ON THE LIMITING STRUCTURE OF SOME CENTRAL BINOMIAL EVALUATIONS

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ABSTRACT. We examine series of the form

$$\sum_{n=0}^{\infty} \binom{2n}{n}^{-1} \frac{(4x^2)^n}{2n+2m+1} \quad \text{and} \quad \sum_{n=0}^{\infty} \binom{2n}{n}^{-1} \frac{(-4x^2)^n}{2n+2m+1}.$$

In each case, there is an evaluation of the form $(p_m(x)f(x)-q_m(x))/x^{2m}$ where $f(x)$ is a transcendental function and $p_m(x)$ and $q_m(x)$ are polynomials with rational coefficients. We prove that for $|x| < 1$,

$$\lim_{m\to\infty} \frac{q_m(x)}{p_m(x)} = f(x).$$

From this result, we derive recurrences for π and for various logarithms.

1. Introduction

Consider the sequence of infinite sums

$$A_m = \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n \frac{1}{2m+2n+1}. \tag{1.1}$$

We have

$$A_1 = 3 - \frac{\pi\sqrt{3}}{2}, \quad A_2 = \frac{3\pi\sqrt{3}}{2} - 8, \quad A_3 = \frac{123}{5} - \frac{9\pi\sqrt{3}}{2},$$

and so on. In general, it can be shown that

$$A_m = (-3)^m \left(\frac{\pi\sqrt{3}}{2} - r_m\right)$$

for some rational number r_m. Moreover, since $\lim_{m\to\infty} A_m = 0$, it follows that $\lim_{m\to\infty} r_m = \frac{\pi\sqrt{3}}{2}$. This is not terribly mysterious; if one introduces a parameter x and writes

$$f_m(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+2m+1},$$

then for $|x| \leq 1$,

$$f_m(x) = \frac{(-1)^m}{x^{2m+1}} \left(\arctan x - \sum_{n=0}^{m-1} (-1)^n \frac{x^{2n+1}}{2n+1} \right).$$

That is, up to the scaling factor $\frac{(-1)^m}{x^{2m+1}}$, $f_m(x)$ differs from $\arctan x$ by a Taylor approximation of $\arctan x$ and $A_m = f_m\left(\frac{1}{\sqrt{3}}\right)$.

The sums in (1.1) are an example of a family

$$A_m = \sum_{n=0}^{\infty} a_{m,n},$$

where for some transcendental constant α, there is an evaluation of the form

$$A_m = r_m \alpha + s_m,$$

with r_m and s_m rational, and $\frac{r_m}{s_m}$ has some limiting behavior as $m \to \infty$. For a subtler example, consider $\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} n^m 2^n$. In [6, p. 456] it is proven that

$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} n^m 2^n = a_m + b_m \pi, \tag{1.2}$$

with a_m and b_m rational. It is stated in that paper that with a_m and b_m so defined,

$$\lim_{m \to \infty} \frac{a_m}{b_m} = \pi, \tag{1.3}$$

though no proof is given. On the other hand, consider the series $\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} n^m (-2)^n$. For $m \geq 2$,

$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} n^m (-2)^n = c_m \sqrt{3} + d_m \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right), \tag{1.4}$$

where c_m and d_m are rational. In this case, the author is unaware of any limiting behavior for $\frac{c_m}{d_m}$, the sequence of quotients being $-\frac{1}{2}$, $\frac{5}{2}$, $-\frac{7}{8}$, $\frac{7}{10}$, $-\frac{89}{58}$, $\frac{13}{196}$, $-\frac{1681}{722}$, $-\frac{97}{278}$, $\ldots$. The series in (1.2) was brought to the author's attention through an undergraduate research project [9], where the limiting behavior (1.3) appeared as a conjecture. From the various summation tables in that project, many other conjectures seem plausible. For example, if

