MULTI-SMOOTH POINTS OF FINITE ORDER

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Abstract. A point x of the unit sphere $S(X)$ of the Banach space X is called a multi-smooth point of order n if there exist exactly n-independent continuous linear functionals $g_1, \dots, g_n$, in $S(X^*)$, the unit sphere of the dual of X, such that $g_i(x) = 1$, for $1 \leq i \leq n$. The object of this paper is to characterize multi-smooth points of some function and operator spaces.

1. Introduction. Let X be a Banach space, X^* be the dual of X, and $S(X)$ the unit sphere of X. The space of bounded linear operators on X will be denoted by $L(X)$. A point $x \in S(X)$ is called a smooth point, if there exists a unique bounded linear functional $g \in S(X^*)$ such that $g(x) = 1$. Holub [5], was the first to consider the problem of characterization of smooth points of compact operators on Hilbert spaces. This was generalized by Heinrich [3], to compact operators on arbitrary Banach spaces. Smooth points of $S(L(\ell^2))$ were studied in [9], while those of $S(L(\ell^p))$, $1 \leq p < \infty$, were studied in [1].

Smooth points are associated with the differentiability of the norm at such points. Indeed, $x \in S(X)$ is smooth if and only if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists. We refer to the monograph of Diestel [2], for a detailed relation of smooth points of $S(X)$ and the geometry of X. Smooth points of the unit sphere of function and Banach spaces were discussed in several papers. We refer the reader to [1, 3, 5, 7, 9, 12, 18, 19] for several interesting results on such points.

Now consider $\ell_2^\infty = (R^2, \|\|_\infty)$. The unit sphere in such a space is the square with corners $(-1,-1)$, $(-1,1)$, $(1,-1)$ and $(1,1)$. We noticed that for each such corner, there exist exactly two independent functionals in $S(\ell_2^1)$ which attain their norm at such a corner. It turned out (Section 5 of this paper), this is the case for any finite dimensional normed space. This observation led us to introduce (Definition 1.1) the concept of multi-smooth points (or smooth points of finite order). It is the object of this paper, to characterize the multi-smooth points of order n for many classical Banach spaces and some operator spaces. In particular, we characterize multi-smooth points of order n for the spaces c_0, ℓ^1, $C(K,X)$, $K(\ell^p)$, the space of compact operators on the Banach space ℓ^p, $L(\ell^p)$, $1 < p < \infty$, $N(\ell^2)$, the trace class operators, and the Calkin algebra $L(\ell^2)/K(\ell^2)$.

2. Smooth Points of Sequence Spaces. In this section we will study multi-smooth points of finite order for c_0, and ℓ^1. Let us start with

Definition 1.1. Let X be a Banach space and $S(X)$ be the unit sphere of X. An element $x \in S(X)$ is called a multi-smooth point of order n if there exists n-independent continuous linear functionals $g_1, \dots, g_n$, on X, each of norm one and $g_i(x) = 1$ for all $1 \leq i \leq n$.

Let c_0 be the space of bounded sequences that converges to zero. With the sup-norm, c_0 is a Banach space.

Theorem 1.1. Let $x \in S(c_0)$. The following are equivalent:

(i) x is a smooth point of order n.

(ii) x has exactly n-coordinates each of absolute value 1.

Proof. (i) $\to$ (ii). Let x be a multi-smooth point of order n. Then there exist $g_1, \dots, g_n$, in the unit ball of ℓ^1such that $< g_i, x >= 1$ for all $1 \leq i \leq n$. If $x = (a_i)$, then

