ON THE RATE OF CONVERGENCE FOR THE CHEBYSHEV SERIES

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Abstract. Let $f(x)$ be a function of bounded variation on $[-1,1]$ and $S_n(f;x)$ the nth partial sum of the expansion of $f(x)$ in a Chebyshev series of the second kind. In this note we give the estimate for the rate of convergence of the sequence $S_n(f;x)$ to $f(x)$ in terms of the modulus of continuity of the total variation of $f(x)$.

1. Introduction. Let $U_n(x)$ be the Chebyshev polynomial of the second kind [4]. Let $f(x)$ be a function of bounded variation on $[-1,1]$ and $S_n(f;x)$ the nth partial sum of the expansion of $f(x)$ in a Chebyshev series of the second kind:

$$\sum_{n=0}^{\infty} a_n U_n(x)$$

with

$$a_n = \frac{2}{\pi}\int_{-1}^{1}(1-y^2)^{1/2} f(y)\frac{\sin(n+1)\arccos y}{\sin \arccos y}dy, \quad (n=0,1,\dots). \tag{1.1}$$

According to the equiconvergence theorem for Jacobi series [4], we know that

$$\lim_{n\to\infty} S_n(f;x) = \frac{1}{2}\big(f(x+0)+f(x-0)\big), \quad x\in(-1,1).$$

In this note we shall find an estimate for the rate of convergence of the sequence $S_n(f;x)$ to $f(x)$. Results of this type for Fourier series of 2π-periodic functions of bounded variation were proved by Bojanic [2].

2. Preliminary Results. Before proving the main theorem we shall state a preliminary result. Al-Khaled [1] has studied the behavior of Chebyshev series for functions of bounded variation on $[-1,1]$ and he proved the following Theorem.

Theorem 2.1. If $f(x)$ is a function of bounded variation on $[-1,1]$. Let

$$A_x(y) = \begin{cases} f(y) - f(x-0), & -1 \leq y < x \\ 0, & y = x \\ f(y) - f(x+0), & x < y \leq 1. \end{cases}$$

Then for every $x \in (-1,1)$ and $n \geq 2$ we have

$$\left|S_n(f;x) - \frac{1}{2}\big(f(x+0) + f(x-0)\big)\right| \leq \frac{9}{n\sqrt{1-x^2}}\left[\frac{1}{1+x}\sum_{k=1}^{n} V_{x-(1+x)/k}^{x}(A_x)\right.$$

$$\left. + \frac{1}{1-x}\sum_{k=1}^{n} V_{x}^{x+(1-x)/k}(A_x)\right] + \frac{4}{n\pi\sqrt{1-x^2}}\left|f(x+0) - f(x-0)\right| \tag{2.1}$$

where $V_a^b A_x$ is the total variation of A_x on $[a,b]$. Since $A_x(y)$ is continuous at $y = x$, the right-hand side of (2.1) converges to zero. For Theorem 2.1, we can make a rough estimate.

Corollary 2.2. Under the assumption of Theorem 2.1, we have

$$\left|S_n(f;x) - \frac{1}{2}\big(f(x+0) + f(x-0)\big)\right| \leq \frac{18}{n(1-x^2)^{3/2}}\sum_{k=1}^{n} V_{x-(1+x)/k}^{x+(1-x)/k}(A_x)$$

$$+ \frac{4}{n\pi\sqrt{1-x^2}}\left|f(x+0) - f(x-0)\right|, \quad x \in (-1,1), \quad n \geq 2. \tag{2.2}$$

Proof. For the quantities in equation (2.1), we note that

$$\frac{1}{1+x}\sum_{k=1}^{n} V_{x-(1+x)/k}^{x}(A_x) = \frac{1-x}{(1-x^2)}\sum_{k=1}^{n} V_{x-(1+x)/k}^{x}(A_x)$$

$$\leq \frac{2}{1-x^2}\sum_{k=1}^{n} V_{x-(1+x)/k}^{x}(A_x).$$

Similarly,

$$\frac{1}{1-x}\sum_{k=1}^{n} V_x^{x+(1-x)/k}(A_x) = \frac{1+x}{(1-x^2)}\sum_{k=1}^{n} V_x^{x+(1-x)/k}(A_x)$$

$$\leq \frac{2}{1-x^2}\sum_{k=1}^{n} V_x^{x+(1-x)/k}(A_x).$$

Combining the above two inequalities we get the required result.

