$$\bigoplus_{i+j=2k+1} \widetilde{H}_i(\Sigma SU(n)/C; \mathbf{Z}/2) \otimes H_j(M(k-1,SU(n)); \mathbf{Z}/2).$$

Using Theorem 3.1 (i), ϕ_q $(q \leq 2k+1)$ in Lemma 3.2 is written as follows:

$$\phi_q : \bigoplus_{i+j=q} \widetilde{H}_i(\Sigma SU(n)/C;\mathbf{Z}/2) \otimes H_j(\Omega_0^3 SU(n);\mathbf{Z}/2) \to H_{q-1}(\Omega_0^3 SU(n);\mathbf{Z}/2), \quad (3.5)$$

where $\phi_q(s\gamma \otimes z) = J_*(\gamma) * z$.
From the exactness, we have

$$H_q(X(k, SU(n));\mathbf{Z}/2) \cong \operatorname{Ker}\phi_q \oplus \operatorname{Coker}\phi_{q+1}.$$

We claim:

$$\operatorname{Ker}\phi_q \cong \Big[\bigoplus_{i=2}^{a} s\alpha_{2i} \otimes I_{2i} \oplus \bigoplus_{i=b}^{n-2} s\beta_{4i+1} \otimes I_{4i+1} \oplus \bigoplus_{i=b}^{n-2} s\beta_{4i+3} \otimes H_*(\Omega_0^3 SU(n);\mathbf{Z}/2)\Big]_q \quad (q \leq 2k) \quad (3.6)$$

and

$$\operatorname{Coker}\phi_{q+1} \cong \left[\frac{H_*(\Omega_0^3 SU(n);\mathbf{Z}/2)}{I_{4n-3}}\right]_q \quad (q \leq 2k). \quad (3.7)$$

Here a and b in (3.6) are defined in Proposition 2.7 and $[\quad]_q$ denotes the subspace consisting of elements of degree q.
It is easy to prove the following:

Lemma 3.8. *Let $\varphi : V \to V'$ be a linear mapping between vector spaces and let W be a subspace of V. Let $\varphi|W : W \to V'$ be the restriction and $\overline{\varphi} : V/W \to V'/\varphi(W)$ be the induced mapping. Then we have*

$$\operatorname{Ker}\varphi \cong \operatorname{Ker}\varphi|W \oplus \operatorname{Ker}\overline{\varphi}.$$

Hereafter, let $q \leq 2k$ and we drop q from all modules and maps.

STEP 1. For φ and W in Lemma 3.8, we take $\varphi = \phi$ in (3.5) and $W = s\alpha_2 \otimes H_*(\Omega_0^3 SU(n);\mathbf{Z}/2)$. Then using Lemma 2.5 (1) (i), we have

$$\operatorname{Ker}\phi \cong 0 \oplus \operatorname{Ker}\overline{\phi},$$

where $\overline{\phi} : \mathbf{Z}/2\{s\alpha_4, \ldots, s\beta_{4n-5}\} \otimes H_*(\Omega_0^3 SU(n);\mathbf{Z}/2) \longrightarrow \frac{H_*(\Omega_0^3 SU(n);\mathbf{Z}/2)}{I_4}$ is the mapping induced from ϕ. ($\mathbf{Z}/2\{s\alpha_4, \ldots, s\beta_{4n-5}\}$ is a free $\mathbf{Z}/2$-module with a $\mathbf{Z}/2$-basis $\{s\alpha_4, \ldots, s\beta_{4n-5}\}$.)

STEP 2. For φ and W in Lemma 3.8, we take $\varphi = \overline{\phi}$ and $W = s\alpha_4 \otimes H_*(\Omega_0^3 SU(n);\mathbf{Z}/2)$. Then

$$\operatorname{Ker}\overline{\phi} \cong s\alpha_4 \otimes I_4 \oplus \operatorname{Ker}\overline{\overline{\phi}},$$

where $\overline{\overline{\phi}} : \mathbf{Z}/2\{s\alpha_6, \ldots, s\beta_{4n-5}\} \otimes H_*(\Omega_0^3 SU(n);\mathbf{Z}/2) \to \frac{H_*(\Omega_0^3 SU(n);\mathbf{Z}/2)}{I_6}$ is the mapping induced from $\overline{\phi}$.

We repeat these steps with respect to $s\alpha_{2i}$ and $s\beta_{2i+1}$. Then we obtain (3.6). (3.7) follows from Lemma 2.5. Now from (3.6), (3.7) and Proposition 2.7, we have Theorem A for $p = 2$. Theorem A for $p \geq 2n+1$ is proved from Proposition 2.9. This completes the proof of Theorem A.

Similarly, Theorem B follows from Propositions 2.12, 2.13 and Theorem 3.1 (ii), and Theorem C follows from Remark 2.10 and Theorem 3.1 (iii).

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