$$< g_i, x >= \sum_{j=1}^{\infty} g_i(j) a_j = 1.$$

Since $\|x\| = 1$ and $\|g_i\| = 1$, it follows that $|a_i| = 1$ whenever $g_i(j) \neq 0$. If $|a_i| = 1$ for more than n-coordinates, say $i_1, \dots, i_m$, with $m > n$, then $< \alpha_k \delta_{i_k}, x >= 1$ for $1 \leq k \leq m$, where (δ_k) is the natural basis of ℓ^1, and $\alpha_k \in R$ is chosen such that $\alpha_k a_k = 1$. So $\{\alpha_1 \delta_{i_1}, \dots, \alpha_m \delta_{i_m}\}$ is a set of m independent unit functionals on c_0, each attains its norm at x. Since $m > n$, we get a contradiction to the assumption on x. Similarly, if there exists k-coordinates of x each of absolute value 1, with $k > n$, then there are only k-independent unit functionals on c_0 each of which attains its norm at x. Thus, x must have only n-coordinates, each is of absolute value 1.

(ii) $\to$ (i). If x is a point in $S(c_0)$, with $n-$coordinates having absolute value 1, say $a_{i1}, \dots, a_{in}$, then, as in the proof of the first part, there are n-independent unit elements in ℓ^1 that attain their norms at x. Further, there is no more and there is no less. So, x is a smooth point of order n.

Theorem 1.2. Let $x \in S(\ell^1)$. The following are equivalent:

(i) x is a smooth point of order n.

(ii) x has n-coordinates each with value zero.

Proof. Let $x \in S(\ell^1)$ with $x = (x_i)$. Now let x have n-zeros, say $x_1, \dots, x_n$. Define

$$\begin{aligned} g_1 &= (0, 0, 0, \dots, 0, sgn(x_{n+1}), \dots) \\ g_2 &= (1, 0, 0, \dots, 0, sgn(x_{n+1}), \dots) \\ &\vdots \\ g_n &= (0, 0, 0, \dots, 1, sgn(x_{n+1}), \dots). \end{aligned}$$

Then $\{g_1, g_2, \dots, g_n\}$ is a set of n-linearly independent unit functionals on ℓ^1, and $g_i(x) = 1$ for all $1 \leq i \leq n$. Further, if g is any unit functional on ℓ^1 such that $< g, x >= 1$ then, $g(i) = sgn(x_i)$ for all $i \geq n$. And thus, g is in the linear span of $g_1, \dots, g_n$. So, x is a smooth point of order n.

If x does not have n-coordinates each of value zero, then, either there is less than n-independent unit functionals on ℓ^1 attaining their norm at x, or there is more than n such functionals. In either case x is not multi-smooth point of order n. This ends the proof.

Remark 1.3. Since the ℓ^p spaces are uniformly convex for all $1 < p < \infty$, it follows that $S(\ell^p)$ has no smooth point of order greater than 1.

Let $C(K, X)$ be the space of all continuous functions on the compact set K with values in the Banach space X. Then, $C(K, X)$ is a Banach space under the sup-norm $\|f\|_\infty = \sup\{\|f(t)\| : t \in K\}$. It is known [12], that $C(K, X) = C(K)\check{\otimes}X$, the completed injective tensor product of $C(K)$ with X. So extreme points of $S(C(K, X))^*$ are functionals of the form $\mu \otimes x^*$, with μ extreme in $S(C(K))^*$ and x^* extreme in $S(X^*)$ [15]. We remark here that the dual of $C(K)$ is $M(K)$, the space of regular Borel measures on K.

Theorem 1.4. Let $f \in C(K, X)$. The following are equivalent:

(i) $f \in S(C(K, X))$ is smooth of order n.

(ii) There exist exactly k elements $\{t_1, \dots, t_k\} \subset K$ with the following properties

(a) $\|f(t_i)\| = 1$ for $1 \leq i \leq k$.

(b) $f(t_i)$ is a multi-smooth point of order m_i, with $m_1 + \cdots + m_k = n$.

