ω-JOINTLY METRIZABLE SPACES

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Abstract. A topological space X is *ω-jointly metrizable* if for every countable collection of metrizable subspaces of X, there exists a metric on X which metrizes every member of this collection. Although the Sorgenfrey line is not jointly partially metrizable [3], we prove that it is *ω-jointly metrizable*.

We show that if X is a regular first countable T_1-space such that X is the union of two subspaces one of which is separable and metrizable, and the other is closed and discrete, then X is *ω-jointly metrizable*.

1. Introduction

Let X be a topological space, and let $\mathcal{F}$ be a family of subspaces of X. We say that X is *jointly metrizable on* $\mathcal{F}$, or $\mathcal{F}$*-metrizable*, if there is a metric d on the set X such that d *metrizes all subspaces of* X *which belong to* $\mathcal{F}$, that is, the restriction of d to A generates the subspace topology on A, for any $A \in \mathcal{F}$ (see [1, 2, 3]). In particular, X is *jointly partially metrizable*, or a *JPM-space*, if there is a metric d on X which metrizes all metrizable subspaces of X.

Theorem 1.1. [3] *Every regular first countable jointly partially metrizable T_1-space X is metrizable.*

Proposition 1.2. [3] *If all metrizable subspaces of a space X are discrete, then X is jointly partially metrizable.*

If X is a space with no non-trivial convergent sequences, then all metrizable subspaces of X are discrete, and hence, X is a JPM-space. Therefore, every extremally disconnected Hausdorff space is a JPM-space. In particular, the Stone-Čech compactification $\beta(\omega)$ of the discrete space ω of natural numbers is jointly partially metrizable.

2. ω-jointly metrizable spaces

We say that a topological space X is *ω-jointly metrizable* if for every countable collection $\{M_i : i = 1, 2, \ldots\}$ of metrizable subspaces of X, there exists a metric d on X which metrizes every M_i for $i = 1, 2, \ldots$.

Clearly, every JPM-space is ω-jointly metrizable.

Proposition 2.1. *Suppose that X is a space and any union of countably many metrizable subspaces of X is metrizable. Then X is ω-jointly metrizable.*

Proof. Let $\Gamma = \{M_i : i = 1, 2, \ldots\}$ be a countable family of metrizable subspaces of X. We show that there is a metric d on X which metrizes each M_i for $i = 1, 2, \ldots$. Let $A = \cup M_i$. Then A is metrizable. Let d_A be the metric on A. We may assume that $d_A(x, y) \leq 1$ for $x, y \in A$. We define a metric d on X as follows: $d(x, y) = d_A(x, y)$ if $x, y \in A$; otherwise, we put $d(x, y) = 1$ if x and y are distinct. It is clear that d is a metric on X which metrizes A and thus it metrizes every subspace M_i in Γ. □

Observe that a subspace of the Sorgenfrey line is metrizable if and only if it is countable. Therefore, it follows from Proposition 2.1 that the Sorgenfrey line is ω-jointly metrizable. On the other hand, the Sorgenfrey line is not jointly partially metrizable by Theorem 1.1, since it is first countable, Tychonoff, and not metrizable.

Example 2.2. Consider the space $A(n)$, the Alexandroff compactification of any uncountable discrete space X [4, Example 3.5.14]. All subspaces of $A(n)$ are of the form $A(m)$ or $D(m)$ with $m \leq n$ and $A(m)$ is metrizable if m is countable.

Let $\Gamma = \{M_i : i = 1, 2, \ldots\}$ be a countable family consisting of metrizable subspaces of $A(n)$. Let B be the union of the metrizable subspaces M_i of the form $A(m)$. Then B is of the form $A(m)$ with m countable and thus it is metrizable. Let e be a metric on B. We may assume that $e(x, y) \leq 1$ for all $x, y \in B$. We define a metric d on $A(n)$ as follows: $d(x, y) = e(x, y)$ if $x, y \in B$; otherwise, we put $d(x, y) = 1$ if x and y are distinct. We can show that d is a metric on $A(n)$ which metrizes B, any subspace M_i of B and metrizes any discrete subspace $D(m)$. Hence, $A(n)$ is ω-jointly metrizable.

A natural question arises: under what conditions a space X is ω-jointly metrizable?

Now we present the main result of this article which gives conditions under which a space X is ω-jointly metrizable.

Theorem 2.3. *Suppose that (X, T) is a regular first countable T_1-space such that $X = A \cup B$. If A is a closed discrete subspace and B is a separable metrizable subspace, then X is ω-jointly metrizable.*

Proof. Let Γ be any countable collection $\{Y_i : i = 1, 2, \ldots\}$ of metrizable subspaces of X.

Let Y be any member of Γ. Let $Y_B = Y \cap B$, and let Y_c be the closure of Y_B in the metrizable space Y and $Y_d = Y \setminus Y_c$.

<u>Claim 1</u>. Y_c is a closed separable metrizable subspace of Y.

