ON FUNCTIONALLY HAUSDORFF SPACES

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ABSTRACT. This paper deals with functionally Hausdorff spaces. Some separation axioms which are introduced recently are studied in this paper. An interesting categorical properties of functionally Hausdorff spaces are given. Finally, we characterize topological spaces for which the functionally Hausdorff-reflection is a spectral space.

1. INTRODUCTION

Let X be a topological space. The ring of all real valued continuous functions defined on X will be denoted by $C(X)$. A *Urysohn function* for A and B, disjoint subsets of X is a mapping f in $C(X)$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Urysohn's Lemma shows that if X is a T_4-space, then any two disjoint closed subsets of X have a Urysohn function and conversely if any two disjoint closed subsets of X have a Urysohn function, then this space is T_4. But the existence of such a function does not guarantee that X is a T_1-space and thus a T_3-space (for this see Example 5 [10]). So the following separation axioms are immediate.

Tychonoff. For any closed subset F of a T_1-space X and $x \notin F$, there is a Urysohn function for F and $\{x\}$.

Functionally Hausdorff. For any two distinct points x and y in X, there is a Urysohn function for $\{x\}$ and $\{y\}$ (We say that x and y are functionally separated or completely separated).

Clearly, Tychonoff implies functionally Hausdorff implies T_2.

Now, let us recall the functionally Hausdorff-reflection of a topological space. Let X be a topological space. We define the equivalence relation $\sim$ on X by $x \sim y$ if and only if $f(x) = f(y)$ for all $f \in C(X)$. Let $FH(X) := X/\sim$ the set of equivalence classes and $\mu_X \colon X \to FH(X)$ be the canonical surjection map assigning to each point of X its equivalence class. The resulting quotient space $FH(X)$ is a functionally Hausdorff space and since every f in $C(X)$ is constant on each equivalence class, we can define $FH(f)\colon FH(X) \to \mathbb{R}$ by $FH(f)(\mu_X(x)) = f(x)$. The construction $FH(X)$ satisfies some categorical properties.

For each functionally Hausdorff space Y and each continuous map $f\colon X \to Y$, there exists a unique continuous map $\widetilde{f}\colon F(X) \to Y$ such that $\widetilde{f} \circ \mu_X = f$. We will say that $FH(X)$ is the *functionally Hausdorff-reflection* of X.

Let $f\colon X \to Y$ be a continuous map. Then there is a unique continuous map $FH(f)\colon FH(X) \to FH(Y)$ such that $FH(f) \circ \mu_X = \mu_Y \circ f$.

Consequently, it is clear that FH is a covariant functor from the category of topological spaces **Top** into the full subcategory **FunHaus** of **Top** whose objects are functionally Hausdorff spaces.

In [4], the authors have introduced some new separation axioms.

Definition 1. *Let i, j be two integers such that $0 \leq i < j \leq 2$. Let us denote by $\mathbf{T_i}$ the functor from* **Top** *to* **Top** *which takes each topological space X to its T_i-reflection (the universal T_i-space associated with X). A topological space X is said to be $T_{(i,j)}$-space if $\mathbf{T}_i(X)$ is a T_j-space (thus we have three new types of separation axioms namely; $T_{(0,1)}$, $T_{(0,2)}$, and $T_{(1,2)}$).*

Definition 2. *Let $\mathbf{C}$ be a category and $\mathbf{F}$, $\mathbf{G}$ two (covariant) functors from $\mathbf{C}$ to itself.*

(1) *An object X of $\mathbf{C}$ is said to be a $T_{(\mathbf{F},\mathbf{G})}$-object if $\mathbf{G}(\mathbf{F}(X))$ is isomorphic with $\mathbf{F}(X)$.*
(2) *Let P be a topological property on the objects of $\mathbf{C}$. An object X of $\mathbf{C}$ is said to be a $T_{(\mathbf{F},P)}$-object if $\mathbf{F}(X)$ satisfies the property P.*

Following Defintion 2, for the functor FH one may define other separation axioms. A space X is called $T_{(0,FH)}$ (resp., $T_{(S,FH)}$) if the space $T_0(X)$ (resp., $S(X)$) is functionally Hausdorff, where $S(X)$ is the sober-reflection of X. Finally, a $T_{(FH,\rho)}$-space is a topological space X such that $FH(X)$ is Tychonoff.