$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{1}{2n+2m+1} = a_m \pi \sqrt{3} - b_m, \tag{1.5}$$

then it appears that

$$\lim_{m \to \infty} \frac{b_m}{a_m} = \pi \sqrt{3}.$$

If

$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(-1)^n}{2n+2m+1} = c_m \sqrt{5} \ln \phi - d_m, \tag{1.6}$$

where ϕ is the golden ratio then

$$\lim_{m \to \infty} \frac{d_m}{c_m} = \sqrt{5} \ln \phi.$$

In [3], sums involving $\frac{n!n!}{(2n)!}$, the reciprocal of the central binomial coefficient $\binom{2n}{n}$, were referred to as "central binomial sums." A proof of the Sherman/Lehmer conjecture (the limiting behavior in (1.3)) is fairly difficult (at least to this author). We provide a proof in a subsequent paper. In this paper, we prove a theorem containing (1.5) and (1.6) as special cases. Our main theorem is as follows.

Theorem 1.1.

a. *If $|x| < 1$, then there is an evaluation of the form*

$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(4x^2)^n}{2n+2m+1} = \frac{\arcsin x}{x^{2m+1}\sqrt{1-x^2}} A_m(x) - \frac{1}{x^{2m}} B_m(x),$$

where $A_m(x)$ and $B_m(x)$ are even polynomials with rational coefficients, and

$$\lim_{m \to \infty} \frac{B_m(x)}{A_m(x)} = \frac{\arcsin x}{x\sqrt{1-x^2}}.$$

b. *Writing* $\operatorname{arcsinh} x$ *for* $\ln\left(x + \sqrt{x^2+1}\right)$ *if $|x| \leq 1$, then,*

$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(-4x^2)^n}{2n+2m+1} = \frac{\operatorname{arcsinh} x}{x^{2m+1}\sqrt{1+x^2}} C_m(x) - \frac{1}{x^{2m}} D_m(x),$$

where $C_m(x)$ and $D_m(x)$ are even polynomials with rational coefficients, and

$$\lim_{m \to \infty} \frac{D_m(x)}{C_m(x)} = \frac{\operatorname{arcsinh} x}{x\sqrt{1+x^2}}.$$

The proof of Theorem 1.1 depends on certain integral representations for the series involved. We present these integral representations in Section 2 along with the definitions of A_m, B_m, C_m, and D_m, and the demonstration that the series in Theorem 1.1 have evaluations of the desired form. The limiting behavior of the evaluations in Theorem 1.1 is proven in Section 3. In Section 4, we use these ideas to obtain recurrences for calculating π and various other transcendental constants.

2. Preliminary Results

In addition to deriving integral representations for the sums in Theorem 1.1, we also need some facts related to elementary integrals. We present these first.

Lemma 2.1. *Let* $a_1 = 1$, $a_n = \frac{(2m-2)(2m-4)\cdots 2}{(2m-1)(2m-3)\cdots 3}$ *if* $m > 1$.

a. $a_m = \int_0^{\pi/2} \sin^{2m-1} x\, dx$,
b. $\int_0^x \sin^{2m-1} u\, du = a_m - \cos x P_m(\sin^2 x)$,
c. $\int_0^x \sinh^{2m-1} u\, du = (-1)^m a_m - (-1)^m \cosh x P_m(-\sinh^2 x)$ *for some polynomial* $P_m(x)$ *of degree* $m-1$ *with positive rational coefficients.*

Proof. These are easy consequences of the reduction formulas

$$\int \sin^n x\, dx - \frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x\, dx$$
$$\int \sinh^n x\, dx = \frac{1}{n} \sinh^{n-1} x \cosh x - \frac{n-1}{n} \int \sinh^{n-2} x\, dx.$$

The formula 2.1(a) is sometimes called Wallis' integral. □

We will need to know the asymptotics of the a_m in Lemma 2.1. This is well-known, but we state it here for completeness.