Proof. (ii) $\rightarrow$ (i). Let $f \in S(C(K, X))$ and $\{t_1, \dots, t_k\} \subset K$ satisfy (a) and (b). By (b), $f(t_i)$ is a multi-smooth point of order m_i. So there exist m_i independent functionals $x^*_{i1}, \dots x^*_{im_i}$ in $S(X^*)$, such that $< f(t_i), x^*_{ir} >= 1$ for all $1 \leq i \leq k$, and $1 \leq r \leq m_i$. Set δ_i to denote the Dirac measure at t_i for $1 \leq i \leq k$. Form the following set of linear functionals on $C(K, X) : F_{ij} = \delta_i \otimes x^*_{is}$ for $1 \leq i \leq k$, $1 \leq s \leq m_j$, and $1 \leq j \leq k$. Since $t_i \neq t_j$, and $x^*_{i1}, \ldots, x^*_{im_i}$ are independent, it follows that F_{im_j} are independent functionals in $S(C(K, X)^*)$ for which $F_{im_j}(f) =< f(t_i), x^*_{is} >= 1$. Hence, there are n-independent functionals in $S(C(K, X)^*)$ and each attains its norm at f. We claim that there are no more independent functionals in $S(C(K, X)^*)$ that attain their norm at f. To see that, let $E(f) = \{F \in S(C(K, X)^*) : F(f) = 1\}$. By the Alaoglu Theorem and the Krein Milman Theorem, $E(f)$ is the w^*-convex hull of its extreme points. Being extremal [11], $extE(f) \subset extS(C(K, X)^*) = \{\delta_t \otimes x^* : x^* \in extS(X^*)\}$. For any such extreme functional, one has $< f, \delta_t \otimes x^* >=< f(t), x^* >= 1$, and so $\|f(t)\| = 1$. Consequently, condition (a) forces t to be in E. Further, x^* must be one of the $x^{*,s}_{is}$, otherwise condition (b) fails to be true. Hence, f is a multi-smooth point of order n.

(i) $\to$ (ii). Let $f \in S(C(K, X))$ be a multi-smooth point of order n. Again, set $E(f) = \{\mu \in S(C(K))^* : \mu(f) = 1\}$. One can easily show that $E(f)$ is w^*-compact. Further $E(f)$ is convex. The Krein-Milman Theorem [14], implies that $E(f)$ is the closed convex hull of its extreme points. Looking at f as a linear functional on $(C(K, X))^*$, we get $E(f)$ an extremal subset of $S(C(K, X))^*$ [11]. Hence [11], $ext(E(f)) \subseteq ext(S(C(K, X))^*) = \{\delta_t \otimes x^* : x^* \in extS(X^*)\}$. Since f is a multi-smooth point of order n, then there are exactly n-independent elements in $ext(E(f))$, say $\delta_{t_1} \otimes x^*_1, \ldots, \delta_{t_n} \otimes x^*_n$. Let $\delta_{t_1}, \ldots, \delta_{t_k}$, $k \leq n$, be the independent elements of $\{\delta_{t_1}, \ldots, \delta_{t_n}\}$. Consequently, $\{t_1, \ldots, t_k\}$ and $\{x^*_1, \ldots, x^*_n\}$ satisfy (a) and (b). This ends the proof of the theorem.

In case X is the set of real numbers, Theorem 1.4 can be stated as follows.

Theorem 1.5. Let $f \in S(C(K))$. The following are equivalent:

(i) f is a multi-smooth point of order n.
(ii) There exist exactly $\{t_1, \ldots, t_n\}$ in K such that $|f(t_i)| = 1$ for $1 \leq i \leq n$.

Remark 1.6. We should remark that for L^1, the space of Lebesgue integrable functions (equivalence classes) on any finite measure space, if $f \in S(L^1)$ is not smooth, then f cannot be a multi-smooth point of any order. This follows from the fact that if f is not smooth, then $Z(f) = \{t : f(t) = 0\}$ has positive measure. So we can write

$$Z(f) = \bigcup_{i=1}^{\infty} D_i,$$

with $\mu(D_i) > 0$. Then

$$\phi_n(t) = \begin{cases} sgnf(t) & \text{if } t \notin Z(f) \\ 1 & \text{if } t \in D_n \\ 0 & \text{if } t \in Z(f) - D_n \end{cases}$$

is a sequence of independent functionals in $S(L^\infty)$ that attains their norm on f.