This paper consists of some investigations concerning functionally Hausdorff spaces. The first one deals with the characterization of $T_{(0,FH)}$, $T_{(S,FH)}$ and $T_{(FH,\rho)}$-spaces. In the second section, we characterize morphisms in **Top** rendered invertible by the functor FH. Finally, we study topological spaces X such that $FH(X)$ is spectral.

2. Functionally Hausdorff Spaces and Separation Axioms

Let G be a covariant functor from **Top** to itself. In order to characterize $T_{(G,FH)}$-spaces let us introduce the following concept.

Definition 3. *A full subcategory $\mathcal{A}$ of* **Top** *is said to be real onto subcategory if it satisfies the following properties*

(a) *$\mathcal{A}$ is reflective in* **Top**.

(b) *The real line* $\mathbb{R}$ *belongs to* $\mathcal{A}$.
(c) *Let* G *be the left adjoint functor of the embedding* $I\colon \mathcal{A} \to TOP$ *and* μ *the unit of the adjunction* (G, I). *Then for each space* X *in* $\mathcal{A}$, $\mu_X : X \to G(X)$ *is onto.*

Example 4.

(a) $\mathbf{Top}_0, \mathbf{Top}_1, \mathbf{Top}_2$ *and* ***FunHaus*** *are real onto subcategories in* **Top**.
(b) *The full subcategory* **Sob** *of sober spaces is not a real onto subcategory.*

Theorem 5. *Let* $\mathcal{A}$ *be a real onto subcategory and* $G\colon TOP \to \mathcal{A}$ *the left adjoint functor of the inclusion functor* $I\colon \mathcal{A} \to TOP$. *Let* μ *be the unit of the adjunction* (G, I). *For a given topological space* X, *the following statements are equivalent.*

(i) X *is a* $T_{(G,FH)}$*-space;*
(ii) *Any two points* x *and* y *in* X *such that* $\mu_X(x) \neq \mu_X(y)$ *have a Urysohn function.*

Proof. $(i) \Rightarrow (ii)$ Let x and y be two points in X such that $\mu_X(x) \neq \mu_X(y)$. Then $\mu_X(x)$ and $\mu_X(y)$ are two distinct points in $G(X)$ which is functionally Hausdorff and thus, there exists a mapping $f \in C(G(X))$ separating $\mu_X(x)$ and $\mu_X(y)$. Clearly $f \circ \mu_X$ is a Urysohn function for x and y. Note that in this implication it is enough to suppose that $\mathcal{A}$ is only reflective.

$(ii) \Rightarrow (i)$ Conversely, since μ_X is onto, let $\mu_X(x)$ and $\mu_X(y)$ be two distinct points in $G(X)$. By (ii) let $f \in C(X)$ such that $f(x) \neq f(y)$. Now, according to the fact that $\mathcal{A}$ is reflective and $\mathbb{R}$ belongs to $\mathcal{A}$, there exists a unique continuous map $\widetilde{f}$ from $G(X)$ to $\mathbb{R}$ such that $f = \widetilde{f} \circ \mu_X$. Clearly, $\widetilde{f}$ is a Urysohn function of $\mu_X(x)$ and $\mu_X(y)$. $\square$

Recall that a topological space X is called a *sober* space if any irreducible closed subset C of X has a unique generic point (there is a unique $a \in X$ such that $C = \overline{\{a\}}$). It is also worth noting that the subcategory SOB of sober spaces is reflective in **Top**.

Let us recall the sober-reflection of a topological space. Let X be a topological space and $\mathbf{S}(X)$ be the set of all nonempty irreducible closed sets of X. Let F be a closed set of X, set $\widetilde{F} = \{G \in \mathbf{S}(X) : G \subseteq F\}$; then $(\widetilde{F}$, F is a closed set of $X)$ is the collection of closed sets of a topology on $\mathbf{S}(X)$.

