A GLIMPSE INTO THE WONDERLAND OF INVOLUTIONS

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1. Introduction. The concept of an involution is fundamental in the theory of groups and algebras. Our interest in this paper is to consider involutions in a more humble setting and to illustrate their use in solving some problems of geometry, theory of functional differential equations, and linear algebra. A function $f(x) \not\equiv x$ that maps a set of real numbers onto itself and satisfies on this set the condition

$$f(f(x)) = x \tag{1}$$

or, equivalently,

$$f(x) = f^{-1}(x), \tag{2}$$

is called an *involution*. In other words, an involution is a mapping that coincides with its own inverse. Identity (1) considered as a functional equation is called the *Babbage equation*. It belongs to the oldest functional equations and was investigated first by Charles Babbage. The simplest examples of involutions are: *reflection*

$$f_1(x) = -x, \qquad x \in (-\infty, \infty)$$

and *inversion* (reciprocation)

$$f_2(x) = \frac{1}{x}, \qquad x \in (0, \infty)\,.$$

Obviously,

$$f_3(x) = c - x$$

is an involution on $(-\infty, \infty)$ where c is an arbitrary real. Other examples of involutions may be found in [1] and [2]. Note that the graph of an involution is always symmetric with respect to the line $y = x$ (since the graph of the inverse of an invertible function is obtained by reflection about $y = x$). This simple geometric notion can serve to introduce students to some interesting self-inverse functions and motivate them to think further about their properties.

2. Properties of Involutions. Involutions on the whole number line are sometimes called *strong involutions* [2]. One of the methods for obtaining strong involutions is the following [3]. Assume that a real function $g\,(x, y)$ is defined on the set of all ordered pairs of real numbers and is such that if $g(x,y) = 0$, then $g(y,x) = 0$. In particular, this condition is satisfied if g is symmetric, i.e., $g(x,y) = g(y,x)$. If to each x there corresponds a single real $y = f(x)$ such that $g(x,y) = 0$, then f is a strong involution. For example, if

$$g(x,y) = x + y - c,$$

then from the equation $x + y - c = 0$ we obtain the involution

$$f(x) = c - x.$$

If we take

$$g(x,y) = x^3 + y^3 - c,$$

the equation $g(x,y) = 0$ generates the involution

$$f(x) = \sqrt[3]{c - x^3}.$$

Furthermore, the symmetric function

$$g(x,y) = cxy - a(x+y) - b$$

originates the important class of *bilinear involutions*,

$$f(x) = \frac{ax+b}{cx-a}. \tag{3}$$

Since most formulas for areas and perimeters are symmetric with respect to the dimensions, bilinear involutions are useful in solving "perimeter equals area" problems [1, 4].

Example 1. For a rectangle of dimensions x and y the equality of its perimeter and area is equivalent to the equation

$$2(x+y) = xy$$

whence

$$y = \frac{2x}{x-2},$$

which is a bilinear involution. To find the integer solutions of the problem, we write

$$y = \frac{2x}{x-2} = 2 + \frac{4}{x-2}$$

and since $x-2$ divides 4 we readily obtain solutions $(4,4)$ and $(3,6)$, and (by symmetry) the solution $(6,3)$. Thus, there exist precisely two integer-sided rectangles that have perimeter and area equal.

Example 2. For a right triangle with legs x and y, "perimeter = area" is expressed by the equation

$$x + y + \sqrt{x^2+y^2} = \frac{1}{2}xy,$$

which leads to the involution

$$y = \frac{4x-8}{x-4} = 4 + \frac{8}{x-4}.$$

In the case of integer solutions, $x-4$ divides 8 and checking

$$x - 4 = 2^k, \quad k = 0,1,2,3$$

yields solutions $(5,12)$ and $(6,8)$. By symmetry of the involution's graph about the line $y = x$, we also have solutions $(12,5)$ and $(8,6)$. Thus, $(5,12)$ and $(6,8)$ are the two solutions that describe all integer-sided right triangles having perimeter and area equal.

