# Theorems of Perron type for uniform exponential dichotomy of linear skew-product semiflows 

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#### Abstract

We study the connections between the uniform exponential dichotomy of a discrete linear skew-product semiflow and the uniform admissibility of the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$. We give necessary and sufficient conditions for uniform exponential dichotomy of linear skew-product semiflows in terms of the uniform admissibility of the pairs $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ and $\left(C_{0}\left(\mathbb{R}_{+}, X\right)\right.$, $C_{00}\left(\mathbb{R}_{+}, X\right)$ ), respectively. We generalize a dichotomy theorem due to Van Minh, Räbiger and Schnaubelt for the case of linear skew-product semiflows.


## 1 Introduction

The concept of exponential dichotomy for differential equations was introduced by Perron, connecting the problem of conditional stability of the system $\dot{x}=A(t) x$ with the existence of bounded solutions for the equation $\dot{x}=A(t) x+f(t, x)$ in a Banach space $X$. These ideas have been continued by Massera and Schäffer ([10]), Coppel ([5]), Daleckii and Krein ([6]), respectively.

A significant step in this direction, in infinite dimensional spaces, has been made by Henry in [7], where he introduced the concept of discrete dichotomy. Thus, he characterized the dichotomy of a sequence of bounded linear operators $\left(T_{n}\right)_{n \in \mathbf{Z}}$ in terms of existence and uniqueness of bounded solutions for $x_{n+1}=T_{n} x_{n}+f_{n}$, for every bounded sequence $\left(f_{n}\right)_{n \in \mathbf{Z}}$. Moreover, Henry presented the connection between the discrete dichotomy and the exponential dichotomy for an evolution family.

[^0]In the last few years significant questions in the theory of evolution equations have been answered using the theory of linear skew-product flows (see [2], [3], [4], [8], [9], [13]-[15], [18], [19]). In [8], Latushkin, Montgomery - Smith and Randolph expressed exponential dichotomy of a linear skew-product flow using a family of weighted shift operators acting on $c_{0}(\mathbf{Z}, X)$. In the same spirit of Henry's theory, Chow and Leiva introduced in [3] the concept of pointwise discrete dichotomy for a skew-product sequence $\left(\Phi_{n}(\theta), \sigma(\theta, n)\right)_{n \in \mathbb{N}}$, over $X \times \Theta$, with $X$ a Banach space and $\Theta$ a compact Hausdorff space, generalizing Henry's theorem for the equations of type $x_{n+1}=\Phi_{n}(\theta) x_{n}+f_{n}$. They presented the relation between the pointwise discrete dichotomy and the uniform discrete dichotomy and emphasized the equivalence between the exponential dichotomy of a linear skew-product semiflow $\pi=(\Phi, \sigma)$ and the discrete exponential dichotomy of the associated skew-product sequence $\hat{\pi}=(\Phi(\sigma(\theta, n), 1), \sigma(\theta, n))_{n \in \mathbb{N}}$ in certain conditions. Other generalizations for the dichotomy and robustness theorems due to Henry, for the case of linear skew-product semiflows have been considered by Pliss and Sell in [18].

Another approach (see [9]) has been presented by Latushkin and Schnaubelt for the case of dichotomy of strongly continuous cocycles over flows, in locally compact spaces. They established the connection between the exponential dichotomy of a strongly continuous cocycle over a flow and the dichotomy of the associated discrete cocycle, employing an evolution semigroup technique. They extended some important theorems in the field of evolution families, proving that the uniform exponential dichotomy of a linear skew-product semiflow is equivalent to the hyperbolicity of its evolution semigroup on $C_{0}(\Theta, X)$. Moreover, in the spirit of Perron's theory they related, in certain conditions, exponential dichotomy of a strongly continuous cocycle $\Phi$ over a flow $\sigma$ to the existence and uniqueness of bounded continuous solutions for the mild inhomogeneous equation

$$
u(\sigma(\theta, t))=e^{-\lambda t} \Phi(\theta, t) u(\theta)+\int_{0}^{t} e^{-\lambda(t-\tau)} \Phi(\sigma(\theta, \tau), t-\tau) g(\sigma(\theta, \tau)) d \tau
$$

on $C_{0}(\Theta, X)$ or $C_{b}(\Theta, X)$. This result can be interpreted as a generalization of a dichotomy theorem due to Van Minh, Räbiger and Schnaubelt (see [16]), given by

Theorem 1.1. An evolution family $\mathcal{U}=\{U(t, s)\}_{t \geq s \geq 0}$ is uniformly exponentially dichotomic if and only if for every $u \in C_{0}\left(\mathbb{R}_{+}, X\right)$ there is $f \in C_{0}\left(\mathbb{R}_{+}, X\right)$ such that

$$
f(t)=U(t, s) f(s)+\int_{s}^{t} U(t, \tau) u(\tau) d \tau, \quad \forall t \geq s \geq 0
$$

and the space

$$
X_{1}=\left\{x \in X: \lim _{t \rightarrow \infty} U(t, 0) x=0\right\}
$$

is closed and complemented in $X$.
The purpose of the present paper is to give other generalizations of this result for the case of linear skew-product semiflows. First, we shall present the connections between the uniform exponential dichotomy of a discrete linear skew-product semiflow $\hat{\pi}$ and the uniform admissibility of the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ for $\hat{\pi}$. Then, we shall prove that the uniform exponential dichotomy of a linear skew-product semiflow is equivalent to the uniform exponential dichotomy of the associated discrete linear skew-product semiflow. We shall study the relation between the uniform exponential dichotomy of a linear skew-product semiflow and the uniform admissibility
of the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ for the associated discrete linear skew-product semiflow. In the spirit of Perron's theory we shall characterize the uniform exponential dichotomy in terms of uniform admissibility of the pair $\left(C_{0}\left(\mathbb{R}_{+}, X\right), C_{00}\left(\mathbb{R}_{+}, X\right)\right)$. In fact, this is a bounded input - bounded output condition for uniform exponential dichotomy of linear skew-product semiflows. In this manner, we shall extend the result of Van Minh, Räbiger and Schnaubelt for the case of linear skew-product semiflows.

## 2 Discrete admissibility and uniform exponential dichotomy for discrete linear skew-product semiflows

In this section we shall present the connections between the uniform admissibility of the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ for a discrete linear skew-product semiflow and its uniform exponential dichotomy.

Let $X$ be a Banach space, let $(\Theta, d)$ be a metric space and let $\mathcal{E}=X \times \Theta$. We shall denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators from $X$ into itself. Throughout the paper, the norm on $X$ and on $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$.

Definition 2.1. A mapping $\hat{\sigma}: \Theta \times \mathbb{Z} \rightarrow \Theta$ is said to be a discrete flow on $\Theta$, if it has the following properties:
(i) $\hat{\sigma}(\theta, 0)=\theta$, for all $\theta \in \Theta$;
(ii) $\hat{\sigma}(\theta, m+n)=\hat{\sigma}(\hat{\sigma}(\theta, m), n)$, for all $(\theta, m, n) \in \Theta \times \mathbb{Z}^{2}$.

Definition 2.2. A pair $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ is called a discrete linear skew-product semiflow on $\mathcal{E}=X \times \Theta$ if $\hat{\sigma}$ is a discrete flow on $\Theta$ and $\hat{\Phi}: \Theta \times \mathbb{N} \rightarrow \mathcal{B}(X)$ satisfies the following conditions:
(i) $\hat{\Phi}(\theta, 0)=I$, the identity operator on $X$, for all $\theta \in \Theta$;
(ii) $\hat{\Phi}(\theta, m+n)=\hat{\Phi}(\hat{\sigma}(\theta, n), m) \hat{\Phi}(\theta, n)$, for all $(\theta, m, n) \in \Theta \times \mathbb{N}^{2}$ (the discrete cocycle identity);
(iii) there are $M \geq 1$ and $\omega>0$ such that

$$
\|\hat{\Phi}(\theta, n)\| \leq M e^{\omega n}, \quad \forall(\theta, n) \in \Theta \times \mathbb{N} .
$$

Example 2.1. Let $X$ be a Banach space, let $\alpha>0$ and let

$$
\Theta:=\left\{T=\left\{T_{n}\right\}_{n \in \mathbb{Z}} \subset \mathcal{B}(X): \sup _{n \in \mathbb{Z}}\left\|T_{n}\right\| \leq \alpha\right\}
$$

endowed with the metric

$$
d(T, S)=\sum_{n=1}^{\infty} \frac{1}{2^{n}} \frac{\sup _{|k| \leq n}\left\|T_{k}-S_{k}\right\|}{1+\sup _{|k| \leq n}\left\|T_{k}-S_{k} \mid\right\|}
$$

We define

$$
\hat{\Phi}(T, 0)=I, \quad \hat{\Phi}\left(T, n_{0}\right)=T_{n_{0}-1} \ldots T_{1} T_{0}
$$

for all $T=\left\{T_{n}\right\}_{n \in \mathbb{Z}}$ and all $n_{0} \in \mathbb{N}^{*}$. If

$$
\hat{\sigma}: \Theta \times \mathbb{Z} \rightarrow \Theta, \quad \hat{\sigma}\left(T, n_{0}\right)=\left\{T_{n+n_{0}}\right\}_{n \in \mathbb{Z}}
$$

then $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ is a discrete linear skew-product semiflow on $\mathcal{E}=X \times \Theta$.
Definition 2.3. A discrete linear skew-product semiflow $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ is said to be uniformly exponentially dichotomic if there exist a family of projections $\{P(\theta)\}_{\theta \in \Theta} \subset$ $\mathcal{B}(X), K \geq 1$ and $\nu>0$ such that
(i) $\hat{\Phi}(\theta, n) P(\theta)=P(\hat{\sigma}(\theta, n)) \hat{\Phi}(\theta, n)$, for all $(\theta, n) \in \Theta \times \mathbb{N}$;
(ii) $\|\hat{\Phi}(\theta, n) x\| \leq K e^{-\nu n}\|x\|$, for all $x \in \operatorname{ImP}(\theta)$ and all $(\theta, n) \in \Theta \times \mathbb{N}$;
(iii) $\|\hat{\Phi}(\theta, n) x\| \geq \frac{1}{K} e^{\nu n}\|x\|$, for all $x \in \operatorname{Ker} P(\theta)$ and all $(\theta, n) \in \Theta \times \mathbb{N}$;
(iv) for every $(\theta, n) \in \Theta \times \mathbb{N}$ the restriction $\hat{\Phi}(\theta, n)_{\mid}: \operatorname{KerP}(\theta) \rightarrow \operatorname{Ker} P(\hat{\sigma}(\theta, n))$ is an isomorphism.