The unit μ of the adjunction (S, I) is defined by $\mu_X(x) = \overline{\{x\}}$ and for any continuous map $f\colon X \to Y$, $S(f)\colon S(X) \to S(Y)$ is given by $S(f)(C) = \overline{C}$ for every element C in $S(X)$, (for more information see [7]).

Corollary 6. *Let X be a topological space. The following statements are equivalent.*

(i) *X is a $T_{(0,FH)}$-space;*
(ii) *X is a $T_{(S,FH)}$-space;*
(iii) *Any two points x and y in X such that $\overline{\{x\}} \neq \overline{\{y\}}$ have a Urysohn function.*

Proof. According to the previous theorem it is sufficient to show $(iii) \Rightarrow (ii)$. Indeed, let C_1 and C_2 be two distinct points in $S(X)$ (that is two distinct irreducible closed subsets in X). Then there exist two distinct points x_1 and x_2, respectively in C_1 and C_2 such that $\overline{\{x_1\}} \neq \overline{\{x_2\}}$. By (iii) let f be a continuous map from X to $\mathbb{R}$ separating x_1 and x_2. Set $g = \mu_{\mathbb{R}}^{-1} \circ S(f) \colon S(X) \to \mathbb{R}$ (where $\mu_{\mathbb{R}} \colon \mathbb{R} \to S(\mathbb{R})$ the map sending each $x \in \mathbb{R}$ to the one point $\{x\}$). Now, let us show that for $i = 1, 2$ we have $g(C_i) = f(x_i)$.

First remark that any continuous image of an irreducible set is irreducible which means that $f(C_i)$ is an irreducible set of $\mathbb{R}$. But any nonempty irreducible subset of a Hausdorff space is a one point, so set $\{a_i\} = f(C_i)$. Hence, $g(C_i) = \mu_{\mathbb{R}}^{-1} \circ S(f)(C_i) = \mu_{\mathbb{R}}^{-1}(\overline{f(C_i)}) = \mu_{\mathbb{R}}^{-1}(\{a_i\}) = a_i$.

On the other hand, since $x_i \in C_i$ for $i = 1, 2$, then $\{f(x_i)\} \subseteq f(C_i) = \{a_i\}$ and thus, $f(x_i) = a_i$, so that $g(C_i) = f(x_i)$. $\square$

Before giving the characterization of $T_{(FH,\rho)}$-spaces let us introduce the following notations and remarks.

Notation 7. *Let X be a topological space.*

(1) *For any $x \in X$, we denote by $d(x) := \cap[f^{-1}(f(\{x\})) : f \in \mathbf{C}(X)]$.*
(2) *For any subset A of X, we denote by $d(A) := \cup[d(a) : a \in A]$.*

The following results are immediate.

Proposition 8. *Let X be a topological space, $a \in X$ and A a subset of X. Then*

(1) $d(A) = \mu_X^{-1}(\mu_X(A))$.
(2) *$d(a)$ is a closed subset of X.*
(3) $A \subseteq d(A) \subseteq \cap[f^{-1}(f(A)) : f \in \mathbf{C}(X)]$.
(4) *For all $f \in \mathbf{C}(X)$, $f(A) = f(d(A))$.*

Now, we give a characterization of functionally Hausdorff spaces and $T_{(0,FH)}$-spaces in term of d.

Proposition 9. *Let X be a topological space. Then the following statements are equivalent.*

(i) *X is a functionally Hausdorff space;*
(ii) *For any set A of X, $d(A) = A$;*

(iii) *For any* $a \in X$, $d(a) = \{a\}$.

Proof. $(i) \Rightarrow (ii)$ If X is a functionally Hausdorff space, then $FH(X) = X$ and μ_X is equal to 1_X and thus, $d(A) = A$.

$(ii) \Rightarrow (iii)$ The proof is straightforward.