Involutions have a number of interesting features, the simplest of which are considered below.

Property 1. Every involution $f(x)$ on a set G is one-to-one.

Proof. Indeed, let $x_1, x_2 \in G$ and $x_1 \neq x_2$. Suppose that $f(x_1) = f(x_2)$, then

$$f(f(x_1)) = f(f(x_2)).$$

By virtue of (1), this means $x_1 = x_2$ which contradicts the assumption.

Property 2. If $f(x)$ is an involution on $(-\infty, \infty)$ with a fixed point p, then the function

$$g(x) = f(x+p) - p$$

is an involution with the fixed point 0. Conversely,

$$f(x) = g(x-p) + p$$

is an involution with the fixed point p, provided that 0 is a fixed point of g.

Proof. In fact, let $f(f(x)) = x$ on $(-\infty, \infty)$ and $f(p) = p$. Then for every x, we have

$$\begin{aligned} g(g(x)) &= g(f(x+p) - p) = f[f(x+p) - p + p] - p \\ &= f(f(x+p)) - p = x + p - p = x. \end{aligned}$$

Moreover, $g(0) = f(p) - p = 0$.

Property 3. Let $f(x)$ be a continuous involution on $(-\infty, \infty)$. Then $f(x)$ is decreasing.

Proof. Since $f(x)$ is one-to-one, it is strictly monotone. By the assumption $f(x) \not\equiv x$, there exists x_0 such that $f(x_0) \neq x_0$. Suppose that $f(x)$ is increasing and let $f(x_0) > x_0$; then $f(f(x_0)) > f(x_0)$, that is, $x_0 > f(x_0)$ which is a contradiction. Now assume $f(x_0) < x_0$; then $f(f(x_0)) < f(x_0)$, that is, $x_0 < f(x_0)$ which is a contradiction. Hence, $f(x)$ is decreasing.

Property 4. Let $f(x)$ be a continuous involution on $(-\infty, \infty)$. Then $f(x)$ has a unique fixed point.

Proof. Consider the function $g(x) = f(x) - x$. It is continuous and decreasing on $(-\infty, \infty)$. Furthermore, since

$$\lim_{x \to -\infty} f(x) = +\infty, \quad \lim_{x \to +\infty} f(x) = -\infty,$$

then also

$$\lim_{x \to -\infty} g(x) = +\infty, \quad \lim_{x \to +\infty} g(x) = -\infty,$$

and hence, there exists a unique point p such that $g(p) = 0$. Thus, $f(p) - p = 0$, or $f(p) = p$.

Property 5. If $f(x)$ is a continuous involution on $(-\infty, \infty)$ and $f(x)$ is odd, then $f(x) = -x$.

Proof. Assume $f(x)$ is an odd function. Since $f(-x) = -f(x)$, we have $f(0) = -f(0)$. Hence, $f(0) = 0$ and $x = 0$ is the unique fixed point of f. Suppose that $f(x) \neq -x$. If $f(x) > -x$, then $f(f(x)) < f(-x)$ because f is decreasing.

Therefore, $x < -f(x)$ or $f(x) < -x$, which contradicts the assumption $f(x) > -x$. If $f(x) < -x$, we have $f(f(x)) > f(-x)$. Hence, $x > -f(x)$ or $f(x) > -x$, which is a contradiction. Thus, we conclude that $f(x) = -x$.

Property 6. Every continuous involution $f(x)$ on $(-\infty, \infty)$ with a fixed point p is of the form

$$f(x) = f_0\,(x - p) + p, \tag{4}$$

where

$$f_0\,(x) = \begin{cases} g(x), & \text{for } x \geq 0 \\ g^{-1}(x), & \text{for } x < 0; \end{cases}$$

$g(x)$ is a continuous function on $(-\infty, \infty)$ such that $g(0) = 0$, $g(x) < 0$ for $x > 0$, $g(x)$ is decreasing for $x > 0$, and $g^{-1}(x)$ is the inverse function to $g(x)$. Conversely, every function of the form (4) is a continuous involution $f(x)$ on the real line with the fixed point p.