Remark 2.1. The condition (i) from Definition 2.3. is equivalent to

$$
\hat{\Phi}(\hat{\sigma}(\theta, n), m-n) P(\hat{\sigma}(\theta, n))=P(\hat{\sigma}(\theta, m)) \hat{\Phi}(\hat{\sigma}(\theta, n), m-n),
$$

for all $(\theta, m, n) \in \Theta \times \mathbb{N}^{2}, m \geq n$.
Proposition 2.1. Let $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ be a discrete linear skew-product semiflow. If $\hat{\pi}$ is uniformly exponentially dichotomic relative to the family of projections $\{P(\theta)\}_{\theta \in \Theta}$, then

$$
\sup _{\theta \in \Theta}\|P(\theta)\|<\infty
$$

Proof. The idea is the same as in [6]. For $\theta \in \Theta$ we define

$$
\delta_{\theta}:=\inf \left\{\left\|x_{1}+x_{2}\right\|: x_{1} \in \operatorname{ImP}(\theta), x_{2} \in \operatorname{Ker} P(\theta),\left\|x_{1}\right\|=\left\|x_{2}\right\|=1\right\}
$$

If $x_{1} \in \operatorname{ImP}(\theta), x_{2} \in \operatorname{Ker} P(\theta)$, with $\left\|x_{1}\right\|=\left\|x_{2}\right\|=1$, then, we have

$$
\begin{gathered}
\left\|x_{1}+x_{2}\right\| \geq \frac{1}{M} e^{-\omega n}\left\|\hat{\Phi}(\theta, n) x_{1}+\hat{\Phi}(\theta, n) x_{2}\right\| \\
\geq \frac{1}{M} e^{-\omega n}\left(\frac{1}{K} e^{\nu n}-K e^{-\nu n}\right), \quad \forall n \in \mathbb{N}
\end{gathered}
$$

where $M, \omega$ are given by Definition 2.2. and $K, \nu$ by Definition 2.3. It follows that there is $c>0$ such that $\delta_{\theta} \geq c$, for all $\theta \in \Theta$.

Let $\theta \in \Theta$ and $x \in X$ with $P(\theta) x \neq 0$ and $(I-P(\theta)) x \neq 0$. Then

$$
\begin{gathered}
\delta_{\theta} \leq\left\|\frac{P(\theta) x}{\|P(\theta) x\|}+\frac{(I-P(\theta)) x}{\|(I-P(\theta)) x\|}\right\|= \\
=\frac{1}{\|P(\theta) x\|}\left\|P(\theta) x+\frac{\|P(\theta) x\|}{\|(I-P(\theta)) x\|}(I-P(\theta)) x\right\|= \\
=\frac{1}{\|P(\theta) x\|}\left\|x+\frac{\|P(\theta) x\|-\|(I-P(\theta)) x\|}{\|(I-P(\theta)) x\|}(I-P(\theta)) x\right\| \leq \frac{2\|x\|}{\|P(\theta) x\|} .
\end{gathered}
$$

It results that $\|P(\theta)\| \leq 2 / \delta_{\theta}$, for all $\theta \in \Theta$, which ends the proof.

Definition 2.4. Let $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ be a discrete linear skew-product semiflow on $\mathcal{E}=X \times \Theta$ and let $(x, \theta) \in \mathcal{E}$. We say that $\hat{\Phi}$ has a negative continuation relative to $(x, \theta)$ if there is a function $\varphi: \mathbf{Z}_{-} \rightarrow X$ such that $\varphi(0)=x$ and

$$
\varphi(m+n)=\hat{\Phi}(\hat{\sigma}(\theta, m), n) \varphi(m), \quad \forall(m, n) \in \mathbf{Z}_{-} \times \mathbb{N} \text { with } m+n \leq 0
$$

For a discrete linear skew-product semiflow $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ on $\mathcal{E}=X \times \Theta$, for every $\theta \in \Theta$, we consider the linear subspaces

$$
\begin{gathered}
\hat{X}_{1}(\theta)=\left\{x \in X: \lim _{n \rightarrow \infty} \hat{\Phi}(\theta, n) x=0\right\} \\
\hat{X}_{2}(\theta)=\{x \in X: \hat{\Phi} \text { has a negative continuation } \varphi \text { relative to }(x, \theta) \\
\text { such that } \left.\lim _{m \rightarrow-\infty} \varphi(m)=0\right\} .
\end{gathered}
$$

Lemma 2.1. If $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ is discrete linear skew-product semiflow then
(i) $\hat{\Phi}(\theta, n) \hat{X}_{1}(\theta) \subset \hat{X}_{1}(\hat{\sigma}(\theta, n))$, for all $(\theta, n) \in \Theta \times \mathbb{N}$;
(ii) $\hat{\Phi}(\theta, n) \hat{X}_{2}(\theta) \subset \hat{X}_{2}(\hat{\sigma}(\theta, n))$, for all $(\theta, n) \in \Theta \times \mathbb{N}$.

Proof. (i) It is obvious.
(ii) Let $(\theta, n) \in \Theta \times \mathbb{N}, x \in \hat{X}_{2}(\theta)$ and let $\varphi$ be a negative continuation of $\hat{\Phi}$ relative to $(x, \theta)$ with $\lim _{m \rightarrow-\infty} \varphi(m)=0$. We denote by $y=\hat{\Phi}(\theta, n) x$ and we define

$$
\psi: \mathbf{Z}_{-} \rightarrow X, \quad \psi(m)=\hat{\Phi}(\hat{\sigma}(\theta, m), n) \varphi(m)
$$

Then, it is easy to see that $\psi$ is a negative continuation of $\hat{\Phi}$ relative to $(y, \hat{\sigma}(\theta, n))$ and $\lim _{m \rightarrow-\infty} \psi(m)=0$, so $y \in \hat{X}_{2}(\hat{\sigma}(\theta, n))$.

Proposition 2.2. Let $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ be a discrete linear skew-product semiflow. If $\hat{\pi}$ is uniformly exponentially dichotomic relative to the family of projections $\{P(\theta)\}_{\theta \in \Theta}$, then
(i) $\operatorname{Im} P(\theta)=\hat{X}_{1}(\theta)$, for all $\theta \in \Theta$;
(ii) $\operatorname{Ker} P(\theta)=\hat{X}_{2}(\theta)$, for all $\theta \in \Theta$.

Proof. (i) It is obvious that $\operatorname{Im} P(\theta) \subset \hat{X}_{1}(\theta)$, for all $\theta \in \Theta$.
Let $\theta \in \Theta$ and $x \in \hat{X}_{1}(\theta)$. If $K, \nu$ are given by Definition 2.3. then

$$
\begin{gathered}
\|x-P(\theta) x\| \leq K e^{-\nu n}\|\hat{\Phi}(\theta, n)(I-P(\theta)) x\| \leq \\
\leq K e^{-\nu n}\left[\|\hat{\Phi}(\theta, n) x\|+K e^{-\nu n}\|P(\theta) x\|\right], \quad \forall n \in \mathbb{N} .
\end{gathered}
$$

It follows that $P(\theta) x=x$, so $x \in \operatorname{Im} P(\theta)$. It results that $\hat{X}_{1}(\theta) \subset \operatorname{ImP}(\theta)$.
(ii) Let $x \in \hat{X}_{2}(\theta)$ and let $\varphi$ be a negative continuation of $\hat{\Phi}$ relative to $(x, \theta)$ with $\lim _{m \rightarrow-\infty} \varphi(m)=0$. We define

$$
\delta: \mathbf{Z}_{-} \rightarrow X, \quad \delta(m)=P(\hat{\sigma}(\theta, m)) \varphi(m) .
$$

If $y=P(\theta) x$ then it is easy to verify that $\delta(0)=y$ and

$$
\delta(m+n)=\hat{\Phi}(\hat{\sigma}(\theta, m), n) \delta(m), \quad \forall(m, n) \in \mathbf{Z}_{-} \times \mathbb{N}, m+n \leq 0
$$

It follows that

$$
\begin{gathered}
\|y\|=\|\hat{\Phi}(\hat{\sigma}(\theta,-n), n) P(\hat{\sigma}(\theta,-n)) \varphi(-n)\| \leq \\
\leq K e^{-\nu n}\|P(\hat{\sigma}(\theta,-n)) \varphi(-n)\| \leq K e^{-\nu n} \sup _{\theta \in \Theta}\|P(\theta)\|\|\varphi(-n)\| \rightarrow 0
\end{gathered}
$$

as $n \rightarrow \infty$, which shows that $x \in \operatorname{Ker} P(\theta)$. So $\hat{X}_{2}(\theta) \subset \operatorname{Ker} P(\theta)$.
Conversely, let $x \in \operatorname{Ker} P(\theta)$. We define

$$
\psi: \mathbf{Z}_{-} \rightarrow X, \quad \psi(m)=\hat{\Phi}(\hat{\sigma}(\theta, m),-m)_{\mid}^{-1} x
$$

Then $\psi(0)=x$. For every $(m, n) \in \mathbf{Z}_{-} \times \mathbb{N}$ with $m+n \leq 0$ we have

$$
\hat{\Phi}(\hat{\sigma}(\theta, m),-m)=\hat{\Phi}(\hat{\sigma}(\theta, m+n),-m-n) \hat{\Phi}(\hat{\sigma}(\theta, m), n)
$$

so

$$
\hat{\Phi}(\hat{\sigma}(\theta, m+n),-m-n)_{\mid}^{-1} x=\hat{\Phi}(\hat{\sigma}(\theta, m), n) \hat{\Phi}(\hat{\sigma}(\theta, m),-m)_{\mid}^{-1} x
$$

which shows that

$$
\psi(m+n)=\hat{\Phi}(\hat{\sigma}(\theta, m), n) \psi(m), \quad \forall(m, n) \in \mathbf{Z}_{-} \times \mathbb{N}, m+n \leq 0
$$

It follows that $\psi$ is a negative continuation for $\hat{\Phi}$. Moreover,

$$
\|\psi(m)\| \leq K e^{\nu m}\|x\| \rightarrow 0, \text { as } m \rightarrow-\infty
$$

and hence $x \in \hat{X}_{2}(\theta)$. We conclude that $\operatorname{Ker} P(\theta) \subset \hat{X}_{2}(\theta)$, which ends the proof.
Remark 2.2. From the previous proposition it follows that if the discrete linear skew-product semiflow $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ is uniformly exponentially dichotomic, then the family of projections $\{P(\theta)\}_{\theta \in \Theta}$ is uniquely determined by the conditions from Definition 2.3.

