DIRECTED GRAPHS, MAGIC SQUARES, AND GROTHENDIECK TOPOLOGIES

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1. Introduction. It is well-known that a graph can be represented by a square matrix by considering its adjacency matrix. One of the goals of this paper is to give an algebraic description of such a correspondence for directed graphs.

A directed graph can be viewed as an order pair (α, β) of mappings from the set of directed edges to the set of vertices in such a way that a directed edge e is the one with the initial vertex $\alpha(e)$ and the terminal vertex $\beta(e)$ [4]. Thus, with an appropriate morphism, we can consider the category of directed graphs whose objects are ordered pairs of mappings of finite sets. More precisely, we consider the category of directed graphs with a fixed set X of vertices whose objects are viewed as the set of ordered pairs (α, β) of mappings $\alpha, \beta: Y \to X$ from various finite sets Y to the given set X. We extend the set of isomorphism classes of the objects in this category to a set which has a ring structure and prove that the resulting ring is isomorphic to the ring of $m \times m$ integral matrices, where m is the number of elements in X (Theorem 1). We also consider a subring of this ring corresponding to regular digraphs and show that it is isomorphic to the ring of generalized magic squares (Theorem 2).

Given a finite set X, the category of single mappings $\phi: Z \to X$ can be regarded as a Grothendieck topology on the category of finite sets [2, 3]. In Section 7, we discuss the action of the ring associated to directed graphs above on the objects of this category.

2. Directed Graphs. A directed graph $G = (V, E)$ consists of a finite set V of vertices and a finite set E of edges, where each edge is an ordered pair of vertices. Let $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_n\}$. If $e_l = (v_i, v_j)$, then v_i is called the initial vertex and v_j is called the terminal vertex of e_l.

To each directed graph $G = (V, E)$, we can associate an ordered pair (α, β) of mappings $\alpha, \beta: E \to V$ such that, for each edge $e \in E$, $\alpha(e)$ is the initial vertex and $\beta(e)$ is the terminal vertex of e [4]. Conversely, to each ordered pair (f, g) of mappings $f, g: Y \to X$ of finite sets, we can associate a directed graph such that X is the set of vertices, Y is the set of edges, and each $y \in Y$ has the initial vertex

$f(y)$ and the terminal vertex $g(y)$. This correspondence leads us to the following definition of the category of directed graphs.

Definition 1. The category $\mathcal{D}G$ of directed graphs is defined by the following:

(i) The objects of $\mathcal{DG}$ are ordered pairs (f, g) of mappings $f, g: Y \to X$ of finite sets.

(ii) A morphism from the pair (f_1, g_1) of mappings $f_1, g_1: Y_1 \to X_1$ to the pair (f_2, g_2) of mappings $f_2, g_2: Y_2 \to X_2$ is a pair (μ, ν) of mappings $\mu: Y_1 \to Y_2$ and $\nu: X_1 \to X_2$ such that $f_2 \circ \mu = \nu \circ f_1$ and $g_2 \circ \mu = \nu \circ g_1$.

Remark 1. It can be easily shown that the pair (μ, ν) in (ii) of the above definition is indeed a morphism in a category with $(\mathrm{id}, \mathrm{id})$ the identity morphism.

Definition 2. Let X be a fixed finite set. The category $\mathcal{DG}_X$ of directed graphs with vertices X is the subcategory of $\mathcal{DG}$ defined as follows:

(i) The objects of $\mathcal{DG}_X$ are triples (Y, f, g), where Y is a finite set and $f, g: Y \to X$ are mappings of Y into X.

(ii) A morphism from the triple (Y_1, f_1, g_1) to the triple (Y_2, f_2, g_2) is a mapping $\varphi: Y_1 \to Y_2$ such that $f_1 = f_2 \circ \varphi$ and $g_1 = g_2 \circ \varphi$.

