A NEW METHOD TO OBTAIN PYTHAGOREAN TRIPLE PRESERVING MATRICES

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Abstract. Another method to obtain Pythagorean Triple Preserving Matrices is proposed and a singular case is put in evidence. Also, a possible connection with physics is sketched by proving that the set of these matrices is a group. In the last section, we generalize our method to Weighted Pythagorean Triple Preserving Matrices. An interesting open problem is generated by the fact that this type of matrix appears as a product of two matrices of order 4 with a form suggesting quaternions.

1. Pythagorean Triple Preserving Matrices. In [2] Palmer, Ahuja, and Tikoo obtained all matrices which convert a Pythagorean Triple into another Pythagorean Triple. In this paper we give a second method which uses the matrix equation of a quadric in real 3-dimensional space.

Recall that a Pythagorean Triple (PT) is a triple (a, b, c) of natural numbers such that $a^2 + b^2 = c^2$ and recall that the general expression of a PT is

$$(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$$

where m and n are two integers. So, a PT represents the coordinates of a point $X \in \mathbb{R}^3$ which belongs to the quadric $\Gamma : x^2 + y^2 - z^2 = 0$. The matrix equation of this quadric is $\Gamma : X^t \cdot S \cdot X = 0$ where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Using [2], define a Pythagorean Triple Preserving Matrix (PTPM)

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}.$$

That is, if $X \in \Gamma$ then $A \cdot X \in \Gamma$. Therefore, $(AX)^t \cdot S \cdot (AX) = 0$ which means $X^t \cdot (A^t SA) \cdot X = 0$. In conclusion, A is a PTPM if and only if there exists a real number ρ such that

$$A^t SA = \rho S. \tag{1.1}$$

A straightforward computation leads to the following form of (1.1).

$$\begin{cases} \alpha_1^2 + \beta_1^2 - \gamma_1^2 = \rho \\ \alpha_2^2 + \beta_2^2 - \gamma_2^2 = \rho \\ \alpha_3^2 + \beta_3^2 - \gamma_3^2 = -\rho \\ \alpha_1\alpha_2 + \beta_1\beta_2 - \gamma_1\gamma_2 = 0 \\ \alpha_2\alpha_3 + \beta_2\beta_3 - \gamma_2\gamma_3 = 0 \\ \alpha_3\alpha_1 + \beta_3\beta_1 - \gamma_3\gamma_1 = 0. \end{cases} \tag{1.2}$$

If we make exactly the choice of [2], namely

$$\begin{cases} r^2 = \frac{\alpha_1+\alpha_3+\gamma_1+\gamma_3}{2}, \qquad s^2 = \frac{\alpha_3-\alpha_1+\gamma_3-\gamma_1}{2} \\ t^2 = \frac{\gamma_1+\gamma_3-(\alpha_1+\alpha_3)}{2}, \qquad u^2 = \frac{\gamma_3-\gamma_1-(\alpha_3-\alpha_1)}{2}. \end{cases} \tag{1.3}$$

then, from (1.2_1), (1.2_3) and (1.2_6) it follows that

$$\begin{cases} \beta_1 + \beta_3 = 2rt \\ -\beta_1 + \beta_3 = 2su \end{cases}$$

which gives

$$\begin{cases} \beta_1 = rt - su \\ \beta_3 = rt + su. \end{cases} \tag{1.4}$$

From (1.3) we have, exactly as in [2], that

$$\begin{cases} \alpha_1 = \frac{(r^2-t^2)-(s^2-u^2)}{2}, \qquad \alpha_3 = \frac{(r^2-t^2)+(s^2-u^2)}{2} \\ \gamma_1 = \frac{(r^2+t^2)-(s^2+u^2)}{2}, \qquad \gamma_3 = \frac{(r^2+t^2)+(s^2+u^2)}{2} \end{cases} \tag{1.5}$$

and then, from (1.2_1) it follows that

$$\rho = (ru - st)^2 . \tag{1.6}$$

Equations (1.2_2), (1.2_4) and (1.2_5) yield

$$\begin{cases} \alpha_2 = rs - tu \\ \beta_2 = ru + st \\ \gamma_2 = rs + tu. \end{cases} \tag{1.7}$$