$(iii) \Rightarrow (i)$ Note that $d(a) = \{a\}$ means that for any $x \in X$ such that $x \neq a$, there exists a continuous map $f : X \to \mathbb{R}$ such that $f(x) \neq f(a)$ and thus X is a functionally Hausdorff space. □

Proposition 10. *Let X be a topological space. Then the following statements are equivalent.*

(i) *X is a $T_{(0,FH)}$-space;*
(ii) *For any $a \in X$, $d(a) = \overline{\{a\}}$.*

Proof. $(i) \Rightarrow (ii)$ Clearly, $d(a)$ is a closed subset of X containing a and thus, $\overline{\{a\}} \subset d(a)$.

Conversely, let $x \in d(a)$, then $f(x) = f(a)$ for any $f \in C(X)$ and thus by Corollary 6 $\overline{\{a\}} = \overline{\{x\}}$ and consequently, $x \in \overline{\{a\}}$.

$(ii) \Rightarrow (i)$ Let x and a be two points in X satisfying $\overline{\{a\}} \neq \overline{\{x\}}$. Then $x \notin \overline{\{a\}}$ or $a \notin \overline{\{x\}}$ that is $x \notin d(a)$ or $a \notin d(x)$ and consequently, there exists a continuous map f from X to $\mathbb{R}$ separating a and x. Finally, Corollary 6 does the job. □

Now, let us introduce the following definition.

Definition 11. *Let X be a topological space and C a subset of X. C is called functionally closed in X (for short F-closed) if $d(C)$ is a closed subset of X.*

Remark 12. *Let X be a topological space.*

(1) *For any subset C of X, C is F-closed if and only if $\mu_X(C)$ is closed in $FH(X)$.*
(2) *Any point set of X is F-closed.*
(3) *If in addition X is an Alexandroff space, then any subset A of X is F-closed.*

Now, we are in a position to characterize $T_{(FH,\rho)}$-spaces.

Theorem 13. *Let X be a topological space. Then the following statements are equivalent.*

(i) *X is a $T_{(FH,\rho)}$-space;*
(ii) *For any F-closed subset C of X and any $x \notin d(C)$, x and C have a Urysohn function.*

Proof. $(i) \Rightarrow (ii)$ Let C be an F-closed subset of X and $x \notin d(C)$. Then $\mu_X(x) \notin \mu_X(C)$. Now, since C is F-closed, then $\mu_X(C)$ is a closed subset of $FH(X)$ which is Tychonoff and consequently, there exists a mapping $g \in C(FH(X))$ such that $g(\mu_X(x)) = 0$ and $g(\mu_X(C)) = \{1\}$. Clearly, $g \circ \mu_X$ is a Urysohn function of $\{x\}$ and C.

$(ii) \Rightarrow (i)$ Let $\mu_X(x) \notin \widetilde{C}$, where $\widetilde{C}$ is a closed subset of $FH(X)$. Then $x \notin \mu_X^{-1}(\widetilde{C})$. Now, since $d(\mu_X^{-1}(\widetilde{C})) = \mu_X^{-1}(\widetilde{C})$, then $\mu_X^{-1}(\widetilde{C})$ is an F-closed subset of X such that $x \notin d(\mu_X^{-1}(\widetilde{C}))$ and thus by (ii) there exists a mapping $f \in C(X)$ such that $f(x) = 0$ and $f(\mu_X^{-1}(\widetilde{C})) = \{1\}$. By universality of functionally Hausdorff-reflection let $\widetilde{f}\colon FH(X) \to \mathbb{R}$ the unique continuous map such that $\widetilde{f} \circ \mu_X = f$. Hence, $\widetilde{f}(\mu_x(x)) = 0$ and $\widetilde{f}(\widetilde{C}) = \widetilde{f}(\mu_X(\mu_X^{-1}(\widetilde{C}))) = f(\mu_X^{-1}(\widetilde{C})) = \{1\}$. Finally, $\widetilde{f}$ is a Urysohn function of $\{\mu_X(x)\}$ and $\widetilde{C}$. □

3. The Class of Continuous Maps Orthogonal to all Functionally Hausdorff Spaces

It is worth noting that reflective subcategories arise throughout mathematics, via several examples such as the free group, free ring, ... functors in algebra, various compactification functors in topology, and completion functors in analysis, (cf. [9, p. 90]).

