A SIMPLE MODULAR PROOF OF FARKAS' ARITHMETIC IDENTITY MODULO 4

Wissam Raji

Abstract. In this paper, we prove an arithmetic identity derived by Farkas [3]. We use generalized eta products and their logarithmic derivatives to determine the arithmetic identity.

1. Introduction. In this paper, we determine an arithmetic identity modulo 4. The group associated to the arithmetic identity is $\Gamma_0(4)$. Notice that the group in question has a fundamental region whose closure has genus 0, so there exists no nontrivial cusp forms of weight 2 and trivial multiplier system, i.e., $S_2(\Gamma_0(4)) = \{0\}$. Notice also that $M_2(\Gamma_0(4))$ has three cusps and hence, $\dim M_2(\Gamma_0(4)) = 2$ (2 Eisenstein series).

2. Arithmetic Identities Modulo 4. Let us define our character here.

$$\chi_4(n) = \left(\frac{-4}{n}\right) = \begin{cases} 1 & n \equiv 1 \mod 4 \\ -1 & n \equiv -1 \mod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Eisenstein series of weight 1 is defined by

$$G_{1,\chi} = \sum_{m=-\infty}^{\infty} \chi(m) G_{1,m},$$

where χ is a nontrivial Dirichlet character on $(\mathbb{Z}/N\mathbb{Z})^*$ and $G_{1,m}$ is given by

$$G_{1,m} = \frac{m}{N} - \frac{1}{2} - \frac{q^m}{1-q^m} + \sum_{\nu=1}^{\infty} \left[\frac{q^{\nu N - m}}{1 - q^{\nu N - m}} - \frac{q^{\nu N + m}}{1 - q^{\nu N + m}}\right]. \tag{1}$$

Now we will consider the Eisenstein series of weight 1 on $\Gamma_0(4)$. When we square it we will get weight 2 Eisenstein series on $\Gamma_0(4)$. But $M_2(\Gamma_0(4))$ is two dimensional. In [6], Vestal determined the basis of $M_2(\Gamma_0(4))$ explicitly. We define now an arithmetic function which will appear in the Fourier expansion of the logarithmic derivative of the generalized Dedekind eta function. Define $\sigma^{(\delta,g)}$ by

$$\sigma^{(\delta,g)}(N) = \sum_{\substack{d \mid N \\ d \equiv g \pmod{\delta}}} d + \sum_{\substack{d \mid N \\ d \equiv -g \pmod{\delta}}} d$$

and define $\delta_{4,2}$ by

$$\delta_{4,2}(n) = \sum_{\substack{d|n \\ d\equiv 1 \pmod 4}} 1 - \sum_{\substack{d|n \\ d\equiv -1 \pmod 4}} 1.$$

In this paper, we prove the following identity using the logarithmic derivative of generalized eta products. The identity is given by

$$2\delta_{4,2}(n) + 4\sum_{j=1}^{n-1} \delta_{4,2}(j)\delta_{4,2}(n-j) = 2\sigma^{(4,1)}(n) + \sigma^{(4,2)}(n).$$

To present the basis, we use generalized Dedekind η-function $\eta_{\delta,g}(\tau)$. Recall that

$$\eta_{\delta,g}(\tau) = e^{\pi i P_2(\frac{g}{\delta})\delta\tau} \prod_{\substack{m>0 \\ m\equiv g \pmod \delta}} (1-x^m) \prod_{\substack{m>0 \\ m\equiv -g \pmod \delta}} (1-x^m),$$

where $x = e^{2\pi i\tau}$, $\tau \in H$, H is the upper half plane, $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second Bernoulli function, and $\{t\} = t - [t]$ is the fractional part of t. Vestal calculated the logarithmic derivative of $\eta_{\delta,g}(\tau)$:

$$\frac{\eta'_{\delta,g}(\tau)}{\eta_{\delta,g}(\tau)} = \pi i\delta P_2(g/\delta) - 2\pi i \sum_{N=1}^{\infty} \sigma^{(\delta,g)}(N)q^N,$$

where $q = e^{2\pi i\tau}$. For simplicity, let $H_2^{(\delta,g)}$ denote the normalization of the above series.

