THE SMALLEST SELF-DUAL EMBEDDABLE GRAPHS IN A PSEUDOSURFACE

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Abstract. A proper embedding of a graph G in a pseudosurface P is an embedding in which the regions of the complement of G in P are homeomorphic to discs and a vertex of G appears at each pinchpoint of P; we say that a proper embedding of G in P is self dual if there exists an isomorphism from G to its topological dual. We determine five possible graphs with 7 vertices and 13 edges that could be self-dual embeddable in the pinched sphere, and we establish, by way of computer-powered methods, that such a self-embedding exists for exactly two of these five graphs.

1. Introduction

This article addresses embeddings of graphs in pseudosurfaces. We assume the reader is familiar with basic topological graph theory, including surfaces, embeddings, and dual graphs; see [8] for more information.

Following [3], a closed, connected *pseudosurface* is a connected topological space obtained from a disjoint union of surfaces via a finite number of point identifications, called *pinches*; the identified points are called *pinchpoints*. Therefore, a surface is a special case of a pseudosurface. A small-enough neighborhood of a pinchpoint is homeomorphic to the union of discs identified at a point; each identified disc is called an *umbrella* of the pinchpoint. A *proper embedding* of a graph G in a pseudosurface P is an embedding in which each of the regions of the complement of G in P is homeomorphic to a disc and a vertex of G appears at each pinchpoint in P. We shall let $G \to P$ denote a proper embedding of G in P; the definitions of the dual graph G^* and the dual embedding $(G \to P)^*$ are immediate natural extensions of the definitions of dual graph and dual embedding for an embedding in a surface. However, as evidenced by Figure 1, $((G \to P)^*)^*$ is not necessarily well-defined since $(G \to P)^*$ is not a proper embedding. Two embeddings of the same graph in the same space are equivalent if there is a homeomorphism of the space that maps one embedding to the other.

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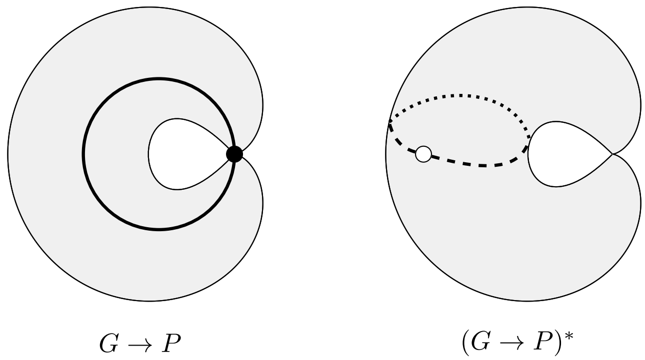


FIGURE 1. An example of a self-dual embedding of a graph in the pinched sphere, and the corresponding dual embedding.

So, in the case that P has at least one pinchpoint, for any graph G and any proper embedding $G \to P$, $G \to P$ cannot be equivalent to $(G \to P)^*$ since there is no homeomorphism that maps a pinchpoint vertex to a non-pinchpoint vertex. Therefore, we give a weaker notion, first put forward in [10], of graph self-duality for pseudosurfaces. For a pseudosurface P with at least one pinchpoint, we say that $G \to P$ is *self dual* if G^* is isomorphic to G, which still requires that the incidence of faces and edges in $G \to P$ is isomorphic to the incidence of vertices and edges in G.

While there is much research on self-dual embeddability of graphs in surfaces ([4] contains several references), questions on the self-dual embeddability of graphs in pseudosurfaces have only recently been explored. The second and third authors in [10] proved that every graph of the form $K_{4m,4n}$ is self-dual embeddable in a pseudosurface, and most of them are embeddable in several different orientable and nonorientable pseudosurfaces. The purpose of this article is to find the smallest self-dual embeddable simple graph(s) in a pseudosurface. Our approach is similar to that of Whitney's planarity criterion in [11] and continued by Abrams and Slilaty in [2, 3]. That is, we are changing a topological question about graph embeddings into a question of combinatorics: we are exploiting the relationship between the edges of an appropritately embedded graph and its topological dual. In order to accomplish this, we build on Abrams' and Slilaty's definition of an algebraic dual, and we use a computer program to analyze all of the permutations of the edges of a graph, looking for those permutations that yield a component-split algebraic dual, which we define in Section 2. Section 3 contains the main results of this article.