A morphism $f\colon A \to B$ and an object X in a category $\mathbf{C}$ are called *orthogonal* [6], if the mapping $\hom_{\mathbf{C}}(f, X)\colon \hom_{\mathbf{C}}(B, X) \to \hom_{\mathbf{C}}(A, X)$ which takes g to gf is bijective. For a class of morphisms Σ (resp., a class of objects $\mathbf{D}$), we denote by $\Sigma^{\perp}$ the class of objects orthogonal to every f in Σ (resp., by $\mathbf{D}^{\perp}$ the class of morphisms orthogonal to all X in $\mathbf{D}$) [6].

The orthogonality class of morphisms $\mathbf{D}^{\perp}$ associated with a reflective subcategory $\mathbf{D}$ of a category $\mathbf{C}$ satisfies the following identity $\mathbf{D}^{\perp\perp} = \mathbf{D}$ [1, Proposition 2.6]. Thus, it is of interest to give explicitly the class $\mathbf{D}^{\perp}$. Note also that, if $\mathbf{I}\colon \mathbf{D} \to \mathbf{C}$ is the inclusion functor and $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ is a left adjoint functor of $\mathbf{I}$, then the class $\mathbf{D}^{\perp}$ is the collection of all morphisms of $\mathbf{C}$ rendered invertible by the functor $\mathbf{F}$ [1, Proposition 2.3].

This section is devoted to the study of the orthogonal class $\mathbf{FunHaus}^{\perp}$ that characterize morphisms rendered invertible by the functor FH.

The following concepts are introduced by O. Echi and S. Lazaar in [5].

Definition 14. *Let $f\colon X \to Y$ be a continuous map.*

(1) *f is said to be ρ-injective (or ρ-one-to-one) if for each $x, y \in X$; x and y are completely separated, then so are $f(x)$ and $f(y)$.*
(2) *f is said to be ρ-surjective (or ρ-onto) if for each $y \in Y$, there exists $x \in X$ such that $f(x)$ and y are not completely separated.*
(3) *f is said to be ρ-bijective if it is both ρ-injective and ρ-surjective.*

Now, we introduce the following definition.

Definition 15. *Let $f\colon X \to Y$ be a continuous map. f is called a functionally closed map (for short F-closed map) if it takes every F-closed subset of X to F-closed subset of Y.*

Theorem 16. *Let $f\colon X \to Y$ be a continuous map. Then the following statements are equivalent.*

(i) *$FH(f)$ is a homeomorphism;*

(ii) *f is both ρ-bijective and F-closed.*

Proof. $(i) \Rightarrow (ii)$

• f is ρ-injective. Let x and y be two separated points in X, that is $\mu_X(x) \neq \mu_X(y)$. Since $FH(f)$ is one-to-one, then $FH(\mu_X(x)) \neq FH(\mu_X(y))$ which means that $\mu_Y(f(x)) \neq \mu_Y(f(y))$ and thus $f(x)$ and $f(y)$ are completely separated. Therefore, f is ρ-injective.

• f is ρ-surjective. Let $y \in Y$. Since $FH(f)$ is onto, there exists $x \in X$ such that $FH(f)(\mu_X(x)) = \mu_Y(y)$. Hence, $\mu_Y(f(x)) = \mu_Y(y)$; and consequently, $f(x)$ and y are not completely separated. Therefore, f is ρ-onto.

• f is F-closed. Let C be an F-closed subset of X, then $\mu_X(C)$ is a closed subset of $FH(X)$ and thus $FH(f)(\mu_X(C))$ is closed in $FH(Y)$ or $FH(f)(\mu_X(C)) = \mu_Y(f(C))$. Therefore, $\mu_Y(f(C))$ is a closed subset of $FH(Y)$ which means that $f(C)$ is an F-closed subset of Y.