Proof. If $f(x)$ is a continuous strong involution with the fixed point p, then by virtue of Property 2, $f_0(0) = 0$ and $f_0(f_0(x)) \equiv x$ which implies $f_0^{-1}(x) = f_0(x)$ and the function $f(x)$ can be written in form (4). Conversely, assume that the function $g(x)$ is continuous on $(-\infty, \infty)$, $g(0) = 0$, $g(x) < 0$, for $x > 0$, and $g(x)$ is decreasing for $x > 0$. Since $g(x) \leq 0$ for $x \geq 0$, we have $f_0(f_0(x)) = g^{-1}(g(x)) \equiv x$. For $x < 0$, we put $u = g^{-1}(x)$, then $g(u) = x < 0$, which implies $g^{-1}(x) = u > 0$. Hence, $f_0(f_0(x)) = g(g^{-1}(x)) \equiv x$ for $x < 0$.

These and other properties of involutions may be found in books [2, 5, 6] where they have been used in the theory of functional and functional differential equations.

3. Differentiating Differential Equations. Differential equations with involutions were introduced for the first time in [7] and [8] and since then have become an important part in the general theory of functional differential equations, with applications to certain biomedical models [9], stability of motion [10], and the pantograph equation [11]. They can be transformed into ordinary differential equations and thus provide an abundant source of relations with analytic solutions, as well as heuristic ideas for equations of more general nature.

Example 3. The solution of the initial-value problem for the differential equation with reflection of the argument,

$$y'(x) = ay(-x), \quad y(0) = y_0 \tag{5}$$

can be obtained very easily by a differentiation of (5). In fact,

$$y''(x) = -ay'(-x)$$

and by (5),

$$y'(-x) = ay(x).$$

Consequently,

$$y''(x) = -a^2 y(x) \quad \text{or} \quad y''(x) + a^2 y(x) = 0.$$

This is a second-order ordinary differential equation, with the general solution

$$y(x) = C_1 \cos ax + C_2 \sin ax.$$

Two initial conditions are available to determine the values of C_1 and C_2. From (5), we have

$$y(0) = y_0 \quad \text{and} \quad y'(0) = ay_0.$$

Therefore, $C_1 = y_0$, $C_2 = y_0$, and the solution of problem (5) is

$$y(x) = y_0(\cos ax + \sin ax). \tag{6}$$

It is instructive to draw the students' attention to the sharp distinction between the qualitative behavior of the solutions to (5), which are periodic, and the solutions of the corresponding ordinary differential equation $y'(x) = ay(x)$, which are monotonic.

Example 4. Silberstein [12] studied the equation

$$y'(x) = y\left(\frac{1}{x}\right), \quad 0 < x < \infty \tag{7}$$

and assumed a solution of the form

$$y(x) = x^k + \lambda x^m,$$

where k, m, and λ are constants. Substituting in (7) gives

$$kx^{k-1} + \lambda m x^{m-1} = x^{-k} + \lambda x^{-m}$$

and

$$k + m = 1, \quad \lambda = k, \quad km = 1.$$

Hence, $k^2 - k + 1 = 0$, and simple computations yield the solution. On the other hand, since $f(x) = 1/x$ is an involution on $(0, \infty)$, differentiating (7) leads to the solution in a very elegant fashion. Indeed,

$$y''(x) = -\frac{1}{x^2} y'\left(\frac{1}{x}\right) = -\frac{1}{x^2} y(x),$$

whence,

$$x^2 y''(x) + y(x) = 0.$$

This is a Cauchy-Euler equation, with the general solution

$$y(x) = \sqrt{x}\left[C_1 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right].$$

Substituting $y(x)$ in (7), we obtain $C_1 = \sqrt{3}C_2$ and

$$y(x) = C\sqrt{x}\cos\left(\frac{\sqrt{3}}{2}\ln x - \frac{\pi}{6}\right). \tag{8}$$

The above examples illustrate the following theorem [2].