3. The Multiplication Operation. We define an operation $\odot$ on the set of objects of the category $\mathcal{DG}_X$ as follows. Given two objects (Y_1, f_1, g_1) and (Y_2, f_2, g_2) of $\mathcal{DG}_X$, let U be the pullback of $g_1: Y_1 \to X$ and $f_2: Y_2 \to X$ and let $p: U \to Y_1$ and $q: U \to Y_2$ be the canonical projection mappings. Thus, we have $U = \{(y_1, y_2) \in Y_1 \times Y_2 \mid g_1(y_1) = f_2(y_2)\}$ and $g_1 \circ p = f_2 \circ q$. We set $(Y_1, f_1, g_1) \odot (Y_2, f_2, g_2) = (U, f_1 \circ p, g_2 \circ q)$. Then the triple $(U, f_1 \circ p, g_2 \circ q)$ is an object of $\mathcal{DG}_X$ and therefore $\odot$ is an operation on $\mathcal{DG}_X$.

Remark 2. In graph-theoretic terms, the product of two graphs $G_1 = (X, E_1)$ and $G_2 = (X, E_2)$ with vertices X and sets of edges E_1 and E_2, respectively, is the graph $G = (X, E)$ with vertices X and edges $E = E_1 \circ E_2$, where "$\circ$" denotes the composition of relations.

Proposition 1. Suppose that (Y_1, f_1, g_1), (Y_2, f_2, g_2) are objects of $\mathcal{DG}_X$ that are isomorphic to the objects (Y_1', f_1', g_1'), (Y_2', f_2', g_2') of $\mathcal{DG}_X$, respectively. Then there is a canonical isomorphism between $(Y_1, f_1, g_1) \odot (Y_2, f_2, g_2)$ and $(Y_1', f_1', g_1') \odot (Y_2', f_2', g_2')$.

Proof. Let $\phi_1: Y_1 \to Y_1'$ and $\phi_2: Y_2 \to Y_2'$ be bijections such that $f_1' \circ \phi_1 = f_1$, $g_1' \circ \phi_1 = g_1$, $f_2' \circ \phi_2 = f_2$, and $g_2' \circ \phi_2 = g_2$. Let $(U, f_1 \circ p, g_2 \circ q) = (Y_1, f_1, g_1) \odot (Y_2, f_2, g_2)$ and $(U', f_1' \circ p', g_2' \circ q') = (Y_1', f_1', g_1') \odot (Y_2', f_2', g_2')$, where $p: U \to Y_1$, $q: U \to Y_2$, $p': U' \to Y_1'$, and $q': U' \to Y_2'$ are projections, and

$$U = \{(y_1, y_2) \in Y_1 \times Y_2 \mid g_1 y_1 = f_2 y_2\}, \quad U' = \{(y_1', y_2') \in Y_1' \times Y_2' \mid g_1' y_1' = f_2' y_2'\}.$$

We define $\phi: U \to U'$ by $\phi(y_1, y_2) = (\phi_1(y_1), \phi_2(y_2))$. If $(y_1, y_2) \in U$, then

$$(f_1' \circ p')\big(\phi(y_1', y_2')\big) = f_1' \circ p'(\phi_1(y_1), \phi_2(y_2)) = f_1' \circ \phi_1(y_1) = f_1(y_1) = f_1 \circ p(y_1, y_2)$$

and

$$(g_2' \circ q')\big(\phi(y_1', y_2')\big) = g_2' \circ q'(\phi_1(y_1), \phi_2(y_2)) = g_2' \circ \phi_2(y_2) = g_2(y_2) = g_2 \circ q(y_1, y_2).$$

Thus, we have $(f_1' \circ p') \circ \phi = f_1 \circ p$ and $(g_2' \circ q') \circ \phi = g_2 \circ q$; hence the isomorphism follows.

Proposition 2. There is a canonical isomorphism

$$\big((Y_1, f_1, g_1) \odot (Y_2, f_2, g_2)\big) \odot (Y_3, f_3, g_3) \cong (Y_1, f_1, g_1) \odot \big((Y_2, f_2, g_2) \odot (Y_3, f_3, g_3)\big)$$

for $(Y_i, f_i, g_i) \in \mathcal{DG}_X$, $i = 1, 2, 3$.