We consider $c_{0}(\mathbb{N}, X)=\left\{s: \mathbb{N} \rightarrow X \mid \lim _{n \rightarrow \infty} s(n)=0\right\}$, which is a Banach space with respect to the norm $\|\|s\|\|=\sup _{n \in \mathbb{N}}\|s(n)\|$. We shall denote by $c_{00}(\mathbb{N}, X):=\{s \in$ $\left.c_{0}(\mathbb{N}, X): s(0)=0\right\}$.

Let $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ be a discrete linear skew-product semiflow on $\mathcal{E}=X \times \Theta$. For every $\theta \in \Theta$ we consider the discrete equation $\left(E_{d}^{\theta}\right)$ given by

$$
\gamma(m)=\hat{\Phi}(\hat{\sigma}(\theta, n), m-n) \gamma(n)+\sum_{j=n+1}^{m} \hat{\Phi}(\hat{\sigma}(\theta, j), m-j) s(j), \forall m, n \in \mathbb{N}, m>n
$$

Remark 2.3. If $\gamma \in c_{0}(\mathbb{N}, X)$ and $s_{1}, s_{2} \in c_{00}(\mathbb{N}, X)$ such that $\left(\gamma, s_{1}\right),\left(\gamma, s_{2}\right)$ verify the equation $\left(E_{d}^{\theta}\right)$, then $s_{1}=s_{2}$. Thus, for every $\theta \in \Theta$ we consider the closed linear operator

$$
Q_{\theta}: c_{0}(\mathbb{N}, X) \rightarrow c_{00}(\mathbb{N}, X), \quad Q_{\theta} \gamma=s
$$

Remark 2.4. It is easy to see that $\operatorname{Ker} Q_{\theta}=\{\gamma: \mathbb{N} \rightarrow X: \gamma(n)=$ $\Phi(\theta, n) \gamma(0)$ and $\left.\gamma(0) \in \hat{X}_{1}(\theta)\right\}$, for all $\theta \in \Theta$.
For every $\theta \in \Theta$ we define

$$
D^{2}\left(Q_{\theta}\right)=\left\{\gamma \in c_{0}(\mathbb{N}, X): \gamma(0) \in \hat{X}_{2}(\theta)\right\} .
$$

Definition 2.5. The pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is said to be uniformly admissible for the discrete linear skew-product semiflow $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ if
(i) for every $\theta \in \Theta$ and every $s \in c_{00}(\mathbb{N}, X)$ there is $\gamma_{\theta, s} \in c_{0}(\mathbb{N}, X)$ such that $\left(\gamma_{\theta, s}, s\right)$ verifies the equation $\left(E_{d}^{\theta}\right)$;
(ii) there is $c>0$ such that $\left\|\left|Q_{\theta} \gamma\| \| c\right|\right\| \gamma\left\|\|\right.$, for all $\gamma \in D^{2}\left(Q_{\theta}\right)$ and all $\theta \in \Theta$.

Remark 2.5. If the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is uniformly admissible for the linear skew-product semiflow $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$, then

$$
\hat{X}_{1}(\theta) \cap \hat{X}_{2}(\theta)=\{0\}, \quad \forall \theta \in \Theta .
$$

Indeed, let $\theta \in \Theta$ and $x \in \hat{X}_{1}(\theta) \cap \hat{X}_{2}(\theta)$. We define

$$
\gamma: \mathbb{N} \rightarrow X, \quad \gamma(n)=\Phi(\theta, n) x
$$

From Remark 2.4. and $x \in \hat{X}_{1}(\theta)$ we have that $\gamma \in \operatorname{Ker} Q_{\theta}$. Since the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is uniformly admissible for $\hat{\pi}$ and $\gamma(0)=x \in \hat{X}_{2}(\theta)$, there is $c>0$ such that $0=\| \| Q_{\theta} \gamma\| \| \geq c\| \| \gamma\| \|$. It results that $\|\|\gamma\|\|=0$ and hence $x=\gamma(0)=0$.

Remark 2.6. If the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is uniformly admissible for the discrete linear skew-product semiflow $\hat{\pi}$ on $\mathcal{E}=X \times \Theta$, then the operator $Q_{\theta \mid D^{2}\left(Q_{\theta}\right)}$ : $D^{2}\left(Q_{\theta}\right) \rightarrow c_{00}(\mathbb{N}, X)$ is injective.

In what follows, we shall present necessary conditions given by the uniform admissibility of the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ for a discrete linear skew-product semiflow.

Theorem 2.1. Let $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ be a discrete linear skew-product semiflow on $\mathcal{E}=X \times \Theta$. If the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is uniformly admissible for $\hat{\pi}$ then there exist $K \geq 1$ and $\nu>0$ such that

$$
\begin{equation*}
\|\hat{\Phi}(\theta, n) x\| \leq K e^{-\nu n}\|x\|, \quad \forall x \in \hat{X}_{1}(\theta), \forall(\theta, n) \in \Theta \times \mathbb{N} \tag{2.1}
\end{equation*}
$$

Proof. From hypothesis there is $\nu \in(0,1)$ sufficiently small such that $2 \nu>e^{\nu}-1$ and

$$
\left\|\mid Q_{\theta} \gamma\right\|\|\geq 2 \nu\|\|\gamma\| \|, \quad \forall \gamma \in D^{2}\left(Q_{\theta}\right), \forall \theta \in \Theta
$$

Let $\theta \in \Theta$ and $x \in \hat{X}_{1}(\theta) \backslash\{0\}$. Let $I_{\theta, x}=\{k \in \mathbb{N}: \hat{\Phi}(\theta, k) x \neq 0\}$. It is possible to have the following two cases:

Case 1. $I_{\theta, x}=\mathbb{N}$. For every $n \in \mathbb{N}^{*}$ we define the sequences

$$
\begin{aligned}
& s_{n}: \mathbb{N} \rightarrow X, \\
& s_{n}(k)=\left\{\begin{array}{cll}
0 & , & k=0 \text { or } k \geq n+1 \\
\frac{\hat{\Phi}(\theta, k) x}{\|\hat{\Phi}(\theta, k) x\|}, & k \in\{1, \ldots, n\}
\end{array}\right. \\
& \gamma_{n}: \mathbb{N} \rightarrow X, \quad \gamma_{n}(k)=\left\{\begin{array}{cl}
0 & k=0 \\
\sum_{j=1}^{k} \frac{\hat{\Phi}(\theta, k) x}{\|\hat{\Phi}(\theta, j) x\|}, & k \in\{1, \ldots, n\} \\
\sum_{j=1}^{n} \frac{\hat{\Phi}(\theta, k) x}{\|\hat{\Phi}(\theta, j) x\|}, & k \geq n+1 .
\end{array}\right.
\end{aligned}
$$

Since $x \in \hat{X}_{1}(\theta)$ it follows that $\gamma_{n} \in c_{00}(\mathbb{N}, X)$ and obviously $s_{n} \in c_{00}(\mathbb{N}, X)$, too. Moreover $\left(\gamma_{n}, s_{n}\right)$ verifies the equation $\left(E_{d}^{\theta}\right)$, for all $n \in \mathbb{N}^{*}$. Thus, we have that

$$
1=\| \| s_{n}\| \|=\left\|\mid Q_{\theta} \gamma_{n}\right\|\|\geq 2 \nu\|\left\|\gamma_{n}\right\| \|, \quad \forall n \in \mathbb{N}^{*}
$$

so

$$
\begin{equation*}
2 \nu \sum_{j=1}^{n} \frac{1}{\|\hat{\Phi}(\theta, j) x\|} \leq \frac{1}{\|\hat{\Phi}(\theta, n) x\|}, \quad \forall n \in \mathbb{N}^{*} \tag{2.2}
\end{equation*}
$$

We consider the sequence

$$
\delta: \mathbb{N}^{*} \rightarrow \mathbb{R}_{+}, \quad \delta(n)=\sum_{j=1}^{n} \frac{1}{\|\hat{\Phi}(\theta, j) x\|}
$$

From (2.2) we deduce that

$$
\frac{1}{\|\hat{\Phi}(\theta, n+1) x\|} \geq 2 \nu \delta(n) \geq\left(e^{\nu}-1\right) \delta(n)
$$

so $\delta(n+1) \geq e^{\nu} \delta(n)$, for all $n \in \mathbb{N}^{*}$. Then, we obtain that

$$
\frac{1}{\|\hat{\Phi}(\theta, n+1) x\|} \geq 2 \nu \delta(n+1) \geq 2 \nu e^{\nu n} \frac{1}{\|\hat{\Phi}(\theta, 1) x\|}, \quad \forall n \in \mathbb{N}^{*}
$$

so

$$
\|\hat{\Phi}(\theta, n+1) x\| \leq L e^{-\nu(n+1)}\|x\|, \quad \forall n \in \mathbb{N}^{*}
$$

where $L=M e^{\omega+\nu} /(2 \nu)$ and $M, \omega$ are given by Definition 2.2. Setting $K=2 L$ we obtain (2.1).
Case 2. $\quad I_{\theta, x}=\{0, \ldots, p\}$. In this case we define the sequences

$$
\begin{aligned}
& s: \mathbb{N} \rightarrow X, \quad s(k)=\left\{\begin{array}{cll}
0 & , & k=0 \text { or } k \geq p+1 \\
\frac{\hat{\Phi}(\theta, k) x}{\|\hat{\Phi}(\theta, k) x\|}, & k \in\{1, \ldots, p\}
\end{array}\right. \\
& \gamma: \mathbb{N} \rightarrow X,
\end{aligned} \quad \gamma(k)=\left\{\begin{array}{cl}
0 & k=0 \\
\sum_{j=1}^{k} \frac{\hat{\Phi}(\theta, k) x}{\|\hat{\Phi}(\theta, j) x\|}, & k \in\{1, \ldots, p\} \\
\sum_{j=1}^{p} \frac{\hat{\Phi}(\theta, k) x}{\|\tilde{\Phi}(\theta, j) x\|}, & k \geq p+1 .
\end{array}\right.
$$

Then we deduce that $s, \gamma \in c_{00}(\mathbb{N}, X)$ and the pair $(\gamma, s)$ verifies the equation $\left(E_{d}^{\theta}\right)$. Using an analogous argument as in Case 1. we obtain the conclusion.