Lemma 2.2. *Let* $a_1 = 1$, $a_m = \frac{(2m-2)(2m-4)\cdots 2}{(2m-1)(2m-3)\cdots 3}$ *if* $m > 1$. *Then*

$$\lim_{m\to\infty} \frac{2ma_m}{\sqrt{m}} = \sqrt{\pi}.$$

Proof. This fact is stated in [1, p. 4]. It follows easily from formula (2) in [6, p. 28] using

$$2ma_m = 2^{2m} \frac{m!m!}{(2m)!}.$$

□

As in Theorem 1.1, we use the trigonometric form arcsinh (x) rather than $\ln\left(x + \sqrt{x^2+1}\right)$ in what follows. Using Lemma 2.1, we now give integral representations for our series.

Lemma 2.3. *If* $|x| < 1$ *then*

a.
$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(4x^2)^n}{2n+2m+1} = \frac{\arcsin x}{x^{2m+1}\sqrt{1-x^2}} - \frac{2m}{x^{2m+1}} \int_0^{\arcsin x} u \sin^{2m-1} u\, du,$$

b.
$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(-4x^2)^n}{2n+2m+1} = \frac{\operatorname{arcsinh} x}{x^{2m+1}\sqrt{1+x^2}} - \frac{2m}{x^{2m+1}} \int_0^{\operatorname{arcsinh} x} u \sinh^{2m-1} u \, du.$$

Proof. For (a), we begin with the evaluation [2, p. 59, but also see 3, 4, 6, 9],
$$(\arcsin x)^2 = \sum_{n=0}^{\infty} \frac{n!n!}{(2n+1)!} \frac{2^{2n} x^{2n+2}}{n+1},$$
which is valid for $|x| \le 1$. Differentiating twice with respect to x, we have
$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} (4x^2)^n = \frac{1}{1-x^2} + \frac{x}{(1-x^2)^{3/2}} \arcsin x. \tag{2.1}$$
Multiplying by x^{2m} and integrating,
$$x^{2m+1} \sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(4x^2)^n}{2n+2m+1} = \int_0^x \left(\frac{1}{1-t^2} + \frac{t}{(1-t^2)^{3/2}} \arcsin t \right) t^{2m} dt$$
$$= \int_0^{\arcsin x} (\sec^2 u + u \tan u \sec^2 u) \sin^{2m} u \cos u \, du$$
$$= \int_0^{\arcsin x} \sin^{2m} u (\sec u + u \tan u \sec u) \, du.$$
Integrating this last expression by parts, using $(\sec u + u \tan u \sec u) \, du = d(u \sec u)$, we have
$$x^{2m+1} \sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(4x^2)^n}{2n+2m+1} = \int_0^{\arcsin x} \sin^{2m} u (\sec u + u \tan u \sec u) \, du$$
$$= \frac{x^{2m}}{\sqrt{1-x^2}} \arcsin x - 2m \int_0^{\arcsin x} u \sin^{2m-1} u \, du,$$
as desired.

The proof of (b) is similar, starting with [5, p. 52],
$$\frac{\operatorname{arcsinh} x}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{2^{2n} x^{2n+1}}{2n+1} (-1)^n,$$
which is valid for $|x| \le 1$. Differentiating with respect to x,
$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} (-4x^2)^n = \frac{1}{1+x^2} + \frac{x}{(1+x^2)^{3/2}} \operatorname{arcsinh} x. \tag{2.2}$$
As before, we multiply by x^{2m}, and integrate. Finally, we integrate by parts using

$$(\operatorname{sech} u - u \tanh u \operatorname{sech} u)\, du = d(u \operatorname{sech} u).$$

□

The integral representations of Lemma 2.3 allow for the evaluation of the sums in Theorem 1.1.