Example 2.1. For a fixed $x \in (-1,1)$, consider a function g of bounded variation on $[-1,1]$, i.e.,

$$g(y) = |y-x|, \qquad y \in (-1,1).$$

Now we have $g(x+0) = g(x-0) = g(x) = 0$ and $A_x(y) = g(y)$, furthermore,

$$V_{x-(1+x)/k}^{x}(A_x) = \frac{1+x}{k}, \qquad V_x^{x+(1-x)/k}(A_x) = \frac{1-x}{k},$$

so, equation (2.1) becomes

$$\left|S_n(g;x)\right| \leq \frac{9}{n\sqrt{1-x^2}}\left(\frac{1}{1+x}\sum_{k=1}^{n}\frac{1+x}{k} + \frac{1}{1-x}\sum_{k=1}^{n}\frac{1-x}{k}\right) = \frac{18}{n\sqrt{1-x^2}}\sum_{k=1}^{n}\frac{1}{k}.$$

But,

$$\sum_{k=1}^{n}\frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \ln n + \gamma + O(1).$$

Therefore, by Theorem 2.1 we get an estimate

$$S_n(g;x) - g(x) = O\left(\frac{\ln n}{n\sqrt{1-x^2}}\right). \tag{2.3}$$

Hereafter, the bounds of the terms "O" are independent of n and x. If we apply the above corollary from $V_{x-(1+x)/k}^{x+(1-x)/k}(A_x) = 2/k$, we shall obtain another estimate

$$S_n(g;x) - g(x) = O\left(\frac{\ln n}{n(1-x^2)\sqrt{1-x^2}}\right). \tag{2.4}$$

Comparing (2.3) with (2.4), we see that when $|x| \to 1$, the estimate (2.3) is more exact than (2.4).

3. The Main Result. Now we state and prove our main result.

Theorem 3.1. If $f(x)$ is a continuous function of bounded variation on $[-1,1]$ and $\omega_{v(f)}(\delta)$ is the modulus of continuity of the total variation $V_{-1}^{t}(f)$, then for $x \in (-1,1)$, $n \geq 2$ we have

$$\begin{aligned} \big|S_n(f;x) - f(x)\big| \leq \frac{9}{n\sqrt{1-x^2}}\left\{\frac{1}{1+x}\omega_{v(f)}(1+x) + \frac{1}{1-x}\omega_{v(f)}(1-x)\right\} \\ + \frac{9}{n\sqrt{1-x^2}}\int_{1/n}^{1}\left\{\frac{\omega_{v(f)}\big((1-x)u\big)}{1-x} + \frac{\omega_{v(f)}\big((1+x)u\big)}{1+x}\right\}\frac{du}{u^2}, \end{aligned} \tag{3.1}$$

especially, when $V_{-1}^{t}(f)$ belongs to the class Lip α $\big(\alpha \in (0,1)\big)$,

$$S_n(f;x) - f(x) = O\left(\frac{1}{n^{\alpha}(1-x^2)^{3/2-\alpha}}\right). \tag{3.2}$$

Further, for the Cesaro mean (c,λ), $\lambda \in (0,1)$:

$$\sigma_n^{\lambda}(f;x) = \frac{1}{(\lambda)_n}\sum_{k=0}^{n}(\lambda-1)_{n-k}S_k(f;x) \tag{3.3}$$

where in general

$$(\beta)_n = \frac{\Gamma(\beta+n+1)}{\Gamma(\beta+1)\Gamma(n+1)}.$$

We have also

$$\sigma_n^\lambda(f;x) - f(x) = O\left(\frac{1}{n^\gamma(1-x^2)^{3/2-\alpha}}\right), \tag{3.4}$$

where $\gamma = \min\{\alpha, 1-\lambda\}$.