Corollary 2.1. Let $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ be a discrete linear skew-product semiflow on $\mathcal{E}=X \times \Theta$. If the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is uniformly admissible for $\hat{\pi}$ then $\hat{X}_{1}(\theta)$ is a closed linear subspace, for all $\theta \in \Theta$.
Proof. Let $\theta \in \Theta$ be fixed and $\left(x_{p}\right) \subset \hat{X}_{1}(\theta)$ converging to $x \in X$. It follows that there is $L>0$ such that $\left\|x_{p}\right\| \leq L$, for all $p \in \mathbb{N}$. If $K, \nu$ are given by Theorem 2.1., we deduce that $\left\|\hat{\Phi}(\theta, n) x_{p}\right\| \leq K L e^{-\nu n}$, for all $n, p \in \mathbb{N}$. Hence we obtain that $\|\hat{\Phi}(\theta, n) x\| \leq K L e^{-\nu n}$, for all $n \in \mathbb{N}$, so $x \in \hat{X}_{1}(\theta)$. It follows that $\hat{X}_{1}(\theta)$ is closed.

Theorem 2.2. Let $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ be a discrete linear skew-product semiflow on $\mathcal{E}=X \times \Theta$. If the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is uniformly admissible for $\hat{\pi}$ then there exist $K \geq 1$ and $\nu>0$ such that

$$
\begin{equation*}
\|\hat{\Phi}(\theta, n) x\| \geq \frac{1}{K} e^{\nu n}\|x\|, \quad \forall x \in \hat{X}_{2}(\theta), \forall(\theta, n) \in \Theta \times \mathbb{N} . \tag{2.3}
\end{equation*}
$$

Proof. Let $\nu \in(0,1)$ such that $2 \nu>e^{\nu}-1$ and

$$
\left\|\left|Q_{\theta} \gamma\right|\right\| \geq 2 \nu\| \| \gamma\| \|, \quad \forall \gamma \in D^{2}\left(Q_{\theta}\right), \forall \theta \in \Theta
$$

Let $\theta \in \Theta$ and $x \in \hat{X}_{2}(\theta) \backslash\{0\}$. From Remark 2.5. it follows that $\hat{\Phi}(\theta, n) x \neq 0$, for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}^{*}$ we consider the sequences

$$
\begin{gathered}
s_{n}: \mathbb{N} \rightarrow X, \quad s_{n}(k)=\left\{\begin{array}{cll}
0 & , & k=0 \text { or } k \geq n+1 \\
-\frac{\hat{\Phi}(\theta, k) x}{\|\hat{\Phi}(\theta, k) x\|} & , & k \in\{1, \ldots, n\}
\end{array}\right. \\
\gamma_{n}: \mathbb{N} \rightarrow X, \quad \gamma_{n}(k)=\left\{\begin{array}{cl}
0 \quad, & k \geq n \\
\sum_{j=k+1}^{n} \frac{\hat{\Phi}(\theta, k) x}{\|\hat{\Phi}(\theta, j) x\|}, & k \leq n-1 .
\end{array}\right.
\end{gathered}
$$

We easily deduce that $\gamma_{n} \in c_{0}(\mathbb{N}, X), s_{n} \in c_{00}(\mathbb{N}, X)$ and $\left(\gamma_{n}, s_{n}\right)$ verifies the equation $\left(E_{d}^{\theta}\right)$, for all $n \in \mathbb{N}^{*}$. Moreover

$$
\gamma_{n}(0)=\sum_{j=1}^{n} \frac{x}{\|\hat{\Phi}(\theta, j) x\|} \in \hat{X}_{2}(\theta),
$$

so $\gamma_{n} \in D^{2}\left(Q_{\theta}\right)$ and $Q_{\theta} \gamma_{n}=s_{n}$, for all $n \in \mathbb{N}^{*}$. It follows that

$$
1=\left|\left\|s_{n}\right\|\|=\|\right| Q_{\theta} \gamma_{n}\left|\|\geq 2 \nu \mid\| \gamma_{n}\| \|, \quad \forall n \in \mathbb{N}^{*} .\right.
$$

Thus, we obtain that

$$
\begin{equation*}
2 \nu \sum_{j=k+1}^{\infty} \frac{1}{\|\hat{\Phi}(\theta, j) x\|} \leq \frac{1}{\|\hat{\Phi}(\theta, k) x\|}, \quad \forall k \in \mathbb{N} . \tag{2.4}
\end{equation*}
$$

Let

$$
f, F: \mathbb{N} \rightarrow \mathbf{R}_{+}^{*}, \quad f(n)=\frac{1}{\|\hat{\Phi}(\theta, n) x\|}, \quad F(n)=\sum_{j=n}^{\infty} f(j)
$$

From (2.4) we deduce that

$$
f(n) \geq 2 \nu F(n+1) \geq\left(e^{\nu}-1\right) F(n+1), \quad \forall n \in \mathbb{N}
$$

so

$$
F(n) \geq e^{\nu} F(n+1), \quad \forall n \in \mathbb{N}
$$

It results that

$$
f(n) \leq F(n) \leq e^{-\nu(n-1)} F(1) \leq \frac{e^{\nu}}{2 \nu} e^{-\nu n} f(0)=\frac{e^{\nu}}{2 \nu} e^{-\nu n} \frac{1}{\|x\|}, \quad \forall n \in \mathbb{N}^{*}
$$

and hence we obtain (2.3) for $K=e^{\nu} / 2 \nu$.
Corollary 2.2. If the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is uniformly admissible for the discrete linear skew-product semiflow $\hat{\pi}$ on $\mathcal{E}=X \times \Theta$, then $\hat{X}_{2}(\theta)$ is a closed linear subspace, for all $\theta \in \Theta$.
Proof. Let $\theta \in \Theta$. If $y \in \hat{X}_{2}(\theta)$ and $\varphi$ is a negative continuation for $\hat{\Phi}$ relative to $(y, \theta)$ with $\lim _{m \rightarrow-\infty} \varphi(m)=0$, then it is easy to see that $\varphi(m) \in \hat{X}_{2}(\hat{\sigma}(\theta, m))$, for all $m \in \mathbf{Z}_{-}$.

Let $\left(x_{p}\right) \subset \hat{X}_{2}(\theta)$ converging to $x \in X$. For every $p \in \mathbb{N}$ there is a negative continuation $\varphi_{p}$ for $\hat{\Phi}$ relative to $\left(x_{p}, \theta\right)$ such that $\lim _{m \rightarrow-\infty} \varphi_{p}(m)=0$. Since

$$
\begin{equation*}
\varphi_{p}(m+n)=\hat{\Phi}(\hat{\sigma}(\theta, m), n) \varphi_{p}(m), \quad \forall(m, n, p) \in \mathbf{Z}_{-} \times \mathbf{N}^{2}, m+n \leq 0 \tag{2.5}
\end{equation*}
$$

for $K, \nu$ given by Theorem 2.2., it follows that

$$
\begin{gather*}
\left\|x_{p}-x_{k}\right\|=\left\|\hat{\Phi}(\hat{\sigma}(\theta,-n), n)\left(\varphi_{p}(-n)-\varphi_{k}(-n)\right)\right\| \geq  \tag{2.6}\\
\geq \frac{1}{K} e^{\nu n}\left\|\varphi_{p}(-n)-\varphi_{k}(-n)\right\|, \quad \forall n, p, k \in \mathbb{N} .
\end{gather*}
$$

Using the fact that $\left(x_{p}\right)_{p \in \mathbb{N}}$ is fundamental, from (2.6) it follows that for every $m \in \mathbf{Z}_{-}$the sequence $\left(\varphi_{p}(m)\right)_{p}$ is fundamental, so it is convergent. We denote by $\varphi(m):=\lim _{p \rightarrow \infty} \varphi_{p}(m)$, for all $m \in \mathbf{Z}_{-}$. Hence $\varphi(0)=x$ and from (2.5) we obtain that

$$
\varphi(m+n)=\hat{\Phi}(\hat{\sigma}(\theta, m), n) \varphi(m), \quad \forall(m, n) \in \mathbf{Z}_{-} \times \mathbb{N}, m+n \leq 0
$$

From (2.6) we deduce that

$$
\|\varphi(-n)\| \leq K e^{-\nu n}\left\|x_{p}-x\right\|+\left\|\varphi_{p}(-n)\right\|, \quad \forall(n, p) \in \mathbf{N}^{2}
$$