In conclusion, from (1.4), (1.5) and (1.7), it follows that the general form of a PTPM is

$$A(r,s,t,u) = \begin{pmatrix} \frac{1}{2}\left(r^2 - t^2 - s^2 + u^2\right) & rs - tu & \frac{1}{2}\left(r^2 - t^2 + s^2 - u^2\right) \\ rt - su & ru + st & rt + su \\ \frac{1}{2}\left(r^2 + t^2 - s^2 - u^2\right) & rs + tu & \frac{1}{2}\left(r^2 + t^2 + s^2 + u^2\right) \end{pmatrix} \tag{1.8}$$

which is exactly the expression given in [2].

A first advantage of the present method (which is of geometrical nature, like PT) is that it uses only 10 variables, namely $(\alpha_i), (\beta_i), (\gamma_i)$ and ρ, instead of 11 variables $(\alpha_i), (\beta_i), (\gamma_i), M, N$ as in [2]. A second advantage is that given in the singular case $\rho = 0$ for relation (1.1) which we will discuss below. A third advantage is that it offers a very quick proof that the set of PTPM, considered with rational entries, is a group with respect to multiplication (see section 3).

We can obtain the pair $(A(r,s,t,u), \rho)$ from the product of two matrices of order 4. Considering

$$\Phi_1 = \begin{pmatrix} r & -s & -t & u \\ t & -u & r & -s \\ r & -s & t & -u \\ t & -u & -r & s \end{pmatrix} \text{ and } \Phi_2 = \begin{pmatrix} r & s & r & s \\ s & -r & -s & r \\ t & u & t & u \\ u & -t & -u & t \end{pmatrix} \tag{1.9}$$

we obtain

$$\frac{1}{2}\Phi_1\cdot\Phi_2=\frac{1}{2}\begin{pmatrix} r^2-s^2-t^2+u^2 & 2\left(rs-tu\right) & r^2+s^2+t^2+u^2 & 0\\ 2\left(rt-su\right) & 2\left(ru+ts\right) & 2\left(rt+su\right) & 0\\ r^2-s^2+t^2-u^2 & 2\left(rs+tu\right) & r^2+s^2+t^2+u^2 & 0\\ 0 & 0 & 0 & -2\left(ru-st\right)\end{pmatrix}$$

$$=\begin{pmatrix} A\left(r,s,t,u\right) & 0\\ 0 & -\sqrt{\rho}\end{pmatrix} \tag{1.10}$$

and this fact, using the expression of Φ_1 and Φ_2 yields the following problem.

Open problem. Does there exist a connection between PTPM and the algebra of quaternions?

As a possible answer, let us note that the matrix (1.8) is close to the matrix from [4] representing the rotations in $\mathbb{R}^3$.

2. The Singular Case. For relation (1.1) the case $\rho=0$ appears as a singular case. From relation (1.6) we have $ru=st$.

Case I. Suppose that one of r or u is zero. Then one of t and s is zero. We make the choice $r=s=0$ and then it follows that

$$A\left(0,0,t,u\right)=\begin{pmatrix}\frac{u^2-t^2}{2} & -tu & \frac{-u^2-t^2}{2}\\ 0 & 0 & 0\\ \frac{t^2-u^2}{2} & tu & \frac{t^2+u^2}{2}\end{pmatrix}=\frac{t^2}{2}\begin{pmatrix}-1 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 1\end{pmatrix}$$

$$+\frac{u^2}{2}\begin{pmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 1\end{pmatrix}+tu\begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

$$=t^2A\left(0,0,1,0\right)+u^2A\left(0,0,0,1\right)+tu\begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}. \tag{2.1}$$