Proof. Since $f(x)$ is a continuous function, we have $A_x(y) = f(y) - f(x)$, so

$$V_x^{x+(1-x)/k}(A_x) = V_x^{x+(1-x)/k}(f) - V_x^x(f) \le \omega_{v(f)}\left(\frac{1-x}{k}\right), \quad 2 \le k \le n$$

and

$$V_{x-(1+x)/k}^x(A_x) \le \omega_{v(f)}\left(\frac{1+x}{k}\right), \quad 2 \le k \le n$$

and

$$V_{-1}^x(A_x) \le \omega_{v(f)}(1+x), \quad V_x^1(A_x) \le \omega_{v(f)}(1-x).$$

Thus,applying Theorem 2.1, then for $x \in (-1,1)$, $n \ge 2$ we obtain

$$\big|S_n(f;x) - f(x)\big| \le \frac{9}{n\sqrt{1-x^2}}\left\{\frac{1}{1+x}\omega_{v(f)}(1+x) + \frac{1}{1-x}\omega_{v(f)}(1-x)\right\}$$

$$+ \frac{9}{n\sqrt{1-x^2}}\left\{\frac{1}{1-x}\sum_{k=2}^{n}\omega_{v(f)}\left(\frac{1-x}{k}\right) + \frac{1}{1+x}\sum_{k=2}^{n}\omega_{v(f)}\left(\frac{1+x}{k}\right)\right\}.$$

From this and noting that

$$\sum_{k=2}^{n}\omega_{v(f)}\left(\frac{1-x}{k}\right) \le \int_{1/n}^{1}\omega_{v(f)}\big((1-x)u\big)u^{-2}du$$

and

$$\sum_{k=2}^{n}\omega_{v(f)}\left(\frac{1+x}{k}\right) \le \int_{1/n}^{1}\omega_{v(f)}\big((1+x)u\big)u^{-2}du$$

we have formula (3.1).

When $V_{-1}^{t}(f) \in \text{Lip}\ \alpha$ $(0 < \alpha < 1)$, we have

$$\omega_{v(f)}\big((1-x)u\big) = O\big((1-x)^{\alpha}u^{\alpha}\big) \quad \text{and} \quad \omega_{v(f)}\big((1+x)u\big) = O\big((1+x)^{\alpha}u^{\alpha}\big).$$

From (3.1), we get (3.2). Now by (3.3) and $(\beta)_n = O(n^{\beta})$, we know for $x \in (-1,1)$, $n \geq 2$ that

$$\begin{aligned}\sigma_n^{\lambda}(f;x) - f(x) &= \frac{1}{(\lambda)_n}\sum_{k=0}^{n}(\lambda-1)_{n-k}\big(S_k(f;x)-f(x)\big)\\ &= \frac{1}{(\lambda)_n}\sum_{k=2}^{n-1}(\lambda-1)_{n-k}\big(S_k(f;x)-f(x)\big)\\ &\quad + O(1/n) + O\left(\frac{1}{n^{\alpha+\lambda}(1-x^2)^{3/2-\alpha}}\right).\end{aligned} \tag{3.5}$$

According to formula (3.2), we get

$$\frac{1}{(\lambda)_n}\sum_{k=2}^{n-1}(\lambda-1)_{n-k}\big(S_k(f;x)-f(x)\big) = O\left(\frac{1}{n^{\lambda}(1-x^2)^{3/2-\alpha}}\right)\sum_{k=2}^{n-1}\frac{1}{k^{\alpha}(n-k)^{1-\lambda}}. \tag{3.6}$$

Let $\gamma = \min\{\alpha, 1-\lambda\}$. By the inequality

$$(a+b)^{\gamma} \leq 2^{\gamma}(a^{\gamma}+b^{\gamma}), a>0,\ b>0,$$

we see that

$$\begin{aligned}\sum_{k=2}^{n-1}\frac{1}{k^{\alpha}(n-k)^{1-\lambda}} &\leq \sum_{k=2}^{n-1}\frac{1}{k^{\gamma}(n-k)^{\gamma}}\\ &= \sum_{k=2}^{n-1}\frac{1}{n^{\gamma}}\left(\frac{1}{k}+\frac{1}{n-k}\right)^{\gamma}\\ &= O\left(\frac{1}{n^{\gamma}}\right)\left\{\sum_{k=2}^{n-1}\frac{1}{k^{\gamma}}+\sum_{k=2}^{n-1}\frac{1}{(n-k)^{\gamma}}\right\}\\ &= O\left(\frac{1}{n^{2\gamma-1}}\right).\end{aligned}$$

From this and (3.5), (3.6), we obtain the formula (3.4). This completes the proof of Theorem 3.1.

A result of the type of equality in Theorem 3.1 for 2π-periodic continuous function of bounded variation on $[-\pi,\pi]$ was proved by Natanson [3].

References

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