It results that $\lim _{m \rightarrow-\infty} \varphi(m)=0$, so $x \in \hat{X}_{2}(\theta)$.
Proposition 2.3. If the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is uniformly admissible for the discrete linear skew-product semiflow $\hat{\pi}$ on $\mathcal{E}=X \times \Theta$ and $\hat{X}_{1}(\theta)+\hat{X}_{2}(\theta)=X$, for all $\theta \in \Theta$, then

$$
\hat{X}_{2}(\hat{\sigma}(\theta, n))=\hat{\Phi}(\theta, n) \hat{X}_{2}(\theta), \quad \forall(\theta, n) \in \Theta \times \mathbb{N}
$$

Proof. Let $(\theta, n) \in \Theta \times \mathbb{N}^{*}$ and $x \in \hat{X}_{2}(\hat{\sigma}(\theta, n))$. We define

$$
s: \mathbb{N} \rightarrow X, \quad s(k)=\left\{\begin{array}{rl}
-x, & k=n \\
0, & k \neq n
\end{array} .\right.
$$

Since $s \in c_{00}(\mathbb{N}, X)$ there is $\gamma \in c_{0}(\mathbb{N}, X)$ such that

$$
\begin{equation*}
\gamma(m)=\hat{\Phi}(\hat{\sigma}(\theta, k), m-k) \gamma(k)+\sum_{j=k+1}^{m} \hat{\Phi}(\hat{\sigma}(\theta, j), m-j) s(j), \forall m>k \tag{2.7}
\end{equation*}
$$

From (2.7), for $k=n$ we deduce that

$$
\gamma(m)=\hat{\Phi}(\hat{\sigma}(\theta, n), m-n) \gamma(n), \quad \forall m>n .
$$

Since $\gamma \in c_{0}(\mathbb{N}, X)$ it follows that $\gamma(n) \in \hat{X}_{1}(\hat{\sigma}(\theta, n))$.
For $m=n$ and $k=0$, from (2.7) we obtain $\gamma(n)=\hat{\Phi}(\theta, n) \gamma(0)-x$. Let $x_{1} \in \hat{X}_{1}(\theta)$ and $x_{2} \in \hat{X}_{2}(\theta)$ such that $\gamma(0)=x_{1}+x_{2}$. Then $\gamma(n)-\hat{\Phi}(\theta, n) x_{1}=$ $\hat{\Phi}(\theta, n) x_{2}-x$. Using Lemma 2.1. and Remark 2.5., it follows that $\hat{\Phi}(\theta, n) x_{2}-x=0$, so $x \in \hat{\Phi}(\theta, n) \hat{X}_{2}(\theta)$. Thus, $\hat{X}_{2}(\hat{\sigma}(\theta, n)) \subset \hat{\Phi}(\theta, n) \hat{X}_{2}(\theta)$. Applying once again Lemma 2.1. we obtain the conclusion.

The main result of this section presents the connection between the uniform exponential dichotomy of a discrete linear skew-product semiflow and the uniform admissibility of the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$.

Theorem 2.3. Let $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ be a discrete linear skew-product semiflow on $\mathcal{E}=X \times \Theta$. The following assertions are equivalent:
(i) $\hat{\pi}$ is uniformly exponentially dichotomic;
(ii) the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is uniformly admissible for $\hat{\pi}$ and $\hat{X}_{1}(\theta)+\hat{X}_{2}(\theta)=$ $X$, for all $\theta \in \Theta$.

Proof. Necessity. Let $\{P(\theta)\}_{\theta \in \Theta}$ be the family of projections given by Definition 2.3. From Proposition 2.2. we have that $\operatorname{ImP}(\theta)=\hat{X}_{1}(\theta), \operatorname{Ker} P(\theta)=\hat{X}_{2}(\theta)$, so $\hat{X}_{1}(\theta)+\hat{X}_{2}(\theta)=X$, for all $\theta \in \Theta$.

Let $\theta \in \Theta$ be fixed. For $s \in c_{00}(\mathbb{N}, X)$ we define the sequence

$$
\begin{align*}
& \gamma_{s}: \mathbb{N}  \tag{2.8}\\
& \rightarrow X, \quad \gamma_{s}(n)=\sum_{k=0}^{n} \hat{\Phi}(\hat{\sigma}(\theta, k), n-k) P(\hat{\sigma}(\theta, k)) s(k) \\
&-\sum_{k=n+1}^{\infty} \hat{\Phi}(\hat{\sigma}(\theta, n), k-n)_{\mid}^{-1}(I-P(\hat{\sigma}(\theta, k))) s(k)
\end{align*}
$$

where for every $k \geq n, \hat{\Phi}(\hat{\sigma}(\theta, n), k-n)_{\mid}^{-1}$ denotes the inverse of the operator $\hat{\Phi}(\hat{\sigma}(\theta, n), k-n)_{\mid}: \operatorname{Ker} P(\hat{\sigma}(\theta, n)) \rightarrow \operatorname{Ker} P(\hat{\sigma}(\theta, k))$.
¿From Proposition 2.1. there exists $L=\sup _{\theta \in \Theta}\|P(\theta)\|<\infty$. It follows that $\gamma_{s} \in c_{0}(\mathbb{N}, X)$ and an easy computation shows that the pair $\left(\gamma_{s}, s\right)$ verifies the equation $\left(E_{d}^{\theta}\right)$. Moreover

$$
\gamma_{s}(0)=-\sum_{k=1}^{\infty} \hat{\Phi}(\theta, k)_{\mid}^{-1}(I-P(\hat{\sigma}(\theta, k))) s(k) \in \hat{X}_{2}(\theta)
$$

so $\gamma_{s} \in D^{2}\left(Q_{\theta}\right)$ and $Q_{\theta} \gamma_{s}=s$.
Let $\gamma \in D^{2}\left(Q_{\theta}\right)$ and $s=Q_{\theta} \gamma$. Using the relation from above and the fact that $Q_{\theta \mid D^{2}\left(Q_{\theta}\right)}$ is injective (see Remark 2.6.) it follows that $\gamma$ is expressed by relation (2.8). If $K, \nu$ are given by Definition 2.3. we obtain that

$$
\|\gamma(n)\| \leq K(L+1) \frac{2}{1-e^{-\nu}}\| \| s\| \|, \quad \forall n \in \mathbb{N} .
$$

Then, for $c=\left(1-e^{-\nu}\right) / 2 K(L+1)$ we have

$$
\left\|\left|Q_{\theta} \gamma\|\geq c \mid\| \gamma\| \|, \quad \forall \gamma \in D^{2}\left(Q_{\theta}\right)\right.\right.
$$

Because $\theta \in \Theta$ was arbitrary and $c$ does not depend on $\theta$, it results that the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is uniformly admissible for $\hat{\pi}$, which ends the necessity.
Sufficiency. From hypothesis, Corollary 2.1., Corollary 2.2. and Remark 2.5. it follows that $\hat{X}_{1}(\theta) \oplus \hat{X}_{2}(\theta)=X$, for all $\theta \in \Theta$. For every $\theta \in \Theta$ let $P(\theta)$ be the projection corresponding to $\hat{X}_{1}(\theta)$, i.e. $\operatorname{ImP}(\theta)=\hat{X}_{1}(\theta)$ and $\operatorname{KerP}(\theta)=\hat{X}_{2}(\theta)$. Using Lemma 2.1. it follows that

$$
\hat{\Phi}(\theta, n) P(\theta)=P(\hat{\sigma}(\theta, n)) \hat{\Phi}(\theta, n), \quad \forall(\theta, n) \in \Theta \times \mathbb{N} .
$$

From Theorem 2.2. and Proposition 2.3. it follows that the restriction $\hat{\Phi}(\theta, n)_{\mid}$: $\operatorname{Ker} P(\theta) \rightarrow \operatorname{Ker} P(\hat{\sigma}(\theta, n))$ is an isomorphism. Finally, using Theorem 2.1. and Theorem 2.2. we obtain that $\hat{\pi}$ is uniformly exponentially dichotomic.

## 3 Admissibility and uniform exponential dichotomy of linear skewproduct semiflows

In what follows we shall establish the connections between the uniform exponential dichotomy of a linear skew-product semiflow and the uniform admissibility of the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$. We shall extend this result for the case of uniform admissibility of the pair $\left(C_{0}\left(\mathbb{R}_{+}, X\right), C_{00}\left(\mathbb{R}_{+}, X\right)\right)$, obtaining a generalization for a theorem due to Van Minh, Räbiger and Schnaubelt.

As in the second section, let X be a Banach space, let $\Theta$ be a metric space and let $\mathcal{E}=X \times \Theta$.

Definition 3.1. A mapping $\sigma: \Theta \times \mathbb{R} \rightarrow \Theta$ is said to be a flow on $\Theta$, if it has the following properties:
(i) $\sigma(\theta, 0)=\theta$, for all $\theta \in \Theta$;
(ii) $\sigma(\theta, s+t)=\sigma(\sigma(\theta, s), t)$, for all $(\theta, s, t) \in \Theta \times \mathbf{R}^{2}$;
(iii) $\sigma$ is continuous.

Definition 3.2. A pair $\pi=(\Phi, \sigma)$ is called a linear skew-product semiflow on $\mathcal{E}=X \times \Theta$ if $\sigma$ is a flow on $\Theta$ and $\Phi: \Theta \times \mathbb{R}_{+} \rightarrow \mathcal{B}(X)$ satisfies the following conditions:
(i) $\Phi(\theta, 0)=I$, the identity operator on $X$, for all $\theta \in \Theta$;
(ii) $\Phi(\theta, t+s)=\Phi(\sigma(\theta, s), t) \Phi(\theta, s)$, for all $(\theta, t, s) \in \Theta \times \mathbf{R}_{+}^{2}$ (the cocycle identity);
(iii) there are $M \geq 1$ and $\omega>0$ such that

$$
\|\Phi(\theta, t)\| \leq M e^{\omega t}, \quad \forall(\theta, t) \in \Theta \times \mathbb{R}_{+}
$$

If, moreover, for every $x \in X$ the mapping $(\theta, t) \mapsto \Phi(\theta, t) x$ is continuous, then $\pi$ is said strongly continuous linear skew-product semiflow.