We have

$$A(0,0,1,0)\begin{pmatrix} m^2-n^2 \\ 2mn \\ m^2+n^2 \end{pmatrix} = \begin{pmatrix} -m^2 \\ 0 \\ m^2 \end{pmatrix} \text{ and } A(0,0,0,1)\begin{pmatrix} m^2-n^2 \\ 2mn \\ m^2+n^2 \end{pmatrix} = \begin{pmatrix} -n^2 \\ 0 \\ n^2 \end{pmatrix}.$$

That is, we obtain the "singular" PT $(-1,0,1)$.

Case II. Suppose that $r,u \neq 0$. Then $s,t \neq 0$. From the relation $s = \frac{ru}{t}$ it follows that

$$A\left(r, \frac{ru}{t}, t, u\right) = \begin{pmatrix} \frac{1}{2}\left(r^2 - t^2 - \frac{r^2u^2}{t^2} + u^2\right) & \frac{r^2u}{t} - tu & \frac{1}{2}\left(r^2 - t^2 + \frac{r^2u^2}{t^2} - u^2\right) \\ rt - \frac{ru^2}{t} & 2ru & rt + \frac{ru^2}{t} \\ \frac{1}{2}\left(r^2 + t^2 - \frac{r^2u^2}{t^2} - u^2\right) & \frac{r^2u}{t} + tu & \frac{1}{2}\left(r^2 + t^2 + \frac{r^2u^2}{t^2} + u^2\right) \end{pmatrix}. \tag{2.2}$$

For example,

$$A(r,ru,1,u) = \begin{pmatrix} \frac{1}{2}\left(r^2 - 1 - r^2u^2 + u^2\right) & u\left(r^2-1\right) & \frac{1}{2}\left(r^2 - 1 + r^2u^2 - u^2\right) \\ r\left(1-u^2\right) & 2ru & r\left(1+u^2\right) \\ \frac{1}{2}\left(r^2 + 1 - r^2u^2 - u^2\right) & u\left(r^2+1\right) & \frac{1}{2}\left(r^2 + 1 + r^2u^2 + u^2\right) \end{pmatrix} \tag{2.3}$$

and then

$$A(r,r,1,1) = \begin{pmatrix} 0 & r^2-1 & r^2-1 \\ 0 & 2r & 2r \\ 0 & r^2+1 & r^2+1 \end{pmatrix} \tag{2.4}$$

which gives

$$A(r,r,1,1)\begin{pmatrix} m^2-n^2 \\ 2mn \\ m^2+n^2 \end{pmatrix} = \begin{pmatrix} \left(r^2-1\right)(m+n)^2 \\ 2r(m+n)^2 \\ \left(r^2+1\right)(m+n)^2 \end{pmatrix}.$$

So,

$$A\,(1,1,1,1)\begin{pmatrix} m^2-n^2 \\ 2mn \\ m^2+n^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2\,(m+n)^2 \\ 2\,(m+n)^2 \end{pmatrix}$$

i.e. the "singular" PT $(0,1,1)$. Comparing the results of this section with a proposition from [2]: "no specific conditions on the nature of r, s, t and u are imposed". In conclusion, relation (1.1) characterizes PTPM yielding "non-singular" PT only for the case $\rho \neq 0$.

3. Connections With Physics. A field of possible applications for the previous results is the $2+1$ Theory of Relativity. Consider $\mathbb{R}^3$ with the Lorentzian metric ([5])

$$<\vec{A},\vec{B}>= a_1b_1 + a_2b_2 - a_3b_3 \tag{3.1}$$

for $\vec{A} = (a_1, a_2, a_3)$, $\vec{B} = (b_1, b_2, b_3) \in \mathbb{R}^3$. In [5], the pair $E^{3,1} = \left(\mathbb{R}^3, <,>\right)$ is called the *Minkowski 3-space*. In this space-time the quadric $\Gamma : x^2 + y^2 - z^2 = 0$ is exactly the *set of null vectors* ([5]). More precisely, Γ is the *null cone* of $E^{3,1}$ because if $\vec{A} \in \Gamma$ then $\lambda\vec{A} \in \Gamma$ for all real λ.