Lemma 2.4. *If $|x| < 1$, then*

a.
$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(4x^2)^n}{2n+2m+1} = \frac{\arcsin x}{x^{2m+1}\sqrt{1-x^2}} A_m(x) - \frac{1}{x^{2m}} B_m(x),$$

where $A_m(x)$ and $B_m(x)$ are even polynomials with rational coefficients of degrees $2m$ and $2m-2$, respectively, defined by

$$A_m(x) = 2m\sqrt{1-x^2}a_m - 2m\sqrt{1-x^2} \int_0^{\arcsin x} \sin^{2m-1} u\, du + x^{2m}$$

and

$$xB_m(x) = 2ma_m \arcsin x - 2m \int_0^{\arcsin x} \int_0^u \sin^{2m-1} t\, dt\, du.$$

b.
$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(-4x^2)^n}{2n+2m+1} = \frac{\operatorname{arcsinh} x}{x^{2m+1}\sqrt{1+x^2}} C_m(x) - \frac{1}{x^{2m}} D_m(x),$$

where $C_m(x)$ and $D_m(x)$ are even polynomials with rational coefficients of degrees $2m$ and $2m-2$, respectively, defined by

$$C_m(x) = (-1)^m 2m\sqrt{1+x^2}a_m - 2m\sqrt{1+x^2} \int_0^{\operatorname{arcsinh} x} \sinh^{2m-1} u\, du + x^{2m}$$

and

$$xD_m(x) = (-1)^m 2ma_m \operatorname{arcsinh} x - 2m \int_0^{\operatorname{arcsinh} x} \int_0^u \sinh^{2m-1} t\, dt\, du.$$

Proof. Integrating $\int_0^{\arcsin x} u \sin^{2m-1} u\, du$ by parts, we have

$$\int_0^{\arcsin x} u \sin^{2m-1} u\, du$$
$$= \arcsin x \int_0^{\arcsin x} \sin^{2m-1} u\, du - \int_0^{\arcsin x} \int_0^u \sin^{2m-1} t\, dt\, du. \tag{2.3}$$

Thus, by Lemma 2.3 (a),

$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(4x^2)^n}{2n+2m+1} = \frac{\arcsin x}{x\sqrt{1-x^2}} - \frac{2m \arcsin x}{x^{2m+1}} \int_0^{\arcsin x} \sin^{2m-1} u \, du$$
$$+ \frac{2m}{x^{2m+1}} \int_0^{\arcsin x} \int_0^u \sin^{2m-1} t \, dt \, du$$
$$= \frac{\arcsin x}{x^{2m+1}\sqrt{1-x^2}} \times$$
$$\left(x^{2m} - 2m\sqrt{1-x^2} \int_0^{\arcsin x} \sin^{2m-1} u \, du + 2m\sqrt{1-x^2}a_m \right)$$
$$+ \frac{1}{x^{2m+1}} \left(2m \int_0^{\arcsin x} \int_0^u \sin^{2m-1} t \, dt \, du - 2ma_n \arcsin x \right)$$
$$= \frac{\arcsin x}{x^{2m+1}\sqrt{1-x^2}} A_m(x) - \frac{1}{x^{2m}} B_m(x).$$

With $P_m(x)$ as in Lemma 2.1,

$$\int_0^{\arcsin x} \sin^{2m-1} u \, du = a_m - \sqrt{1-x^2} P_m(x^2).$$

Consequently,

$$A_m(x) = x^{2m} + 2m\sqrt{1-x^2}a_m - 2m\sqrt{1-x^2} \left(a_m - \sqrt{1-x^2} P_m(x) \right)$$
$$= x^{2m} - 2m(1-x^2)P_m(x^2)$$

is an even polynomial with rational coefficients. Since the coefficient of x^{2m} in $P_m(x)$ is $\frac{1}{2m-1}$, it follows that $A_m(x)$ has degree $2m$.

Similarly,

$$B_m(x) = \frac{1}{x} \left(-2m \int_0^{\arcsin x} \int_0^u \sin^{2m-1} t \, dt \, du + 2ma_m \arcsin x \right)$$
$$= \frac{1}{x} \left(-2m \int_0^{\arcsin x} (a_m - \cos u P_m(\sin^2 u)) \, du + 2ma_m \arcsin x \right)$$
$$= \frac{2m}{x} \int_0^x P_m(u^2) \, du,$$

is an even polynomial with rational coefficients, and degree $2m-2$.