Remark 3.1. If $\pi=(\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E}=X \times \Theta$ then one can associate to $\pi$ a discrete linear skew-product semiflow $\hat{\pi}=(\hat{\Phi}, \hat{\sigma})$ by $\hat{\Phi}(\theta, n)=\Phi(\theta, n)$ and $\hat{\sigma}(\theta, m)=\sigma(\theta, m)$, for all $(\theta, n, m) \in \Theta \times \mathbb{N} \times \mathbf{Z}$.

Example 3.1. Let $\Theta=\mathbb{R}, \sigma(\theta, t)=\theta+t$ and let $\mathcal{U}=\{U(t, s)\}_{t \geq s}$ be an evolution family on the Banach space $X$. We define $\Phi(\theta, t)=U(t+\theta, \theta)$, for all $(\theta, t) \in \mathbf{R} \times \mathbb{R}_{+}$. Then $\pi=(\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E}=X \times \Theta$ called the linear skew-product semiflow generated by the evolution family $\mathcal{U}$.

Example 3.2. Let $\Theta$ be a locally compact metric space, let $\sigma$ be a flow on $\Theta$ and let $\mathbf{T}=\{T(t)\}_{t \geq 0}$ be a $C_{0}$-semigroup on $X$. If $\Phi_{T}(\theta, t)=T(t)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_{+}$ then the pair $\pi_{T}=\left(\Phi_{T}, \sigma\right)$, is a strongly continuous linear skew-product semiflow on $\mathcal{E}=X \times \Theta$, which is called the linear skew-product semiflow generated by the $C_{0}$-semigroup $\mathbf{T}$ and the flow $\sigma$.

Example 3.3. Let $X$ be a Banach space. We consider $C(\mathbf{R})$, the space of all continuous functions $f: \mathbb{R} \rightarrow \mathbf{R}$, which is metrizable with the metric

$$
d(f, g)=\sum_{n=1}^{\infty} \frac{1}{2^{n}} \frac{d_{n}(f, g)}{1+d_{n}(f, g)}
$$

where $d_{n}(f, g)=\sup _{t \in[-n, n]}|f(t)-g(t)|$.
Let $a: \mathbb{R} \rightarrow \mathbb{R}_{+}$be a continuous function such that there exists $\lim _{t \rightarrow \pm \infty} a(t) \in \mathbb{R}$. If we denote by $a_{s}(t)=a(t+s)$ and by $\Theta$ the closure of $\left\{a_{s}: s \in \mathbb{R}_{+}\right\}$in $C(\mathbf{R})$, then

$$
\sigma: \Theta \times \mathbb{R} \rightarrow \Theta, \quad \sigma(\theta, t)(s):=\theta(t+s)
$$

is a flow on $\Theta$ and for

$$
\Phi: \Theta \times \mathbb{R}_{+} \rightarrow \mathcal{B}(X), \quad \Phi(\theta, t) x=\exp \left(\int_{0}^{t} \theta(\tau) d \tau\right) x
$$

we have that $\pi=(\Phi, \sigma)$ is a strongly continuous linear skew-product semiflow on $\mathcal{E}=X \times \Theta$.

Definition 3.3. A linear skew-product semiflow $\pi=(\Phi, \sigma)$ is said to be uniformly exponentially dichotomic if there exist a family of projections $\{P(\theta)\}_{\theta \in \Theta} \subset$ $\mathcal{B}(X), K \geq 1$ and $\nu>0$ such that
(i) $\Phi(\theta, t) P(\theta)=P(\sigma(\theta, t)) \Phi(\theta, t)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_{+}$;
(ii) $\|\Phi(\theta, t) x\| \leq K e^{-\nu t}\|x\|$, for all $x \in \operatorname{Im} P(\theta)$ and all $(\theta, t) \in \Theta \times \mathbb{R}_{+}$;
(iii) $\|\Phi(\theta, t) x\| \geq \frac{1}{K} e^{\nu t}\|x\|$, for all $x \in \operatorname{Ker} P(\theta)$ and all $(\theta, t) \in \Theta \times \mathbb{R}_{+}$;
(iv) for every $(\theta, t) \in \Theta \times \mathbb{R}_{+}$, the restriction $\Phi(\theta, t)_{\mid}: \operatorname{Ker} P(\theta) \rightarrow \operatorname{Ker} P(\sigma(\theta, t))$ is an isomorphism.

Remark 3.2. As in Proposition 2.1. it follows that if $\pi=(\Phi, \sigma)$ is uniformly exponentially dichotomic relative to the family of projections $\{P(\theta)\}_{\theta \in \Theta}$, then

$$
\sup _{\theta \in \Theta}\|P(\theta)\|<\infty
$$

Definition 3.4. Let $\pi=(\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E}=$ $X \times \Theta$. We say that $\Phi$ has a negative continuation relative to $(x, \theta)$ if there is a function $h: \mathbf{R}_{-} \rightarrow X$ such that $h(0)=x$ and

$$
h(s+t)=\Phi(\sigma(\theta, s), t) h(s), \quad \forall(s, t) \in \mathbf{R}_{-} \times \mathbb{R}_{+} \text {with } s+t \leq 0
$$

For a linear skew-product semiflow $\pi=(\Phi, \sigma)$ on $\mathcal{E}=X \times \Theta$, for every $\theta \in \Theta$, we consider the linear subspaces

$$
X_{1}(\theta)=\left\{x \in X: \lim _{t \rightarrow \infty} \Phi(\theta, t) x=0\right\}
$$

$$
\begin{aligned}
& X_{2}(\theta)=\{x \in X: \Phi \text { has a negative continuation } h \text { relative to }(x, \theta) \\
& \text { such that } \left.\lim _{s \rightarrow-\infty} h(s)=0\right\} .
\end{aligned}
$$

Lemma 3.1. If $\pi=(\Phi, \sigma)$ is linear skew-product semiflow on $\mathcal{E}=X \times \Theta$ then
(i) $\Phi(\theta, t) X_{1}(\theta) \subset X_{1}(\sigma(\theta, t))$, for all $(\theta, t) \in \Theta \times \mathbb{R}_{+}$;
(ii) $\Phi(\theta, t) X_{2}(\theta) \subset X_{2}(\sigma(\theta, t))$, for all $(\theta, t) \in \Theta \times \mathbb{R}_{+}$.

Proof. It follows in an analogous manner as in Lemma 2.1.
Lemma 3.2. If $\pi=(\Phi, \sigma)$ is linear skew-product semiflow on $\mathcal{E}=X \times \Theta$ and $\hat{\pi}$ is the associated discrete linear skew-product semiflow then
(i) $X_{1}(\theta)=\hat{X}_{1}(\theta)$, for all $\theta \in \Theta$;
(ii) $X_{2}(\theta)=\hat{X}_{2}(\theta)$, for all $\theta \in \Theta$.

Proof. (i) It is a simple exercise.
(ii) Let $\theta \in \Theta$. It is sufficient to show that $\hat{X}_{2}(\theta) \subset X_{2}(\theta)$, the other inclusion being trivial.

Therefore, let $x \in \hat{X}_{2}(\theta)$ and let $\varphi$ be a negative continuation for $\hat{\Phi}$ relative to $(x, \theta)$, with $\lim _{m \rightarrow-\infty} \varphi(m)=0$. We define

$$
h: \mathbf{R}_{-} \rightarrow X, \quad h(t)=\Phi(\sigma(\theta,[t]), t-[t]) \varphi([t])
$$

Then a simple computation shows that

$$
h(s+t)=\Phi(\sigma(\theta, s), t) h(s), \quad \forall(s, t) \in \mathbf{R}_{-} \times \mathbb{R}_{+}, s+t \leq 0
$$

Moreover $\lim _{s \rightarrow-\infty} h(s)=0$. Hence $x \in X_{2}(\theta)$, which ends the proof.

Theorem 3.1. Let $\pi=(\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E}=X \times \Theta$ and let $\hat{\pi}$ be the discrete linear skew-product semiflow associated to $\pi$. Then $\pi$ is uniformly exponentially dichotomic if and only if $\hat{\pi}$ is uniformly exponentially dichotomic.
Proof. Necessity is obvious.
Sufficiency. Since $\hat{\pi}$ is uniformly exponentially dichotomic there are a family of projections $\{P(\theta)\}_{\theta \in \Theta}$ and two constants $K \geq 1$ and $\nu>0$ such that

1. $\|\Phi(\theta, n) x\| \leq K e^{-\nu n}\|x\|$, for all $x \in \operatorname{Im} P(\theta)$ and all $(\theta, n) \in \Theta \times \mathbb{N}$;
2. $\|\Phi(\theta, n) x\| \geq \frac{1}{K} e^{\nu n}\|x\|$, for all $x \in \operatorname{Ker} P(\theta)$ and all $(\theta, n) \in \Theta \times \mathbb{N}$;
3. the restriction $\Phi(\theta, n)_{\mid}: \operatorname{Ker} P(\theta) \rightarrow \operatorname{Ker} P(\sigma(\theta, n))$ is an isomorphism for all $(\theta, n) \in \Theta \times \mathbb{N}$.
From Proposition 2.2. and Lemma 3.2. it follows that $\operatorname{Im} P(\theta)=X_{1}(\theta)$ and $\operatorname{Ker} P(\theta)=X_{2}(\theta)$, for all $\theta \in \Theta$. Using Lemma 3.1. we obtain that

$$
\Phi(\theta, t) P(\theta)=P(\sigma(\theta, t)) \Phi(\theta, t), \quad \forall(\theta, t) \in \Theta \times \mathbb{R}_{+}
$$