For a vertex v of G, $G - v$ shall denote the subgraph of G induced by all edges not incident to v. For a subset $X \subset E(G)$, $G[X]$ shall denote the subgraph of G induced by X. For two graphs G_1 and G_2, $G_1 + G_2$ shall denote the join of G_1 and G_2. For more terminology and notation, see [6].

2. Algebraic Duals and Euler Characteristic

We begin by providing an algebraic representation of a graph embedding and its topological dual. The definitions in this section are adapted for our purposes from [2, 3, 6]. Let G be a simple graph, and let $\mathcal{E}(G)$ be the $\mathbb{Z}_2$ vector space of sums of edges of G. For $v \in V(G)$, let $\mathrm{star}(v)$ denote the set of all edges incident to v. Lastly, let $Z(G)$ be the *cycle space* of G, which is the subspace of $\mathcal{E}(G)$ generated by the cycles in G.

Definition 2.1. [2, 3] *A graph G^* is an* algebraic dual *of another graph G if there exists a bijection $\phi\colon E(G^*) \to E(G)$ such that $\phi(star(v^*)) \in Z(G)$ for all $v^* \in V(G^*)$; we say that such a bijection ϕ is an* algebraic duality correspondence (ADC) *between G^* and G.*

Definition 2.2 will allow us to algebraically capture the relationship between a properly embedded graph and its topological dual.

Definition 2.2. [2, 3] *An algebraic dual G^* of G with ADC ϕ is a* component-split *algebraic dual of G if for all $v^* \in V(G^*)$, $G[\phi(star(v^*))]$ is connected.*

We say that a pseudosurface P is *face-connected* if for any two faces f and f' of a 2-complex homeomorphic to P, there is a sequence of faces $f = f_1 f_2 \cdots f_n = f'$ such that any two consecutive faces have a common boundary edge. If a connected graph G has a self-dual embedding in P, then P must be face connected.

For a pseudosurface P, we let $\chi(P)$ denote the Euler characteristic of P which, as an invariant of P, does not depend on a cellular decomposition of P. For $G \to P$, we let $F(G \to P)$ denote the *faces* of $G \to P$ which are the regions of the complement of G in P. Therefore, for $G \to P$,

$$\chi(P) = |V(G)| - |E(G)| + |F(G \to P)|. \tag{2.1}$$

Following [3, Construction 3.1], given a simple graph G and an algebraic dual graph G^* with ADC ϕ, we may construct a 2-complex $K(G, G^*)$. The 0-cells and 1-cells of $K(G, G^*)$ are the vertices and edges of G, respectively. The 2-cells of $K(G, G^*)$ appear as follows: for each $v^* \in V(G^*)$, we let $F_1, F_2, \ldots, F_k$ denote the components of $\phi(\mathrm{star}(v^*))$; to each component

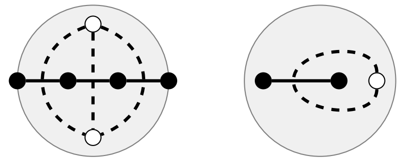


Figure 2. Examples of a nonsimple dual graph resulting from an embedded graph containing vertices of degree one or two. The dual graph is drawn with dashed edges joining white vertices.

F_i, we make a choice of facial boundary walk, and we glue a 2-cell following our chosen facial boundary walk. The resulting 2-complex may have pinchpoints, but it is not necessarily face connected. Moreover, if G^* is a component-split algebraic dual of G, then G^* is isomorphic to the topological dual of G in $K(G, G^*)$.

Lemma 2.3. [9, Theorem 1.2] *For any face-connected pseudosurface P with h handles, c crosscaps, and p pinches needed to produce P from a surface S with h handles and c crosscaps,*

$$\chi(P) = 2 - 2h - c - p.$$

We will work specifically on the case that $\chi(P) = 1$, so by Lemma 2.3, to differentiate between the pinched sphere and the projective plane, it will suffice to discern the existence of a pinchpoint in P.