$$H_2^{(\delta,g)} = 1 - \frac{2}{\delta P_2(g/\delta)} \sum_{N=1}^{\infty} \sigma^{(\delta,g)}(N)q^N.$$

Then $H_2^{(\delta,g)}(\tau) \in M_2(\Gamma_0(\delta))$. Note that $M_2(\Gamma_0(4))$ is two dimensional, so the basis of $\Gamma_0(4)$ [6] consists of

$$H_2^{(4,1)} = 1 + 24\sum_{N=1}^{\infty} \sigma^{(4,1)}(N)q^N$$

and

$$H_2^{(4,2)} = 1 + 6\sum_{N=1}^{\infty} \sigma^{(4,2)}(N)q^N.$$

Vestal proceeds to find that

$$\theta^4(\tau) = \frac{1}{3}H_2^{(4,1)}(\tau) + \frac{2}{3}H_2^{(4,2)}(\tau),$$

where

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 \tau}.$$

Notice that $\theta(\tau) \in M_{1/2}(\Gamma_0(4))$. We relate now the above basis to our Eisenstein series of weight 1. With the odd character defined above, the author in [5] derives the following form of $G_{1,\chi}$ explicitly from (1) and thus, we get

$$G_{1,\chi} = -\frac{1}{2} - 2\sum_{n=1}^{\infty} \sum_{d|n} \chi(d) q^n.$$

As a result, we get

$$G_{1,\chi} = -\frac{1}{2} - 2\sum_{n=1}^{\infty} \delta_{4,2}(n) q^n,$$

where

$$\delta_{4,2} = \sum_{d|n} \chi(d).$$

Notice that $\delta_{4,2}$ is the difference between the number of divisors of n congruent to 1 mod 4 and the number of divisors of n congruent to -1 modulo 4. By a classical result that goes back to Jacobi

$$\theta^2(z) = 1 + 4\sum_{n=1}^{\infty} \delta_{4,2} z^n.$$

Hence,

$$\theta^2(\tau) = -2G_{1,\chi}(\tau).$$

Therefore,

$$4G_{1,\chi}^2(\tau) = \frac{1}{3} H_2^{(4,1)}(\tau) + \frac{2}{3} H_2^{(4,2)}(\tau).$$

As a result, we get

$$4\left(-\frac{1}{2} - 2\sum_{n=1}^{\infty} \delta_{4,2}(n) q^n\right)^2 = \frac{1}{3} H_2^{(4,1)}(\tau) + \frac{2}{3} H_2^{(4,2)}(\tau).$$

This leads to the following identity

$$8\delta_{4,2}(n) + 16\sum_{j=1}^{n-1} \delta_{4,2}(j)\delta_{4,2}(n-j) = 8\sigma^{(4,1)}(n) + 4\sigma^{(4,2)}(n).$$

Consequently,

$$2\delta_{4,2}(n) + 4\sum_{j=1}^{n-1}\delta_{4,2}(j)\delta_{4,2}(n-j) = 2\sigma^{(4,1)}(n) + \sigma^{(4,2)}(n).$$

This is a new proof of another identity due to Farkas [3].

References

1. T. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer-Verlag, New York, 1989.
2. H. Farkas, "On an Arithmetical Function," *The Ramanujan J.*, 8 (2004), 309–315.
3. H. Farkas, *On an Arithmetical Function II*, Complex Analysis and Dynamical Systems II, 121–130, Contemp. Math., 382, Amer. Math. Soc., Providence, RI, 2005.
4. P. Guerzhoy and W. Raji, "A Modular Look at Farkas' Identities," submitted for publication.
5. T. Miyake, *Modular Forms*, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
6. D. Vestal, *Generalized Dedekind Eta Functions with Applications to Additive Number Theory*, Ph.D. Thesis, University of Colorado, 1998.
7. E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis*, Cambridge Mathematical Library, U. K., 2002.

Mathematics Subject Classification (2000): 11F11, 11F20

Wissam Raji
Department of Mathematics
Temple University
Philadelphia, PA 19122
email: wissam@temple.edu