It is clear that Y_c is closed and metrizable. It remains to show that it is separable. Since B is separable metrizable, B is second countable. It follows that Y_B is second countable. Thus, Y_B is separable, that is, there exists a countable subset W of Y_B such that $\overline{W} \supseteq Y_B$. Therefore, $\overline{W} \supseteq Y_c$. Hence, Y_c is separable.

Next, the following statements are easily verified.

<u>Claim 2</u>. Y_d is a closed discrete subspace of Y.

<u>Claim 3</u>. $Y_c \cap Y_d = \emptyset$ and $Y_c \cup Y_d = Y$.

<u>Claim 4</u>. Y_c and Y_d are disjoint open subspaces of the space Y.

Now, let $A_Y = Y_c \cap A$.

<u>Claim 5</u>. The set A_Y is countable.

By Claim 1, the space Y_c is second countable. Thus, A_Y is second countable so that it is separable. But A_Y is a discrete space being a subspace of A. Hence, A_Y is countable.

Let C be a countable subset of A and let S be a base for the topology T of X. Let $S_C = S \cup \{\{x\} : x \in A \setminus C\}$. The family S_C is a base of some topology T_C on the set X.

<u>Claim 6</u>. The space (X, T_C) is metrizable.

It is clear that (X, T_C) is a T_1-space. We show that (X, T_C) is regular. Let $x \in X$. Then $x \in B$ or $x \in C$ or $x \in A \setminus C$. If $x \in B$ or $x \in C$, then given any neighborhood U of x, there exists a neighborhood V of x such that $\overline{V} \subset U$ since (X, T) is regular. However, if $x \in A \setminus C$, then given any neighborhood U of x, for the neighborhood $\{x\}$ we have $\{x\} = \overline{\{x\}} \subset U$. Hence, (X, T_C) is regular. Let V_1 be a countable base for B. At every point $x \in C$, take a countable local base V_2^x. Let $V_2 = \cup\{V_2^x : x \in C\}$. Then V_2 is countable. For $A \setminus C$, let V_3 be the family consisting of all singletons $\{x\}$ where $x \in A \setminus C$. Let $V = V_1 \cup V_2 \cup V_3$. Then V is a σ-discrete base for the space (X, T_C). Hence, the space (X, T_C) is metrizable by the Bing Metrization Theorem [4, Theorem 4.4.8].

<u>Claim 7</u>. Suppose that $A_Y \subset C$. Then the topology T of X induces on Y the same topology as the topology T_C does.

Let T_Y denote the topology of Y induced by T and let T_{C_Y} denote the topology of Y induced by T_C. From the definition of T_C, it follows that $T_Y \subset T_{C_Y}$. Now we show that $T_{C_Y} \subset T_Y$. Take any $x \in A \setminus C$; then $\{x\}$ is a basis element in T_{C_Y}. Then $x \in Y_d$ and $x \notin Y_c$. There exists a basis element $U \in T$ containing x such that $(U \cap Y) \cap Y_B = \emptyset$. That is, $(U \cap Y) \cap B = \emptyset$. Thus, $x \in U \cap Y \subset A$. Then $U \cap Y$ is an open discrete subspace of (Y, T_Y). Hence, $\{x\} \in T_Y$. Therefore, $T_{C_Y} \subset T_Y$. Hence, the claim is proved.

Now let d be a metric on X that induces the topology T_C of X. Let $C = \cup\{A_{Y_i} : i \in \omega\}$. Then C is countable. Using Claim 7, we deduce that d is the metric on X which metrizes each metrizable subspace Y_i of Γ. Hence, X is ω-jointly metrizable. $\square$

The following are examples of ω-jointly metrizable spaces.

Example 2.4. Let X be the Niemytzky plane [4, Example 1.2.4]. Then X is a first countable Tychonoff space and $X = A \cup B$, where A is the closed discrete bottom line and B is the separable metrizable open half plane. Therefore, X is ω-jointly metrizable, by Theorem 2.3.

Example 2.5. Let X be the Mrowka space [4, Exercise 3.6.I]. Then X is a first countable Tychonoff space and $X = A \cup B$, where A is an uncountable closed discrete subspace of X and B is a countable open discrete subspace of X. Therefore, it follows from Theorem 2.3 that X is ω-jointly metrizable.

Example 2.6. Consider the set of real numbers $\mathbb{R}$ with the Rational Sequence Topology [5, Example 65]. With this topology, $\mathbb{R} = \mathbb{P} \cup \mathbb{Q}$, where $\mathbb{P}$ is the set of irrationals and $\mathbb{Q}$ is the set of rationals, is regular T_1-space and first countable. $\mathbb{P}$ is a closed discrete subspace and $\mathbb{Q}$ is an open separable metrizable subspace. Therefore, it follows from Theorem 2.3 that $\mathbb{R}$ with the Rational Sequence Topology is ω-jointly metrizable.

We shall now give an example of a space which is not ω-jointly metrizable.