3. Smooth Points In Operator Spaces. Let X be a Banach space and $L(X)$ be the space of bounded linear operators on X. Let $K(X)$ be the compact operators in $L(X)$. The trace class operators on a Hilbert space H is denoted by $N(H)$ [13]. In this section, we characterize the multi-smooth points of order n of $S(K(\ell^p))$, $S(L(\ell^p))$ and $S(N(H))$, $1 \leq p < \infty$. Further, we use the idea in [9] to prove that the Calkin Algebra $L(H)/K(H)$ has no multi-smooth points of any order.

We start with the following lemma which is a generalization of the result in [19].

Lemma 2.1. Let $T, T_i \in S(L(X,Y))$, for $i = 1, \dots, k+1$ such that $T = T_1 + \cdots + T_{k+1}$. If $\|T_i \pm T_j\| \leq 1$ for all $i \neq j$, then T cannot be smooth of order r for any $r \leq k$.

Proof. Since $\|T_i\| = 1$, there exists linear functionals $F_i \in S(L(X,Y))^*$, such that $F_i(T_i) = 1$ for $1 \leq i \leq k+1$. For $i \neq j$, we have $|F_i(T_i \pm T_j)| = |1 + F_i(T_j)| \leq \|F_i\| \, \|T_i \pm T_j\| \leq 1$. So $F_i(T_j) = 0$ for all $1 \leq i \neq j \leq k+1$. But this implies that $F_i(T) = 1$ for all $1 \leq i \leq k+1$. Further, the functionals $F_1, \dots, F_{k+1}$ are linearly independent. Indeed, if $\sum a_i F_i = 0$, then evaluating at T_j we get $a_j = 0$. Thus, T cannot be a multi-smooth point of order $r \leq k$.

Now, we prove the following theorem.

Theorem 2.2. Let $T \in S(K(\ell^p))$, $1 < p < \infty$. The following are equivalent:

(i) T is a multi-smooth point of order k.
(ii) T attains its norm at exactly k-linearly independent elements, say $x_1, \dots, x_k$.

Proof. (i) $\to$ (ii). Let T be a multi-smooth point of order k, and $E(T) = \{F \in S(K(\ell^p))^* : F(T) = 1\}$. A usual argument, using the Alaoglu Theorem and the

Krein Milman Theorem, implies that $E(T)$ is the closed (w^*-) convex hull of its extreme points. Being an extremal set [11], $extE(T) \subset extS(K(\ell^p))^*$. But

$$(K(\ell^p))^* = \ell^p \hat{\otimes} \ell^q, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

where $\ell^p \hat{\otimes} \ell^q$ is the completed projective tensor product of ℓ^p with ℓ^q [12]. So, the extreme points of $E(T)$ are of the form $x \otimes y$, with $\|x\| = \|y\| = 1$. Using the assumption, there are only k independent extreme points $x_1 \otimes y_1, \ldots, x_k \otimes y_k$ in $E(T)$. But then, we have $< T, x_i \otimes y_i >=< Tx_i, y_i >= 1$ for $1 \leq i \leq k$. Since $\|T\| = \|x_i\| = \|y_i\| = 1$, and every unit vector in ℓ^p is a smooth point of $S(\ell^p)$, it follows that $< Tx_i, y_i >= \|Tx_i\| = 1$ for $1 \leq i \leq k$. Since the extreme points are independent, it follows that $x_1, \ldots, x_k$ are independent.