Again, (b) is very similar. After an integration by parts,

$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(-4x^2)^n}{2n+2m+1} = \frac{\operatorname{arcsinh} x}{x\sqrt{1+x^2}} - \frac{2m \operatorname{arcsinh} x}{x^{2m+1}} \int_0^{\operatorname{arcsinh} x} \sinh^{2m-1} u \, du$$

$$+ \frac{2m}{x^{2m+1}} \int_0^{\operatorname{arcsinh} x} \int_0^u \sinh^{2m-1} t \, dt \, du$$

$$= \frac{\operatorname{arcsinh} x}{x^{2m+1}\sqrt{1+x^2}} \times$$

$$\left(x^{2m} - 2m\sqrt{1+x^2} \int_0^{\operatorname{arcsinh} x} \sinh^{2m-1} u \, du + (-1)^m 2m \sqrt{1+x^2} a_m \right)$$

$$+ \frac{1}{x^{2m+1}} \left(2m \int_0^{\operatorname{arcsinh} x} \int_0^u \sinh^{2m-1} t \, dt \, du - (-1)^m 2m a_m \operatorname{arcsinh} x \right)$$

$$= \frac{\operatorname{arcsinh} x}{x^{2m+1}\sqrt{1+x^2}} C_m(x) - \frac{1}{x^{2m}} D_m(x).$$

The proof that $C_m(x)$ and $D_m(x)$ are polynomials with the proper degree is entirely analogous. This completes the proof of the lemma. □

3. A Proof of Theorem 1.1

To complete the proof of Theorem 1.1, we must show that with $A_m(x)$, $B_m(x)$, $C_m(x)$, and $D_m(x)$ as in Lemma 2.4,

$$\lim_{m\to\infty} \frac{B_m(x)}{A_m(x)} = \frac{\arcsin x}{x\sqrt{1-x^2}},$$

or

$$\lim_{m\to\infty} \frac{xB_m(x)}{A_m(x)} = \frac{\arcsin x}{\sqrt{1-x^2}},$$

and that

$$\lim_{m\to\infty} \frac{xD_m(x)}{C_m(x)} = \frac{\operatorname{arcsinh} x}{\sqrt{1+x^2}}.$$

These will both follow from the following lemma.

Lemma 3.1. *If $|x| < 1$, then the five quantities*

$$\frac{x^{2m}}{a_m}, \quad \frac{\int_0^{\arcsin x} \sin^{2m-1} u \, du}{a_m}, \quad \frac{\int_0^{\arcsin x} \int_0^u \sin^{2m-1} t \, dt \, du}{a_m},$$

$$\frac{\int_0^{\operatorname{arcsinh} x} \sinh^{2m-1} u \, du}{a_m}, \quad \frac{\int_0^{\operatorname{arcsinh} x} \int_0^u \sinh^{2m-1} t \, dt \, du}{a_m}$$

each have a limit of 0 as $m \to \infty$.

Proof. First, note that for $|x|<1$, if $0\le u\le \arcsin x$, then $0\le \sin u\le x$. Thus,

$$\int_0^{\arcsin x}\sin^{2m-1}u\,du \le \int_0^{\arcsin x}x^{2m-1}\,du = x^{2m-1}\arcsin x,$$
$$\int_0^{\arcsin x}\int_0^u \sin^{2m-1}t\,dt\,du \le \int_0^{\arcsin x}\int_0^u x^{2m-1}\,dt\,du = \frac{1}{2}(\arcsin x)^2x^{2m-1}.$$

Similarly, for $|x|<1$, if $0\le u\le \operatorname{arcsinh} x$, then $0\le \sinh u\le x$, so

$$\int_0^{\operatorname{arcsinh} x}\sinh^{2m-1}u\,du \le x^{2m-1}\operatorname{arcsinh} x,$$
$$\int_0^{\operatorname{arcsinh} x}\int_0^u \sinh^{2m-1}t\,dt\,du \le \frac{1}{2}(\operatorname{arcsinh} x)^2x^{2m-1}.$$