Example 2.7. Consider the Michael line [4, Example 5.1.32]. Let $Y = \mathbb{P} \cup \{q\}$ be a subset of the real line where $\mathbb{P}$ is the set of irrationals and q is a rational number. Then Y is a metrizable subspace of the real line and thus $\{q\}$ is a G_δ-set in Y. Using Exercise 5.5.2 in [4], we conclude that Y is a metrizable subspace of the Michael line.

<u>Claim</u>. If x is a real number and A is a subset of the Michael line such that $x \in \overline{A}$, then either $x \in \overline{A \cap \mathbb{P}}$ or $x \in \overline{A \cap \mathbb{Q}}$, where $\mathbb{Q}$ is the set of rationals.

Note that $A = A \cap \mathbb{R} = A \cap (\mathbb{P} \cup \mathbb{Q})$. Then $A = (A \cap \mathbb{P}) \cup (A \cap \mathbb{Q})$. Thus, $\overline{A} = \overline{(A \cap \mathbb{P}) \cup (A \cap \mathbb{Q})} = \overline{(A \cap \mathbb{P})} \cup \overline{(A \cap \mathbb{Q})}$. Hence, if $x \in \overline{A}$, then either $x \in \overline{A \cap \mathbb{P}}$ or $x \in \overline{A \cap \mathbb{Q}}$.

Now we show that the Michael line is not an ω-jointly metrizable. Assume the contrary. Consider the countable family of metrizable subspaces $\Gamma = \{\mathbb{P} \cup \{q\} : q \in \mathbb{Q}\} \cup \{\mathbb{Q}\}$. Then there exists a metric d on the Michael line metrizing each member of Γ. Next, we show that d metrizes the Michael line. Let x be any real number and A any subset of the Michael line. We shall show that $x \in \overline{A}$ if and only if $d(x, A) = 0$.

First, assume that $x \in \overline{A}$. By the above Claim, either $x \in \overline{A \cap \mathbb{P}}$ or $x \in \overline{A \cap \mathbb{Q}}$. Suppose that $x \in \overline{A \cap \mathbb{P}}$. Let $C = (A \cap \mathbb{P}) \cup \{x\}$. If either x is rational or irrational, then C is metrized by d since C is subset of $\mathbb{P} \cup \{x\}$. Thus, $d(x, A \cap \mathbb{P}) = 0$. Hence, $d(x, A) = 0$. Suppose that $x \in \overline{A \cap \mathbb{Q}}$. Let $C = (A \cap \mathbb{Q}) \cup \{x\}$. Since $\overline{A \cap \mathbb{Q}} \subset \overline{\mathbb{Q}}$ and $\mathbb{Q}$ is closed, it follows that $x \in \mathbb{Q}$. Thus, C is metrized by d since C is subset of $\mathbb{Q}$. Therefore, $d(x, A \cap \mathbb{Q}) = 0$. Hence, $d(x, A) = 0$.

Conversely, assume that $d(x, A) = 0$. We shall show that $x \in \overline{A}$. Assume that $x \notin \overline{A}$. Then $x \notin A$. For each positive $n \in \omega$, fix $a_n \in A$ such that $d(x, a_n) < 1/n$. Let $B = \{a_n : n \in \omega\}$. Then B is an infinite subset of A and $d(x, B) = 0$. Note that $B = (B \cap \mathbb{P}) \cup (B \cap \mathbb{Q})$ is infinite. Then either $B \cap \mathbb{P}$ or $B \cap \mathbb{Q}$ is an infinite subset of B. Assume that $B \cap \mathbb{P}$ is an infinite subset of B. Then $d(x, B \cap \mathbb{P}) = 0$. Let $C = (B \cap \mathbb{P}) \cup \{x\}$. If either x is rational or irrational, C is metrized by d since C is subset of $\mathbb{P} \cup \{x\}$. Since $B \cap \mathbb{P} \subset B \subset A$ and $x \notin \overline{A}$, it follows that $x \notin \overline{B \cap \mathbb{P}}$. Therefore, $d(x, B \cap \mathbb{P}) > 0$ which is a contradiction. Hence, $x \in \overline{A}$. Assume that $B \cap \mathbb{Q}$ is an infinite subset of B. Then $d(x, B \cap \mathbb{Q}) = 0$. Let $C = (B \cap \mathbb{Q}) \cup \{x\}$. If x is rational, then C is metrized by d since $C \subset \mathbb{Q}$. Since $B \cap \mathbb{Q} \subset B \subset A$ and $x \notin \overline{A}$, it follows that $x \notin \overline{B \cap \mathbb{Q}}$. Therefore, $d(x, B \cap \mathbb{Q}) > 0$ which is a contradiction. Hence, $x \in \overline{A}$. However, if x is irrational, then x is isolated in the Michael line. Thus, x is isolated in every subspace of the Michael line containing x. Therefore, $d(x, B \cap \mathbb{Q}) = \inf\{d(x, y) : y \in B \cap \mathbb{Q}\} > 0$ which is a contradiction. Hence, $x \in \overline{A}$.

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