$(ii) \Rightarrow (i)$

• $FH(f)$ is injective. Let $x, y \in X$ be such that $FH(f)(\mu_X(x)) = FH(f)(\mu_X(y))$. Then $\mu_Y(f(x)) = \mu_Y(f(y))$. Hence, $f(x)$ and $f(y)$ are not completely separated. Since f is ρ-injective, we conclude that x and y are not completely separated, and thus, $\mu_X(x) = \mu_X(y)$. Therefore, $FH(f)$ is one-to-one.

• $FH(f)$ is surjective. Let $y \in Y$. Since f is ρ-onto, there exists $x \in X$ such that $f(x)$ and y are not completely separated. Hence, $\mu_Y(f(x)) = \mu_Y(y) = FH(f)(\mu_X(x))$. Thus, $FH(f)$ is onto.

• $FH(f)$ is closed. Let $\widetilde{C}$ be a closed subset of $FH(X)$. Since $d(\mu_X^{-1}(\widetilde{C})) = \mu_X^{-1}(\widetilde{C})$, then $\mu_X^{-1}(\widetilde{C})$ is an F-closed subset of X and by (ii), $f(\mu_X^{-1}(\widetilde{C}))$ is an F-closed subset of Y. On the other hand,

$$FH(f)(\widetilde{C}) = FH(f)(\mu_X(\mu_X^{-1}(\widetilde{C}))) = (\mu_Y \circ f)(\mu_X^{-1}(\widetilde{C}))) = \mu_Y(f(\mu_X^{-1}(\widetilde{C}))).$$

Finally, $\mu_Y^{-1}(FH(f)(\widetilde{C})) = d(f(\mu_X^{-1}(\widetilde{C})))$ is a closed subset of Y (since $f(\mu_X^{-1}(\widetilde{C}))$ is F-closed in Y) which means that $FH(f)(\widetilde{C})$ is closed and consequently, $FH(f)$ is a closed map.

Now, $FH(f)$ is a bijective continuous closed map; so that it is a homeomorphism. □

4. Functionally Hausdorff Spectral Spaces

Recall that a topological space X is said to be *spectral* [8] if the following axioms hold.

(i) X is a sober space.
(ii) X is compact and has a basis of compact open sets.
(iii) The family of compact open sets of X is closed under finite intersections.

Let $Spec(R)$ denote the set of prime ideals of a commutative ring R with identity. Recall that, the *Zariski topology* or the *hull-kernel topology* for $Spec(R)$ is defined by letting $C \subseteq Spec(R)$ be closed if and only if there exists an ideal $\mathcal{A}$ of R such that $C = \{\mathcal{P} \in Spec(R) : \mathcal{P} \supseteq \mathcal{A}\}$. Hochster [8] has proved that a topological space is homeomorphic to the prime spectrum of some ring equipped with the Zariski topology if and only if it is spectral. In lattice theory, a spectral space is characterized by the fact that it is homeomorphic to the prime spectrum of a bounded (with a 0 and a 1) distributive lattice.

Spectral spaces are of interest not only in topological ring and lattice theory, but also in computer science, in particular, in domain theory.

Note that if X is a T_1-space, then X is spectral if and only if X is compact and totally disconnected (i.e., X is a Stone space [8]).

Now, by [11, Lemma 29.6] a compact T_2-space is totally disconnected if and only if whenever $x \neq y$ in X, there is a clopen set of X containing x not containing y.

Recently, some authors (see for example [2, 3, 5]) have been interested in particular types of spectral spaces.

Pursuing this kind of investigation for spectral spaces, we are interested in topological spaces such that the functionally Hausdorff -reflection is spectral.

Definition 17. *A topological space X is said to be functionally Hausdorff Spectral, if $FH(X)$ is a spectral space.*

First, we introduce some concepts.