Consequently, we need only establish the limit

$$\lim_{m\to\infty}\frac{x^{2m}}{a_m}=0.$$

Now

$$2ma_m=\frac{2m}{2m-1}\frac{2m-2}{2m-3}\cdots\frac{2}{1}>1,$$

so $a_m>\frac{1}{2m}$. Thus, for $|x|<1$,

$$0\le \frac{x^{2m}}{a_m}\le 2mx^{2m},$$

and

$$\lim_{m\to\infty}2mx^{2m}=0.$$

This completes the proof of the lemma. $\square$

We now complete the proof of the theorem. As expected, cases (a) and (b) are very similar, so we only give a proof of (b). Assume first that $|x|<1$. By Lemma 2.4 (b), we are interested in the quantity

$$\begin{aligned}
&\frac{xD_m(x)}{C_m(x)}\\
&=\frac{(-1)^m2ma_m\operatorname{arcsinh} x-2m\int_0^{\operatorname{arcsinh} x}\int_0^u\sinh^{2m-1}t\,dt\,du}{(-1)^m2m\sqrt{1+x^2}a_m-2m\sqrt{1+x^2}\int_0^{\operatorname{arcsinh} x}\sinh^{2m-1}u\,du+x^{2m}}\\
&=\frac{\operatorname{arcsinh} x-\left((-1)^m\int_0^{\operatorname{arcsinh} x}\int_0^u\sinh^{2m-1}t\,dt\,du\right)/a_m}{\sqrt{1+x^2}+\frac{(-1)^mx^{2m}}{2ma_m}-\left((-1)^m\int_0^{\operatorname{arcsinh} x}\sinh^{2m-1}u\,du\right)/a_m},
\end{aligned}$$

and by the previous lemma, in the limit, this becomes $\frac{\operatorname{arcsinh} x}{\sqrt{1+x^2}}$, as desired. We note that since $\frac{x^{2m}}{a_m} \to 0$ essentially as $x^{2m} \to 0$, the convergence is rapid for small x (but strictly first order).

Finally, in the case of part (b), we must show that the result is true for $x = \pm 1$. By Abel's Theorem [8, p. 174], we have

$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(-4)^n}{2n+2m+1} = C_m(1) \frac{\operatorname{arcsinh} 1}{\sqrt{2}} - D_m(1),$$

with

$$C_m(1) = (-1)^m 2m\sqrt{2} a_m + 1 - 2m\sqrt{2} \int_0^{\operatorname{arcsinh} 1} \sinh^{2m-1} u \, du$$

and

$$D_m(1) = (-1)^m 2m a_m \operatorname{arcsinh} 1 - 2m \int_0^{\operatorname{arcsinh} 1} \int_0^u \sinh^{2m-1} t \, dt \, du.$$

It follows from an easy induction that

$$2m \int_0^{\operatorname{arcsinh} 1} \sinh^{2m-1} u \, du < 2.$$

However, $C_m(1)$ and $D_m(1)$ both grow like $\sqrt{\pi m}$ (by Lemma 2.2). Since

$$\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(-4)^n}{2n+2m+1} \to 0$$

as $m \to \infty$,

$$C_m(1) \frac{\operatorname{arcsinh} 1}{\sqrt{2}} - D_m(1) \to 0$$

as well. This obviously forces

$$\lim_{m\to\infty} \frac{D_m(1)}{C_m(1)} = \frac{\operatorname{arcsinh} 1}{\sqrt{2}} = \frac{\ln(1+\sqrt{2})}{\sqrt{2}},$$

as desired.