Therefore, a PT represents a point in the null cone, with natural coordinates and then a PTPM is a linear transformation of $E^{3,1}$ which preserves the points of natural coordinates from the null cone of $E^{3,1}$.

Using (1.1) results in the fact that the set of PTPM with rational entries is a group with respect to multiplication. Indeed, A is the unit matrix for $r = u = 1$, $s = t = 0$ and if A_1 and A_2 are PTPM with corresponding ρ_1 and ρ_2, then (1.1) yields

$$(A_1A_2)^t\, S\,(A_1A_2) = A_2^t\left(A_1^tSA_1\right)A_2 = \rho_1 A_2^tSA_2 = \rho_1\rho_2 S$$

which means that A_1A_2 is a PTPM with corresponding $\rho_1\rho_2$. With *MAPLE* it is easy to obtain the relation

$$\begin{aligned} &A\,(r_1,s_1,t_1,u_1)\cdot A\,(r_2,s_2,t_2,u_2) \\ &= A\,(r_1r_2 + t_2s_1, r_1s_2 + u_2s_1, r_2t_1 + t_2u_1, t_1s_2 + u_1u_2) \end{aligned} \tag{3.2}$$

which implies

$$A^2\left(r,s,t,u\right)=A\left(r^2+ts,\left(r+u\right)s,\left(r+u\right)t,ts+u^2\right) \tag{3.3}$$

$$A^{-1}\left(r,s,t,u\right)=A\left(\frac{u}{ru-st},\frac{-s}{ru-st},\frac{-t}{ru-st},\frac{r}{ru-st}\right) \tag{3.4}$$

for $ru\neq st$ (see the previous section). Other properties of $A\left(r,s,t,u\right)$ which are obtained with *MAPLE* are

(i) The trace is

$$TrA=r^2+u^2+ru+st. \tag{3.5}$$

(ii) The eigenvalues are

$$\lambda_1=ru-st \tag{3.6a}$$

$$\lambda_2=\frac{1}{2}\left(r^2+u^2\right)+ts+\frac{1}{2}\sqrt{r^4-2r^2u^2+4r^2st+u^4+4u^2st+8rstu} \tag{3.6b}$$

$$\lambda_3=\frac{1}{2}\left(r^2+u^2\right)+ts-\frac{1}{2}\sqrt{r^4-2r^2u^2+4r^2st+u^4+4u^2st+8rstu}. \tag{3.6c}$$

4. Weighted Pythagorean Triple Preserving Matrices. A Weighted Pythagorean Triple (WPT) is a triple (x,y,z) of natural numbers such that

$$p^2x^2+q^2y^2=p^2q^2z^2 \tag{4.1}$$

where p and q are two natural numbers. So, a WPT represents the coordinates of a point $X\in\mathbb{R}^3$ which belongs to the quadric $\Gamma: p^2x^2+q^2y^2-p^2q^2z^2=0$. The matrix equation of this quadric is $\Gamma: X^t\cdot S(p,q)\cdot X=0$ where

$$X=\begin{pmatrix}x\\y\\z\end{pmatrix} \text{ and } S(p,q)=\begin{pmatrix}p^2&0&0\\0&q^2&0\\0&0&-p^2q^2\end{pmatrix}.$$

In this section we find the general form of a Weighted Pythagorean Triple Preserving Matrix (WPTPM) A, i.e., if $X\in\Gamma$ then $A\cdot X\in\Gamma$. Using the same argument as in the first section results in the fact that A is a WPTPM if and only if there exists a real number ρ such that

$$A^t \cdot S(p,q) \cdot A = \rho S\left(p,q\right). \tag{4.2}$$

A straightforward computation leads to the following form of (4.2).