Let now $t>0$ and $n=[t]$. Since

$$
\Phi(\theta, t) x=\Phi(\sigma(\theta, n), t-n) \Phi(\theta, n) x, \quad \forall x \in \operatorname{Ker} P(\theta)
$$

it follows that it is sufficient to show that $\Phi(\sigma(\theta, n), t-n)_{\mid}: \operatorname{Ker} P(\sigma(\theta, n)) \rightarrow$ $\operatorname{Ker} P(\sigma(\theta, t))$ is an isomorphism. But,

$$
\Phi(\sigma(\theta, n), 1) x=\Phi(\sigma(\theta, t), n+1-t) \Phi(\sigma(\theta, n), t-n) x, \forall x \in \operatorname{Ker} P(\sigma(\theta, n))
$$

so for every $x \in \operatorname{Ker} P(\sigma(\theta, n))$

$$
x=\Phi(\sigma(\theta, n), 1)_{\mid}^{-1} \Phi(\sigma(\theta, t), n+1-t) \Phi(\sigma(\theta, n), t-n) x
$$

which means that $\Phi(\sigma(\theta, n), t-n)$ has a left-inverse.
Because for every $x \in \operatorname{Ker} P(\sigma(\theta, t-1))$ we have

$$
\Phi(\sigma(\theta, t-1), 1) x=\Phi(\sigma(\theta, n), t-n) \Phi(\sigma(\theta, t-1), n+1-t) x
$$

we deduce that for every $x \in \operatorname{Ker} P(\sigma(\theta, t))$

$$
x=\Phi(\sigma(\theta, n), t-n) \Phi(\sigma(\theta, t-1), n+1-t) \Phi(\sigma(\theta, t-1), 1)_{\mid}^{-1} x,
$$

so $\Phi(\sigma(\theta, n), t-n)$ has a right-inverse.
Hence, $\Phi(\theta, t)_{\mid}: \operatorname{Ker} P(\theta) \rightarrow \operatorname{Ker} P(\sigma(\theta, t))$ is an isomorphism, for all $(\theta, t) \in$ $\Theta \times \mathbb{R}_{+}$.

Let $(\theta, t) \in \Theta \times \mathbb{R}_{+}$and $n=[t]$. If $x \in \operatorname{Im} P(\theta)$ and $M, \omega$ are given by Definition 3.2., we have that

$$
\|\Phi(\theta, t) x\| \leq M e^{\omega}\|\Phi(\theta, n) x\| \leq M K e^{\omega+\nu} e^{-\nu t}\|x\|
$$

If $x \in \operatorname{Ker} P(\theta)$, we deduce that

$$
\frac{1}{K} e^{\nu(n+1)}\|x\| \leq\|\Phi(\theta, n+1) x\| \leq M e^{\omega}\|\Phi(\theta, t) x\|
$$

so

$$
\|\Phi(\theta, t) x\| \geq \frac{1}{M K e^{\omega}} e^{\nu t}\|x\| .
$$

Setting $N=M K e^{\omega+\nu}$ we obtain the conclusion.

Remark 3.3. The result from above has been obtained by Latushkin and Schnaubelt in [9] for the case of strongly continuous linear skew-product semiflows. In [9] this theorem is proved using the equivalence between the uniform exponential dichotomy of a linear skew-product semiflow $\pi=(\Phi, \sigma)$ on $\mathcal{E}=X \times \Theta$ and the hyperbolicity of the associated evolution semigroup on $C_{0}(\Theta, X)$.

Corollary 3.1. Let $\pi=(\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E}=X \times \Theta$. Then the following assertions are equivalent:
(i) $\pi$ is uniformly exponentially dichotomic;
(ii) the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is uniformly admissible for the discrete linear skew-product semiflow $\hat{\pi}$ associated to $\pi$ and $X_{1}(\theta)+X_{2}(\theta)=X$, for all $\theta \in \Theta$.

Proof. It is immediate from Theorem 3.1., Lemma 3.2. and Theorem 2.3.
Let $C_{0}\left(\mathbb{R}_{+}, X\right)$ be the space of all continuous functions $f: \mathbb{R}_{+} \rightarrow X$ with the property that $\lim _{t \rightarrow \infty} f(t)=0$, which is a Banach space with respect to the norm $\left\|\left||f| \|=\sup _{t \in \mathbb{R}_{+}} f(t)\right.\right.$. We shall denote by $C_{00}\left(\mathbb{R}_{+}, X\right)=\left\{f \in C_{0}\left(\mathbb{R}_{+}, X\right): f(0)=0\right\}$.

Let $\pi=(\Phi, \sigma)$ be a strongly continuous linear skew-product semiflow on $\mathcal{E}=$ $X \times \Theta$. For every $\theta \in \Theta$ we consider the integral equation $\left(E_{c}^{\theta}\right)$ given by

$$
f(t)=\Phi(\sigma(\theta, s), t-s) f(s)+\int_{s}^{t} \Phi(\sigma(\theta, \tau), t-\tau) u(\tau) d \tau, \quad t \geq s \geq 0
$$

with $u, f \in C_{0}\left(\mathbb{R}_{+}, X\right)$.
Remark 3.4. If $u_{1}, u_{2} \in C_{00}\left(\mathbb{R}_{+}, X\right)$ and $f \in C_{0}\left(\mathbb{R}_{+}, X\right)$ such that the pairs $\left(f, u_{1}\right),\left(f, u_{2}\right)$ verify the equation $\left(E_{c}^{\theta}\right)$, then $u_{1}=u_{2}$. Hence, for every $\theta \in \Theta$ we consider the linear subspace

$$
D\left(A_{\theta}\right)=\left\{f \in C_{0}\left(\mathbb{R}_{+}, X\right): \exists u \in C_{00}\left(\mathbb{R}_{+}, X\right) \text { such that }(f, u) \text { verifies }\left(E_{c}^{\theta}\right)\right\}
$$

and the linear operator

$$
A_{\theta}: D\left(A_{\theta}\right) \rightarrow C_{00}\left(\mathbb{R}_{+}, X\right), \quad A_{\theta} f=u
$$

We shall denote by $D^{2}\left(A_{\theta}\right)=\left\{f \in D\left(A_{\theta}\right): f(0) \in X_{2}(\theta)\right\}$.
Definition 3.5. The pair $\left(C_{0}\left(\mathbb{R}_{+}, X\right), C_{00}\left(\mathbb{R}_{+}, X\right)\right)$ is said to be uniformly admissible for the strongly continuous linear skew-product semiflow $\pi=(\Phi, \sigma)$ on $\mathcal{E}=X \times \Theta$ if
(i) for every $\theta \in \Theta$ and every $u \in C_{00}\left(\mathbb{R}_{+}, X\right)$ there is $f_{\theta, u} \in C_{0}\left(\mathbb{R}_{+}, X\right)$ such that $\left(f_{\theta, u}, u\right)$ verifies the equation $\left(E_{c}^{\theta}\right)$;
(ii) there is $c>0$ such that $\left|\left\|A_{\theta} f\left|\|\geq c \mid\| f\| \|\right.\right.\right.$, for all $f \in D^{2}\left(A_{\theta}\right)$ and all $\theta \in \Theta$.

Remark 3.5. It is easy to see that for every $\theta \in \Theta, \operatorname{Ker} A_{\theta}=\left\{f \in D\left(A_{\theta}\right)\right.$ : $f(t)=\Phi(\theta, t) f(0)$ with $\left.f(0) \in X_{1}(\theta)\right\}$. Using this fact we observe that if condition (i) from Definition 3.5. holds for the strongly continuous linear skew-product semiflow $\pi=(\Phi, \sigma)$ on $\mathcal{E}=X \times \Theta$ and $X_{1}(\theta)+X_{2}(\theta)=X$, for all $\theta \in \Theta$, then for every $\theta \in \Theta$ and every $u \in C_{00}\left(\mathbb{R}_{+}, X\right)$ there exists $f_{\theta, u} \in D^{2}\left(A_{\theta}\right)$ such that $\left(f_{\theta, u}, u\right)$ verifies the equation $\left(E_{c}^{\theta}\right)$.

Before presenting the next theorem of the paper, we shall prove a necessary condition given by the uniform exponential dichotomy of a strongly continuous linear skew-product semiflow.

Proposition 3.1. Let $\pi=(\Phi, \sigma)$ be a strongly continuous linear skew-product semiflow on $\mathcal{E}=X \times \Theta$. If $\pi$ is uniformly exponentially dichotomic relative to the family of projections $\{P(\theta)\}_{\theta \in \Theta}$, then
(i) for every $(\theta, t) \in \Theta \times \mathbf{R}_{+}^{*}$ and every $x \in \operatorname{KerP}(\sigma(\theta, t))$ the mapping $s \rightarrow$ $\Phi(\sigma(\theta, s), t-s)_{\mid}^{-1} x$ is continuous on $[0, t]$;
(ii) for every $(x, \theta) \in \mathcal{E}$ the mapping $t \rightarrow P(\sigma(\theta, t)) x$ is continuous on $\mathbb{R}_{+}$.

Proof. (i) Let $t>0, \theta \in \Theta$ and $x \in \operatorname{Ker} P(\sigma(\theta, t))$. There is $y \in \operatorname{Ker} P(\theta)$ such that $x=\Phi(\theta, t) y$.