4. Recurrences and Other Results

Using the fact that for fixed $|x| < 1$,

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(4x^2)^n}{2n+2m+1} = 0$$

and

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(-4x^2)^n}{2n+2m+1} = 0,$$

it follows from Lemma 2.2 that

$$\lim_{m\to\infty} \frac{2m}{x^{2m+1}} \int_0^{\arcsin x} u \sin^{2m-1} u \, du = \frac{\arcsin x}{x\sqrt{1-x^2}} \tag{4.1}$$

and

$$\lim_{m\to\infty} \frac{2m}{x^{2m+1}} \int_0^{\operatorname{arcsinh} x} u \sinh^{2m-1} u \, du = \frac{\operatorname{arcsinh} x}{x\sqrt{1+x^2}}. \tag{4.2}$$

Given Lemma 2.3, we may rewrite our results in terms of limits of recurrence relations. To do this, we make use of the reduction formulas:

$$\begin{aligned}\int x \sin^{2m-1} x \, dx = &-\frac{1}{2m-1} x \sin^{2m-2} x \cos x + \frac{1}{(2m-1)^2} \sin^{2m-1} x \\ &+ \frac{2m-2}{2m-1} \int x \sin^{2m-3} x \, dx,\end{aligned} \tag{4.3}$$

$$\begin{aligned}\int x \sinh^{2m-1} x \, dx = &\frac{1}{2m-1} x \sinh^{2m-2} x \cosh x - \frac{1}{(2m-1)^2} \sinh^{2m-1} x \\ &- \frac{2m-2}{2m-1} \int x \sinh^{2m-3} x \, dx.\end{aligned} \tag{4.4}$$

Letting $f_m(x)$ and $g_m(x)$ be defined by

$$f_m(x) = \sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(4x^2)^n}{2n+2m+1},$$

and

$$g_m(x) = \sum_{n=0}^{\infty} \frac{n!n!}{(2n)!} \frac{(-4x^2)^n}{2n+2m+1}$$

we have

$$f_m(x) = -\frac{1}{2m-1} \frac{\arcsin x}{x\sqrt{1-x^2}} - \frac{2m}{2m-1} \frac{1}{x^2} + \frac{2m}{2m-1} \frac{1}{x^2} f_{m-1}(x), \tag{4.5}$$

and

$$g_m(x) = -\frac{1}{2m-1} \frac{\operatorname{arcsinh} x}{x\sqrt{1+x^2}} + \frac{2m}{(2m-1)^2} \frac{1}{x^2} - \frac{2m}{2m-1} \frac{1}{x^2} g_{m-1}(x). \tag{4.6}$$

Comparing with Lemma 2.4, this gives the following theorem.

Theorem 4.1. *With A_m, B_m, C_m, D_m as in Lemma 2.4*

(a)

$$A_0 = 1, \quad A_m(x) = \frac{2m}{2m-1} A_{m-1}(x) - \frac{x^{2m}}{2m-1} \quad \textit{for} \quad m \geq 1,$$

(b)

$$B_0 = 0, \quad B_m(x) = \frac{2m}{2m-1} B_{m-1}(x) + \frac{2m}{(2m-1)^2} x^{2m-2} \quad \textit{for} \quad m \geq 1,$$

(c)

$$C_0 = 1, \quad C_m(x) = -\frac{2m}{2m-1} C_{m-1}(x) - \frac{x^{2m}}{2m-1} \quad \textit{for} \quad m \geq 1,$$

(d)

$$D_0 = 0, \quad D_m(x) = -\frac{2m}{2m-1} D_{m-1}(x) - \frac{2m}{(2m-1)^2} x^{2m-2} \quad \textit{for} \quad m \geq 1.$$

In particular, by Theorem 1.1, with polynomials A_m, B_m, C_m, D_m defined as in (a), (b), (c), (d), we have

$$\lim_{m \to \infty} \frac{B_m(x)}{A_m(x)} = \frac{\arcsin x}{x\sqrt{1-x^2}}, \tag{4.7}$$

for $|x| < 1$ and

$$\lim_{m \to \infty} \frac{D_m(x)}{C_m(x)} = \frac{\ln\left(x + \sqrt{1+x^2}\right)}{x\sqrt{1+x^2}}, \tag{4.8}$$

for $|x| \leq 1$.