(ii) $\rightarrow$ (i). Let T attain its norm at k-independent unit vectors, say $x_1, \ldots, x_k$. So $\|Tx_i\| = 1$ for $1 \leq i \leq k$. Since $Tx_i \in \ell^p$, there exists a unique $y_i \in S(\ell^q)$ such that $< Tx_i, y_i >= \|Tx_i\| = 1$ for $1 \leq i \leq k$. Hence, $x_1 \otimes y_1, \ldots, x_k \otimes y_k$ are k-independent functionals in $S(K(\ell^p))^*$ for which $< T, x_i \otimes y_i >= 1$, for $1 \leq i \leq k$. The set $E(T)$ is extremal. Since $(K(\ell^p))^* = N(\ell^q) = \ell^p \hat{\otimes} \ell^q$, and $ext(\ell^p \hat{\otimes} \ell^q) = \{x \otimes y : \|x\|_p = \|y\|_q = 1\}$, then (i) implies that $E(T)$ cannot have more than k-independent extreme points. Thus, $\{x_1 \otimes y_1, \ldots, x_k \otimes y_k\}$ is a maximal independent set of unit functionals in $(K(\ell^p))^*$ that attain their norm at T, and T is a multi-smooth point of order k.

For a set $A \subset X$, we set $A^{\perp} = \{x^* \in X^* : x^*(a) = 0 \text{ for all } a \in A\}$. Now for $L(\ell^p)$, we have the following theorem.

<u>Theorem 2.3</u>. Let $T \in S(L(\ell^p))$. The following are equivalent:

(i) T is a multi-smooth operator of order k.

ii) $d(T, K(\ell^p)) < 1$, and T attains its norm at exactly k-independent unit vectors in ℓ^p.

<u>Proof</u>. (ii) $\rightarrow$ (i). Let $x_1, \ldots, x_k$ be independent vectors in $S(\ell^p)$ such that $\|Tx_i\| = 1$ for $1 \leq i \leq k$. Hence, there exists a unique $y_i \in S(\ell^q)$ such that $< Tx_i, y_i > = \|Tx_i\| = 1$ for $1 \leq i \leq k$, and $x_1 \otimes y_1, \ldots, x_k \otimes y_k$ are k-independent functionals in $S(K(\ell^p))^*$ for which $< T, x_i \otimes y_i >= 1$, for $1 \leq i \leq k$. We claim that $\{x_1 \otimes y_1, \ldots, x_k \otimes y_k\}$ is a maximal independent set of unit functionals in $(L(\ell^p))^*$ that attain their norm at T. So let $E(T) = \{F \in (L(\ell^p))^* : F(T) = 1\}$. But [4], $(L(\ell^p))^* = (K(\ell^p))^* \oplus (K(\ell^p))^{\perp}$, where the sum is a direct summand, in the sense it is a direct sum and if $F = F_1 + F_2$, then $\|F\| = \|F_1\| + \|F_2\|$. This implies

that extreme points of $E(T)$ are either extreme in $(K(\ell^p))^*$ or extreme points in $(K(\ell^p))^\perp$. If possible let $F \in extE(T) \cap (K(\ell^p))^\perp$. Since $d(T, K(\ell^p)) < 1$, there exists $B \in K(\ell^p)$ such that $\|T - B\| < 1$. Then $F(T) = F(T - B) \leq \|T - B\| < 1$, which contradicts $F(T) = 1$. Consequently, $extE(T) \subset (K(\ell^p))^*$. But now we can use the same type of argument in Theorem 2.2 to conclude that T is a multi-smooth point of order k.

(ii) $\to$ (i). First we show that $d(T, K(\ell^p)) < 1$. If possible let $d(T, K(\ell^p)) \geq 1$. Since $\|T\| = 1$ and $0 \in K(\ell^p)$, it follows that $d(T, K(\ell^p)) = 1$. Define the sequence of contractive finite rank projections

$$P_n = \sum_{i=1}^{n} \delta_i \otimes \delta_i \in L(\ell^p).$$

Then $\|I - P_n\| = 1$, and TP_n is compact for all n. Further, $1 = d(T, K(\ell^p)) \leq \|T - TP_n\| \leq \|T\| \, \|I - P_n\| = 1$, and so $\|T - TP_n\| = 1$ for all n. Since P_n converges to I strongly, we get:

$$1 = \|T - TP_n\| \leq \underline{\lim}_n \|TP_n - TP_m\| \leq \overline{\lim}_n \|TP_n - TP_m\| \leq \overline{\lim}_n \|P_n - P_m\| \leq 1.$$