Let $s_{0} \in[0, t]$. It is easy to see that

$$
\begin{aligned}
& \Phi(\sigma(\theta, s), t-s)_{\mid}^{-1} x-\Phi\left(\sigma\left(\theta, s_{0}\right), t-s_{0}\right)_{\mid}^{-1} x= \\
& \quad=\Phi(\theta, s) y-\Phi\left(\theta, s_{0}\right) y \rightarrow 0, \text { as } s \rightarrow s_{0}
\end{aligned}
$$

(ii) Let $(x, \theta) \in \mathcal{E}$. Let $t_{0}>0$. We have that

$$
\begin{gathered}
\left\|P(\sigma(\theta, t)) x-P\left(\sigma\left(\theta, t_{0}\right)\right) x\right\| \leq\left\|P(\sigma(\theta, t)) x-P(\sigma(\theta, t)) \Phi\left(\sigma\left(\theta, t_{0}\right), t-t_{0}\right) x\right\|+ \\
+\left\|\Phi\left(\sigma\left(\theta, t_{0}\right), t-t_{0}\right) P\left(\sigma\left(\theta, t_{0}\right)\right) x-P\left(\sigma\left(\theta, t_{0}\right)\right) x\right\| \leq \\
\leq \sup _{\theta \in \Theta}\|P(\theta)\|\left\|\Phi\left(\sigma\left(\theta, t_{0}\right), t-t_{0}\right) x-x\right\|+ \\
+\left\|\Phi\left(\sigma\left(\theta, t_{0}\right), t-t_{0}\right) P\left(\sigma\left(\theta, t_{0}\right)\right) x-P\left(\sigma\left(\theta, t_{0}\right)\right) x\right\| \rightarrow 0
\end{gathered}
$$

as $t \searrow t_{0}$, so the mapping $P(\sigma(\theta, \cdot)) x$ is right-continuous in $t_{0}$.
Let $t<t_{0}$. Since

$$
\begin{aligned}
& (I-P(\sigma(\theta, t))) x=\Phi\left(\sigma(\theta, t), t_{0}-t\right)_{\mid}^{-1} \Phi\left(\sigma(\theta, t), t_{0}-t\right)(I-P(\sigma(\theta, t))) x= \\
= & \Phi\left(\sigma(\theta, t), t_{0}-t\right)_{\mid}^{-1}\left(I-P\left(\sigma\left(\theta, t_{0}\right)\right)\right) \Phi\left(\sigma(\theta, t), t-t_{0}\right) x \rightarrow\left(I-P\left(\sigma\left(\theta, t_{0}\right)\right)\right) x,
\end{aligned}
$$

as $t \nearrow t_{0}$, we obtain that the mapping $P(\sigma(\theta, \cdot)) x$ is left-continuous in $t_{0}$.
The next result of this section presents the connection between the uniform exponential dichotomy of a strongly continuous linear skew-product semiflow and the uniform admissibility of the pairs $\left(C_{0}\left(\mathbb{R}_{+}, X\right), C_{00}\left(\mathbb{R}_{+}, X\right)\right)$ and $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ respectively.

Theorem 3.2. Let $\pi=(\Phi, \sigma)$ be a strongly continuous linear skew-product semiflow on $\mathcal{E}=X \times \Theta$. The following assertions are equivalent:
(i) $\pi$ is uniformly exponentially dichotomic;
(ii) the pair $\left(C_{0}\left(\mathbb{R}_{+}, X\right), C_{00}\left(\mathbb{R}_{+}, X\right)\right)$ is uniformly admissible for $\pi$ and $X_{1}(\theta)+$ $X_{2}(\theta)=X$, for all $\theta \in \Theta$;
(iii) the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is uniformly admissible for the discrete linear skew-product semiflow $\hat{\pi}$ associated to $\pi$ and $X_{1}(\theta)+X_{2}(\theta)=X$, for all $\theta \in \Theta$.

Proof. (i) $\Longrightarrow$ (ii) Let $\{P(\theta)\}_{\theta \in \Theta}$ be the family of projections corresponding to the fact that $\pi$ is uniformly exponentially dichotomic. From Lemma 3.2. and Proposition 2.2. we have that $X_{1}(\theta)=\operatorname{ImP}(\theta)$ and $X_{2}(\theta)=\operatorname{Ker} P(\theta)$, for all $\theta \in \Theta$, so $X_{1}(\theta)+X_{2}(\theta)=X$, for all $\theta \in \Theta$.

Let $\theta \in \Theta$ and $u \in C_{00}\left(\mathbb{R}_{+}, X\right)$. We define the function

$$
\begin{aligned}
f_{u}: \mathbb{R}_{+} & \rightarrow X, \quad f_{u}(t)=\int_{0}^{t} \Phi(\sigma(\theta, s), t-s) P(\sigma(\theta, s)) u(s) d s- \\
& -\int_{t}^{\infty} \Phi(\sigma(\theta, t), s-t)_{\mid}^{-1}(I-P(\sigma(\theta, s))) u(s) d s
\end{aligned}
$$

where $\Phi(\sigma(\theta, t), s-t)_{\mid}^{-1}$ denotes the inverse of the operator $\Phi(\sigma(\theta, t), s-t)_{\mid}$: $\operatorname{Ker} P(\sigma(\theta, t)) \rightarrow \operatorname{Ker} P(\sigma(\theta, s))$. It follows that $f_{u} \in C_{0}\left(\mathbb{R}_{+}, X\right)$ and the pair $\left(f_{u}, u\right)$ verifies the equation $\left(E_{c}^{\theta}\right)$. Moreover

$$
f_{u}(0)=-\int_{0}^{\infty} \Phi(\theta, s)_{\mid}^{-1}(I-P(\sigma(\theta, s))) u(s) d s \in X_{2}(\theta)
$$

Using an analogous argument as in the necessity of Theorem 2.3., for $N, \nu$ given by Definition 3.3. we obtain that

$$
\left\|\left|\left\|A_{\theta} f \mid\right\| \geq c\| \| f\| \|, \quad \forall f \in D^{2}\left(A_{\theta}\right), \forall \theta \in \Theta\right.\right.
$$

where $c=\nu / 2 N(L+1)$ and $L:=\sup _{\theta \in \Theta}\|P(\theta)\|$.
$(i i) \Longrightarrow(i i i)$ Let $c>0$ such that

$$
\begin{equation*}
\left\|\left\|A_{\theta}\right\|\right\| \geq c\| \| f\| \|, \quad \forall f \in D^{2}\left(A_{\theta}\right), \forall \theta \in \Theta \tag{3.1}
\end{equation*}
$$

We consider a continuous function $\alpha:[0,1] \rightarrow[0,2]$ with the support contained in $(0,1)$ and

$$
\int_{0}^{1} \alpha(\tau) d \tau=1
$$

Let $\theta \in \Theta$ and $s \in c_{00}(\mathbb{N}, X)$. We define

$$
u: \mathbb{R}_{+} \rightarrow X, \quad u(t)=\Phi(\sigma(\theta,[t]), t-[t]) s([t]) \alpha(t-[t])
$$

Then $u$ is continuous and $u(0)=0$. If $M, \omega$ are given by Definition 3.2., we observe that

$$
\|u(t)\| \leq 2 M e^{\omega}\|s([t])\|, \quad \forall t \geq 0
$$

so $u \in C_{00}\left(\mathbb{R}_{+}, X\right)$. From hypothesis and Remark 3.5. there is $f_{u} \in D^{2}\left(A_{\theta}\right)$ such that

$$
f_{u}(t)=\Phi(\sigma(\theta, s), t-s) f_{u}(s)+\int_{s}^{t} \Phi(\sigma(\theta, \tau), t-\tau) u(\tau) d \tau, \quad \forall t \geq s \geq 0
$$

Let $m, n \in \mathbb{N}, m>n$. Then

$$
\begin{aligned}
& f_{u}(m)=\Phi(\sigma(\theta, n), m-n) f_{u}(n)+\int_{n}^{m} \Phi(\sigma(\theta, \tau), m-\tau) u(\tau) d \tau= \\
& \quad=\Phi(\sigma(\theta, n), m-n) f_{u}(n)+\sum_{j=n}^{m-1} \int_{j}^{j+1} \Phi(\sigma(\theta, \tau), m-\tau) u(\tau) d \tau .
\end{aligned}
$$

But, it is easy to see that

$$
\Phi(\sigma(\theta, \tau), m-\tau) u(\tau)=\Phi(\sigma(\theta, j), m-j) s(j) \alpha(\tau-j), \quad \forall \tau \in[j, j+1)
$$

and thus we deduce that

$$
f_{u}(m)=\Phi(\sigma(\theta, n), m-n) f_{u}(n)+\sum_{j=n}^{m-1} \Phi(\sigma(\theta, j), m-j) s(j), \forall m, n \in \mathbb{N}, m>n
$$

Let

$$
\begin{equation*}
\gamma: \mathbb{N} \rightarrow X, \quad \gamma(n)=f_{u}(n)+s(n) . \tag{3.2}
\end{equation*}
$$

It follows that

$$
\gamma(m)=\Phi(\sigma(\theta, n), m-n) \gamma(n)+\sum_{j=n+1}^{m} \Phi(\sigma(\theta, j), m-j) s(j), \forall m, n \in \mathbb{N}, m>n
$$

Because $f_{u} \in C_{0}\left(\mathbb{R}_{+}, X\right)$, we have that $\gamma \in c_{0}(\mathbb{N}, X)$ and from above $(\gamma, s)$ verifies the equation $\left(E_{d}^{\theta}\right)$. Moreover $\gamma(0)=f_{u}(0) \in X_{2}(\theta)$, so $\gamma \in D^{2}\left(Q_{\theta}\right)$.

Let $\theta \in \Theta, \gamma \in D^{2}\left(Q_{\theta}\right)$ and $s=Q_{\theta} \gamma$. Since $Q_{\theta}: D^{2}\left(Q_{\theta}\right) \rightarrow c_{00}(\mathbb{N}, X)$ is injective it follows that $\gamma$ is given by (3.2). So using (3.1) we have that

$$
\||\gamma|\| \leq\left\|\left|\left|f_{u}\| \|+\left\|\left|| s | \left\|\leq \frac{1}{c}\left|\| u \| \left\|+\left|\left\|s \left|\left\|\leq\left(\frac{2}{c} M e^{\omega}+1\right)|\|s \mid\| .\right.\right.\right.\right.\right.\right.\right.\right.\right.\right.\right.\right.
$$

Setting $c^{\prime}=c /\left(2 M e^{\omega}+c\right)$ we finally obtain that

$$
\left\|\left\|Q_{\theta} \gamma\right\|\right\| \geq c^{\prime}\| \| \gamma\| \|, \quad \forall \gamma \in D^{2}\left(Q_{\theta}\right), \forall \theta \in \Theta
$$

so, the pair $\left(c_{0}(\mathbb{N}, X), c_{00}(\mathbb{N}, X)\right)$ is uniformly admissible for the discrete linear skewproduct semiflow $\hat{\pi}$.
$(i i i) \Longrightarrow(i)$ It follows from Corollary 3.1.
Remark 3.6. The equivalence $(i) \Longleftrightarrow$ (ii) represents an extension of the dichotomy theorem due to Van Minh, Räbiger and Schnaubelt (see [16]), for the case of linear skew-product semiflows.

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