Many variations of these formulas exist. In particular, we may scale away the x^{2m} term and multiply the limits (4.7) and (4.8) by any given fixed r. If we also introduce a factor of $(-1)^m$ for C and D, then with

$$a_0 = 1, \quad a_m(x) = \frac{2m}{x^2(2m-1)} a_{m-1}(x) - \frac{1}{2m-1} \quad \text{for} \quad m \geq 1,$$

$$b_0 = 0, \quad b_{m,r}(x) = \frac{2m}{x^2(2m-1)} b_{m-1,r}(x) + \frac{2mr}{x^2(2m-1)^2} \quad \text{for} \quad m \geq 1,$$

$$c_0 = 1, \quad c_m(x) = \frac{2m}{x^2(2m-1)} c_{m-1}(x) - (-1)^m \frac{1}{2m-1} \quad \text{for} \quad m \geq 1,$$

$$d_0 = 0, \quad d_{m,r}(x) = \frac{2m}{x^2(2m-1)} d_{m-1,r}(x) - (-1)^m \frac{2m}{x^2(2m-1)^2} \quad \text{for} \quad m \geq 1,$$

we have

$$\lim_{m \to \infty} \frac{b_{m,r}(x)}{a_m(x)} = \frac{r \arcsin x}{x\sqrt{1-x^2}}, \tag{4.9}$$

and

$$\lim_{m \to \infty} \frac{d_{m,r}(x)}{c_m(x)} = \frac{r \ln\left(x + \sqrt{1+x^2}\right)}{x\sqrt{1+x^2}}. \tag{4.10}$$

One may select r so as to simplify the look of special cases of (4.9) and (4.10). For example, if $x = \frac{1}{2}$ and $r = \frac{9}{2}$, then $\lim_{m\to\infty} \frac{b_{m,r}(x)}{a_m(x)} = \pi\sqrt{3}$. If $x = \frac{1}{2}$, $r = \frac{5}{4}$, then $\lim_{m\to\infty} \frac{d_{m,r}(x)}{c_m(x)} = \sqrt{5}\ln\phi$, where ϕ is the golden ratio.

With (4.10), if we let $x = \frac{3}{4}$ and $r = \frac{15}{16}$ then $\lim_{m\to\infty} \frac{d_{m,r}(x)}{c_m(x)} = \ln 2$. More generally, if $p^2 - q^2 < 2pq$, then with $x = \frac{p^2-q^2}{2pq}$, $r = \frac{p^4-q^4}{4p^2q^2}$, we obtain $\lim_{m\to\infty} \frac{d_{m,r}(x)}{c_m(x)} = \ln\left(\frac{p}{q}\right)$.

If in (4.9) we replace x by $\sin\left(\frac{\pi}{n}\right)$ then we have

$$\lim_{m\to\infty} \frac{b_{m,r}(x)}{a_m(x)} = \frac{\pi r}{n \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)} = \frac{2\pi r}{n \sin\left(\frac{2\pi}{n}\right)}. \tag{4.11}$$

In particular, if we let $r = \frac{1}{2} n \sin\left(\frac{2\pi}{n}\right)$, then

$$\lim_{m\to\infty} \frac{b_{m,r}(x)}{a_m(x)} = \pi. \tag{4.12}$$

The simplest case of (4.12) is with $n = 4$, so $x = \frac{1}{\sqrt{2}}$, and $r = 2$. In this case, with

$$\begin{gathered} a_0 = 1, \quad a_m = \frac{4m}{2m-1} a_{m-1} - \frac{1}{2m-1} \quad \text{for} \quad m \geq 1, \\ b_0 = 0, \quad b_m = \frac{4m}{2m-1} b_{m-1} + \frac{8m}{(2m-1)^2} \quad \text{for} \quad m \geq 1, \\ \lim_{m\to\infty} \frac{b_m}{a_m} = \pi. \end{gathered} \tag{4.13}$$

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