3. The Smallest Self-dual Embeddable Graphs in a Pseudosurface

3.1. **The candidate graphs.** Lemma 3.1 and Proposition 3.2 provide a classification of the smallest possible self-dual embeddable graphs in a pseudosurface with at least one pinchpoint.

Lemma 3.1. *In terms of the size of its vertex and edge sets, the smallest simple graphs self-dual embeddable in a pseudosurface P with at least one pinchpoint have seven vertices and thirteen edges, with one vertex of degree* 6 *and all other vertices having minimum degree* 3.

Proof. Consider $G \to P$ for a face-connected pseudosurface P with at least one pinchpoint. There are at least two umbrellas of any pinchpoint vertex v of G. It is easy to deduce (after considering Figure 2) that each umbrella must intersect at least three edges of a properly embedded graph in order for that embedding to have a simple dual graph. It follows that a simple graph that is self-dual embeddable in a pseudosurface must have at least one vertex of degree at least six and six other vertices of degree at least three.

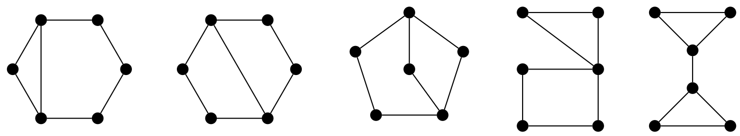


FIGURE 3. All possible seven-edge graphs on six vertices with minimum degree two.

If we let G_1 be a simple graph with exactly one vertex of degree six and six other vertices of degree three, then G_1 has exactly seven vertices and twelve edges. If G_1 is self-dually embedded in a face-connected pseudosurface P_1, then equation 2.1 and Lemma 2.3 imply that

$$\chi(P_1) = 7 - 12 + 7 = 2 = 2 - 2 \cdot 0 - 0 - 0,$$

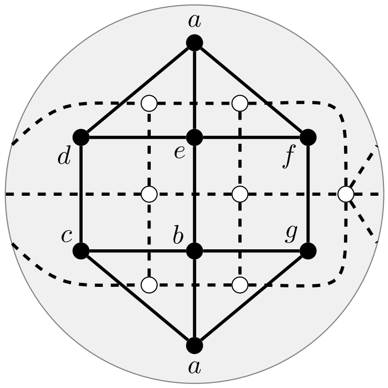
and so P_1 must really be a sphere. To add an additional vertex to G_1 would mean adding at least three additional edges. So, we conclude that a self-dual embeddable graph in P must have the same vertex degree requirements as G_1, but it must have at least thirteen edges. $\square$

For the remainder of this article we let F_i denote a graph of the form described in Lemma 3.1; Proposition 3.2 describes all possible F_i's.

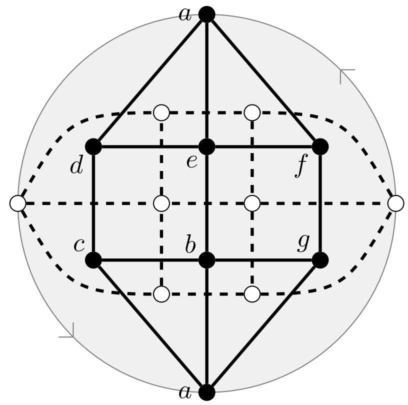
Proposition 3.2. *There are exactly five graphs F_i, and each one is the result of joining one of the graphs H_i in Figure 3 to K_1.*

Proof. Let v be the vertex of degree six in a graph of the form F_i, and let $H_i = F_i - v$. The graph H_i has six vertices of minimum degree two, has seven edges, and is connected. We leave it to the reader to verify that there are five graphs of the form H_i, which are given in Figure 3. Thus, the five candidate graphs for a self-dual embedding in the pinched sphere are $F_i = H_i + K_1$ for $1 \leq i \leq 5$. $\square$