Theorem 1. Let the equation

$$y'(x) = F(x, y(x), y(f(x))) \tag{9}$$

satisfy the following hypotheses.

(i) The function $f(x)$ is a continuously differentiable strong involution with a fixed point x_0.

(ii) The function F is defined and is continuously differentiable in the whole space of its arguments.

(iii) The given equation is uniquely solvable with respect to $y(f(x))$:

$$y(f(x)) = G(x, y(x), y'(x)). \tag{10}$$

Then the solution of the ordinary differential equation

$$y''(x) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y(x)}y'(x) + \frac{\partial F}{\partial y(f(x))}f'(x)F(f(x), y(f(x)), y(x)) \tag{11}$$

(where $y(f(x))$ is given by expression (10)) with the initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = F(x_0, y_0, y_0) \tag{12}$$

is a solution of (9) with the initial condition

$$y(x_0) = y_0. \tag{13}$$

Proof. Equation (11) is obtained by differentiating (9). Indeed, we have

$$y''(x) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y(x)} y'(x) + \frac{\partial F}{\partial y(f(x))} y'(f(x)) f'(x),$$

but from (9) and the relation $f(f(x)) = x$ it follows that

$$y'(x) = F(f(x), y(f(x)), y(x)).$$

The second of the initial conditions (12) is a compatibility condition and is found from (9), with regard to (13) and $f(x_0) = x_0$.

It is especially clear to see the role of involutions in equations the right-hand sides of which do not contain x and $y(x)$ explicitly. In this case

$$y'(x) = F(y(f(x))). \tag{14}$$

Theorem 2. Assume that in (14) the function $f(x)$ is a continuously differentiable strong involution with a fixed point x_0 and the function $F(x)$ is defined, continuously differentiable, and strictly monotone on $(-\infty, \infty)$. Then the solution of the ordinary differential equation

$$y''(x) = F'(y(f(x))) F(y(x)) f'(x),$$

$$y(f(x)) = F^{-1}(y'(x)),$$

with the initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = F(y_0)$$

is a solution of (14) with the initial condition $y(x_0) = y_0$.

Example 5. By differentiating the equation

$$y'(x) = \frac{1}{y(a-x)} \tag{15}$$

and taking into account that

$$y'(a-x) = \frac{1}{y(x)}\ ,$$

we obtain the ordinary differential equation

$$\frac{d^2y}{dx^2} = \frac{1}{y(x)}\left(\frac{dy}{dx}\right)^2. \tag{16}$$

The fixed point of the involution $f(x) = a - x$ is $x_0 = a/2$. The initial condition for (15) is

$$y\left(\frac{a}{2}\right) = y_0;$$

the corresponding conditions for (16) are

$$y\left(\frac{a}{2}\right) = y_0, \quad y'\left(\frac{a}{2}\right) = \frac{1}{y_0}.$$

Equation (16) is integrable in quadratures.

$$y(x) = y_0 \exp\left(\frac{x - a/2}{y_0^2}\right).$$

This is a solution of the original equation (15).

In conclusion, let us consider the differential equation with reflection

$$y'(x) = ay(x) + by(-x) \tag{17}$$

and denote

$$z(x) = y(-x).$$

Then $z'(x) = -y'(-x)$ and, by virtue of (17), we have

$$-y'(-x) = -ay(-x) - by(x). \tag{18}$$

Combining (17) and (18) produces a linear system of ordinary differential equations

$$\frac{dy}{dx} = ay + bz, \quad \frac{dz}{dx} = -by - az, \tag{19}$$

with the initial condition $y(0) = z(0)$. From the qualitative analysis of the solutions of system (19) the student can derive qualitative information about the solutions of the equation with reflection. Furthermore, the matrix of the system,

$$A = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix},$$

has a remarkable property

$$A^2 = \left(a^2 - b^2\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This observation naturally leads the student to the concept of an *involutory matrix* the square of which is an identity matrix.

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