$$\begin{cases} p^2\alpha_1^2 + q^2\beta_1^2 - p^2q^2\gamma_1^2 = \rho p^2 \\ p^2\alpha_2^2 + q^2\beta_2^2 - p^2q^2\gamma_2^2 = \rho q^2 \\ p^2\alpha_3^2 + q^2\beta_3^2 - p^2q^2\gamma_3^2 = -\rho p^2 q^2 \\ p^2\alpha_1\alpha_2 + q^2\beta_1\beta_2 - p^2q^2\gamma_1\gamma_2 = 0 \\ p^2\alpha_2\alpha_3 + q^2\beta_2\beta_3 - p^2q^2\gamma_2\gamma_3 = 0 \\ p^2\alpha_3\alpha_1 + q^2\beta_3\beta_1 - p^2q^2\gamma_3\gamma_1 = 0. \end{cases} \tag{4.3}$$

With the choice

$$\begin{cases} r^2 = \frac{1}{2q^2}\left[pq\left(\gamma_3 + q\gamma_1\right) + p\left(\alpha_3 + q\alpha_1\right)\right], s^2 = \frac{1}{2q^2}\left[pq\left(\gamma_3 - q\gamma_1\right) + p\left(\alpha_3 - q\alpha_1\right)\right] \\ t^2 = \frac{1}{2q^2}\left[pq\left(\gamma_3 + q\gamma_1\right) - p\left(\alpha_3 + q\alpha_1\right)\right], u^2 = \frac{1}{2q^2}\left[pq\left(\gamma_3 - q\gamma_1\right) - p\left(\alpha_3 - q\alpha_1\right)\right] \end{cases} \tag{4.4}$$

it follows that the solution

$$A\left(r,s,t,u\right) = \begin{pmatrix} \frac{q}{2p}\left(r^2 - t^2 - s^2 + u^2\right) & \frac{q^2}{p^2}\left(rs - tu\right) & \frac{q^2}{2p}\left(r^2 - t^2 + s^2 - u^2\right) \\ rt - su & \frac{q}{p}\left(ru + st\right) & q\left(rt + su\right) \\ \frac{1}{2p}\left(r^2 + t^2 - s^2 - u^2\right) & \frac{q}{p^2}\left(rs + tu\right) & \frac{q}{2p}\left(r^2 + t^2 + s^2 + u^2\right) \end{pmatrix}. \tag{4.5}$$

Also,

$$\rho = \frac{q^2}{p^2}\left(ru - st\right)^2. \tag{4.6}$$

Returning to (4.1) with $x = qa$ and $y = pb$ results in $a^2 + b^2 = z^2$, i.e. (a, b, z) is a PT and therefore, we have the general form of a WPT.

$$(x, y, z) = \left(q\left(m^2 - n^2\right), 2pmn, m^2 + n^2\right). \tag{4.7}$$

Finally, consider the system

$$A\left(r,s,t,u\right)\cdot\begin{pmatrix} q\left(m^2-n^2\right) \\ 2pmn \\ m^2+n^2 \end{pmatrix} = \begin{pmatrix} q\left(M^2-N^2\right) \\ 2pMN \\ M^2+N^2 \end{pmatrix} \tag{4.8}$$

with solution

$$M^2 = \frac{q}{p}\left(mr+ns\right)^2, \quad N^2 = \frac{q}{p}\left(mt+ns\right)^2. \tag{4.9}$$

This yields the following proposition.

Proposition. The results of this section are true only for the case

$$q = p\cdot\alpha^2 \tag{4.10}$$

with α a natural number. Then (4.1) becomes

$$x^2+\alpha^4y^2 = p^2\alpha^4z^2. \tag{4.11}$$

Obviously, for $p=q=1$ we reobtain the results of the first section.

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