Hence, $\lim_n \|TP_n - TP_m\| = 1$. Thus, there exists an increasing subsequence of $Q_r = P_{n_{r+1}} - P_{n_r}$ such that $\|TQ_r\| > 1 - \frac{1}{1+r}$ for all $r \in N$. Clearly

$$\sum_{r=0}^{\infty} Q_r x = x$$

for all $x \in \ell^p$. Consider the following $n + 1$ operators

$$J_k x = \sum_{r=0}^{\infty} Q_{rn+(r+k-1)} x.$$

Note that $J_1 + \cdots + J_{n+1} = I$. Set $T_i = TR_i$ for $1 \leq i \leq n + 1$. Then

$$\sum_{i=1}^{n+1} T_i = T.$$

Further, $\|T_i\| = 1$ for $1 \leq i \leq n+1$. Indeed, $\|T_i\| = \|TR_i\| \leq 1$. On the other hand

$$\begin{aligned} \|T_i\| &\geq \|T_i Q_{rn+r+i-1}\| \\ &= \|TR_i Q_{rn+r+i-1}\| \\ &= \|TQ_{rn+r+i-1}\|, \quad (\text{since } R_i Q_{rn+r+i-1} = Q_{rn+r+i-1}) \\ &> 1 - \frac{1}{1+rn+r+i-1} = 1 - \frac{1}{rn+r+i}. \end{aligned}$$

Since this is true for all $r \in N$, we get $\|T_i\| = 1$. Since $\|J_i \pm J_j\| \leq 1$, it follows that $\|T_i \pm T_j\| \leq 1$. Lemma 2.1 now implies that $d(T, K(\ell^p)) < 1$. But now, one can use a similar argument as in (ii) $\to$ (i) of Theorem 2.2 to get the result. This ends the proof.

We turn now to operators on Hilbert spaces.

4. Smooth Points In L(H). For a Hilbert space H, let $L(H)$ be the space of bounded linear operators on H and $N(H)$ be the space of trace class operators [17]. The quotient algebra $J = L(H)/K(H)$ is called the Calkin algebra. In [13], it was shown that the unit ball of the Calkin algebra has no smooth points. In this section we show that the unit ball of the Calkin algebra has no multi-smooth points of finite order. Multi-smooth points of finite order of $S(N(H))$ are also characterized in this section.

Theorem 3.1. Let $T \in S(N(H))$. The following are equivalent:

(i) T is a multi-smooth point of order $n = \binom{2m}{r}$.

(ii) T has a Schmidt representation

$$T = \sum \lambda_i e_i \otimes \theta_i$$

with

$$\sum \lambda_i = 1,$$

where either (e_i) needs r-elements to form an orthonormal basis and (θ_i) needs m-elements to form an orthonormal basis, or visa-versa.

Proof. (i) $\to$ (ii). Let $E(T) = \{F \in S(L(H)) : F(T) = 1\}$. Then $E(T)$ is a w^*-compact set in $L(H)$, and hence, the closed convex hull of its extreme points.

Since the extreme points of $S(L(H))$ are the isometries and the co-isometries [4], then (i) implies that $E(T)$ has $n = \binom{2m}{r}$ independent isometries or co-isometries, say $\{F_1, \ldots, F_n\}$. Now if $F_i(T) = F_j(T)$ for some i and j, then

$$\sum \lambda_k < F_i(e_k), \theta_k >= \sum \lambda_k < F_j(e_k), \theta_k > .$$

Consequently, $F_i(e_k) = F_j(e_k)$ and $F_i^*(\theta_k) = F_j^*(\theta_k)$ for all k. So, if (e_k) and (θ_k) are both complete then $F_i = F_j$, contradicting (i). So both (e_k) and (θ_k) are not complete. Let $[u_1, u_2, \ldots]$ denote the closed linear span of $\{u_1, u_2, \ldots\}$ in H. Consider the orthogonal decomposition of the Hilbert space H into orthogonal subspaces:

$$H = [e_1, e_2, \ldots] \oplus M$$
$$H = [\theta_1, \theta_2, \ldots] \oplus W.$$