Definition 18. *Let X be a topological space and C a subset of X.*

(1) *C is called a strongly functionally closed subset of X (for short s-F-closed) if C is a closed subset of X such that $C = d(C)$.*
(2) *C is called a strongly functionally open subset of X (for short s-F-open) if C is an open subset of X such that $C = d(C)$.*
(3) *C is called a strongly functionally clopen subset of X (for short s-F-clopen) if C is a clopen subset of X such that $C = d(C)$.*

Remark 19. *Clearly, an s-F-closed (resp., s-F-open) subset of a topological space is both closed (resp., open) and F-closed (resp., F-open). The converse* *does not hold. Indeed, let* $X := \{0, 1\}$ *the Sierpinski space. It is easily seen that* $FH(X)$ *is a one point and thus* $d(1) = d(0) = X$*. So that* $\{1\}$ *(resp.,* $\{0\}$*) is both closed (resp., open) and F-closed (resp., F-open) which is not s-F-closed (resp., s-F-open).*

Now, we give the main result of this section

Theorem 20. *Let* X *be a topological space. Then the following statements are equivalent.*

(1) $FH(X)$ *is spectral;*
(2) X *satisfies the following properties.*
  (i) *Every s-F-open cover of* X *has a finite subcover.*
  (ii) *For each completely separated points* $x, y \in X$*, there exists an s-F-clopen subset of* X *containing one of the* x, y *and not containing the other.*

Proof of Theorem 20. (1) $\Rightarrow$ (2)

(i) Let $\{V_i : i \in X\}$ an s-F-open cover of X. Since for any $i \in I$, V_i is s-F-open, then $V_i = \mu_X^{-1}(\mu_X(V_i))$ for any $i \in I$.

On the other hand, the equality $X = \cup[V_i : i \in I]$ implies that $FH(X) = \mu_X(X) = \cup[\mu_X(V_i) : i \in I]$ and thus, $\{\mu_X(V_i) : i \in I\}$ is an open cover of X. Now, since $FH(X)$ is spectral and consequently a compact space, there exists a finite subset J of I such that $FH(X) = \cup[\mu_X(V_i) : i \in J]$ and consequently, $X = \cup[\mu_X^{-1}(\mu_X(V_i)) : i \in J]$. Therefore, X has a finite subcover.

(ii) Let x and y be two completely separated points in X. Then $\mu_X(x) \neq \mu_X(y)$. Since $FH(X)$ is a spectral T_1-space then, it is totally disconnected. On the other hand $FH(X)$ is a spectral T_2-space and consequently, there exists a clopen subset $\widetilde{U}$ of $FH(X)$ containing $\mu_X(x)$ and not containing $\mu_X(y)$. Set $U = \mu_X^{-1}(\widetilde{U})$. Clearly, U is an s-F-clopen subset of X containing x and not containing y.

(2) $\Rightarrow$ (1) First let us remark that (i) means that $FH(X)$ is compact. Indeed, let $\{V_i : i \in I\}$ be an open cover of $FH(X)$. Then $\{\mu_X^{-1}(V_i) : i \in I\}$ is an s-f-open cover of X and thus by (i) there exists a finite subset J of I such that $X = \cup[\mu_X^{-1}(V_i) : i \in J]$. Hence, $FH(X) = \mu_X(X) = \cup[\mu_X(\mu_X^{-1}(V_i)) : i \in J] = \cup[V_i : i \in J]$. Therefore, $FH(X)$ is a compact space.

Now, since $FH(X)$ is a compact T_1-space, then $FH(X)$ is spectral if and only if it is totally disconnected. And thus if and only if for any two distinct points of $FH(X)$, there is a clopen set containing one not the other (because $FH(X)$ is a T_2 compact space). So, let $x, y \in X$ such that $\mu_X(x) \neq \mu_X(y)$. Then x and y are completely separated and by (ii), there exists an s-F-clopen set U of X containing x not y. Therefore, $\mu_X(U)$ is a clopen set of $FH(X)$ containing $\mu_X(x)$ and not containing $\mu_X(y)$. $\square$

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