3.2. **The algorithm.** By the discussion in Section 2, if we can find, among all permutations of the edges of a graph, an ADC $\phi\colon E(F_i) \to E(F_i)$ making F_i a component-split algebraic dual of itself, then we know that F_i can be made the 1-skeleton of a 2-complex $K(F_i, F_i^*)$ for which F_i is both the 1-skeleton and isomorphic to the topological dual F_i^*. However, we do not know if the facial boundary walks of $K(F_i, F_i^*)$ and the relevant choices that may be made in constructing it may produce a pseudosurface with at least one pinchpoint. It is straightforward enough to reconstruct the embeddings for these small graphs (indeed, see Figures 4-6), but the presence of a pinchpoint vertex can also be determined by checking for the



$F_2 \to P$



$F_2 \to P^2$

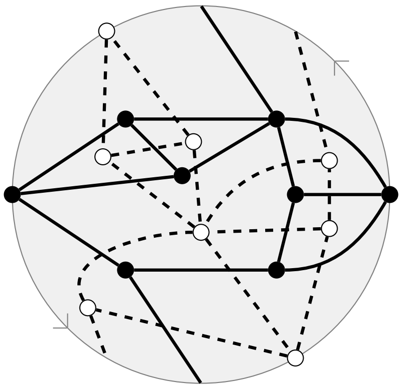
FIGURE 4. Self-dual embeddings of F_2 in the pinched sphere P and the projective plane P^2; the two black vertices labeled a in the left-hand embedding correspond to the points that are identified to produce a pinched sphere.

existence of what Bruhn and Diestel call a *cluster* or a *local cluster* (see [5] for more details).

3.3. **The results.** For each graph F_i, we first tested all 13! possible permutations of the edges of F_i to determine the set of ADCs that make F_i a component-split algebraic dual of itself. For any such maps, we then checked for the existence of a cluster or local cluster to determine whether the reconstructed embedding is in the pinched sphere or the projective plane. The results of our search are summarized in the following theorem. We advise the reader that the code we developed and the results we obtained may be found at [1]; the reader should consult the file README.txt before trying to read the code, the graph files, or the results.

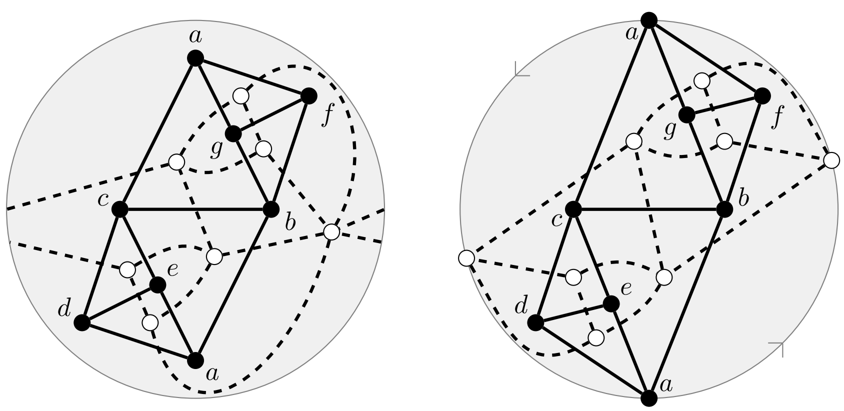
Theorem 3.3. *Let $F_i = H_i + K_1$, where H_i is one of the graphs from Figure 3.*

- *The graphs F_2 and F_5 have a self-dual embedding in the pinched sphere, and thus are the smallest graphs to have a self-dual embedding in any pseudosurface with at least one pinchpoint (see Figures 4 and 6).*
- *The graphs F_2, F_4, and F_5 have a self-dual embedding in the projective plane (see Figures 4-6).*



$F_4 \to P^2$

FIGURE 5. A self-dual embedding of F_4 in the projective plane P^2.



$F_5 \to P$ $\qquad$ $F_5 \to P^2$

FIGURE 6. Self-dual embeddings of F_5 in the pinched sphere P and the projective plane P^2; the two black vertices labeled a in the left-hand embedding correspond to the points that are identified to produce a pinched sphere.

- *The graphs F_1 and F_3 have no self-dual embeddings in any surface or pseudosurface.*

Remark 3.1. *The self-dual embedding of F_i $(i = 2, 5)$ in the projective plane can be obtained from the self-dual embedding of F_i in the pinched sphere via a surgery of Edmonds* [7]*; see* [10] *for another application of this surgery.*

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