If the dimension of M is infinite, then any linear isometric operator A defined on $[e_1, e_2, \ldots]$ has an infinite number of independent extensions. Similarly, assume W is infinite dimensional. Consequently, T is not multi-smooth of finite order. Thus, M and W are of finite dimension, say

$$M = [g_1, \ldots, g_r] \text{ and } W = [h_1, \ldots, h_m].$$

We assume that $r \leq m$ (if $r \geq m$ then we go to the adjoint operator of A). Since $A(g_i) = h_i$, and $A(g_i) = -h_i$ gives two independent extensions of A, $A + g_1 \otimes h_1$ and $A - g_1 \otimes h_1$ are independent. So there are $\binom{2m}{r}$ isometries (or co-isometries) attaining their norm at T, and T is multi-smooth of order $n = \binom{2m}{r}$.

For (ii) $\to$ (i), the proof is clear.

Problem. Characterization of smooth points of $N(\ell^p) = \ell^q \hat{\otimes} \ell^p$, the nuclear operators on ℓ^p, $1 < p \neq 2 < \infty$, is an open problem [7]. This is due to the fact that characterization of extreme points of $S(L(\ell^p))$, $1 < p \neq 2 < \infty$, is still an open problem [8].

Theorem 3.2. The unit ball of $J = L(H)/C_\infty(H)$ has no multi-smooth points of finite order.

Proof. For $T \in L(H)$, we let $\|T\|_e$ denote the norm of T in J. Then there exists an orthonormal sequence (x_n) in H such that $\lim_{n\to\infty} \|Tx_n\| = 1 = \|T\|_e$ [12]. Let A be any infinite subset of $\mathbb{N}$, the set of natural numbers. Let M be the closure of

$span\{x_i : i \in A\}$, and $\hat{M}$ be the closure of $span\{x_i : i \in \mathbb{N}\backslash A\}$. Set P to denote the orthogonal projection onto M, $B = TP$, and $S = T(I - P)$. So $B + S = T$. Using the argument in [13] and the Hahn-Banach Theorem, we get two independent unit functionals in $(L(H))^*$, say f_1 and f_2, such that $f_1(B) = 1$, $f_1(D) = 0$ for all $D \in span\{C_\infty, S\}$ and $f_2(S) = 1$, $f_2(D) = 0$ for all $D \in span\{C_\infty, B\}$. Consequently $f_1(B + S) = f_1(B) = 1$, and $f_2(B + S) = f_2(S) = 1$. This implies that for each infinite subset of $\mathbb{N}$, we have two unit independent functionals attaining their norm at T. Since different infinite subsets of $\mathbb{N}$ give different independent pairs of unit functionals on $L(H)$ attaining their norm at T, it follows that T cannot be a multi-smooth point of finite order. This ends the proof.

5. Smooth Points Related To Extreme Points. There is a very nice relation between multi-smooth points of order n and extreme points. Such a relation is given in the following theorem.

Theorem 4.1. Let X be a finite dimensional Banach space and $x \in S(X)$. If x is a smooth point of order $n = \dim(X)$, then x is an extreme point of the unit ball of X.

Proof. Let x be a smooth point of order n of $S(X)$. Thus, there exist n independent unit functionals on X, say $f_1, \ldots, f_n$ such that $f_i(x) = 1$ for all $1 \le i \le n$. If possible assume x is not an extreme point. So there exists a $y \neq z$ in $S(X)$ such that $x = \frac{1}{2}(y + z)$. It follows that $f_i(y) = f_i(z) = 1$ for all $1 \le i \le n$. Thus, $y - z$ is a non zero element in the kernel of n-independent linear functionals in a space X with dimension n. This cannot be true since $\cap \ker(f_i) = \{0\}$. Thus, x is an extreme point.

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