Proof. Let $(U, p, g) = (Y_1, f_1, g_1) \odot (Y_2, f_2, g_2)$ and $(V, p', q') = (Y_2, f_2, g_2) \odot (Y_3, f_3, g_3)$. Thus,

$$U = \{(y_1, y_2) \in Y_1 \times Y_2 \mid g_1 y_1 = f_2 y_2\}, \quad V = \{(y_2, y_3) \in Y_2 \times Y_3 \mid g_2 y_2 = f_3 y_3\}$$

and $p: U \to Y_1$, $q: U \to Y_2$, $p': V \to Y_2$, $q': V \to Y_3$ are the natural projections. We have

$$\begin{aligned}
(U, p, q) \odot (Y_3, f_3, g_3) &= \{\big((y_1, y_2), y_3\big) \in U \times Y_3 \mid g_2 \circ q(y_1, y_2) = f_3 y_3\} \\
&= \{\big((y_1, y_2), y_3\big) \in U \times Y_3 \mid g_2 y_2 = f_3 y_3\} \\
&= \{\big((y_1, y_2), y_3\big) \in (Y_1 \times Y_2) \times Y_3 \mid g_1 y_1 = f_2 y_2,\ g_2 y_2 = f_3 y_3\}.
\end{aligned}$$

Similarly, $(Y_1, f_1, g_1) \odot (V, p', q')$ is equal to

$$\{\big(y_1, (y_2, y_3)\big) \in Y_1 \times (Y_2 \times Y_3) \mid g_1 y_1 = f_2 y_2,\ g_2 y_2 = f_3 y_3\}.$$

Thus, the mapping $((y_1, y_2), y_3) \mapsto (y_1, (y_2, y_3))$ determines a canonical isomorphism.

We denote by M_X the set of isomorphism classes $M_X = \mathrm{Obj}(\mathcal{DG}_X)/\cong$ of objects of $\mathcal{DG}_X$. By Proposition 1, the operation $\odot$ on $\mathcal{DG}_X$ induces an operation on M_X which will also be denoted by $\odot$.

Proposition 3. If 1_X is the identity mapping on X, then the set M_X together with the operation $\odot$ is a monoid with $(X, 1_X, 1_X)$ the identity element.

Proof. Obviously, $(X, 1_X, 1_X)$ is an identity with respect to the operation $\odot$. The operation $\odot$ is associative by Proposition 2; hence the proposition follows.

4. The Sum Operation. Given two directed graphs (Y_1, f_1, g_1) and (Y_2, f_2, g_2) in $\mathcal{DG}_X$, we define $Y_1 \amalg Y_2$ to be the disjoint union of Y_1 and Y_2 with inclusions $i_1: Y_1 \to Y_1 \amalg Y_2$ and $i_2: Y_2 \to Y_1 \amalg Y_2$. We denote by $f_1 \amalg f_2, g_1 \amalg g_2: Y_1 \amalg Y_2 \to X$ the mappings that satisfy

$$(f_1 \amalg f_2) \circ i_1 = f_1, \quad (f_1 \amalg f_2) \circ i_2 = f_2, \quad (g_1 \amalg g_2) \circ i_1 = g_1, \quad (g_1 \amalg g_2) \circ i_2 = g_2.$$

Then, we define the operations $\oplus$ on $\mathcal{DG}_X$ by $(Y_1, f_1, g_1) \oplus (Y_2, f_2, g_2) = (Y_1 \amalg Y_2, f_1 \amalg f_2, g_1 \amalg g_2)$.

Remark 3. In graph-theoretic terms, the sum $G_1 \oplus G_2$ of the graphs $G_1 = (X, E_1)$ and $G_2 = (X, E_2)$ is (X, E), where E is just the disjoint union of the sets of edges E_1 and E_2.

Lemma 1. Suppose that (Y_1, f_1, g_1) and (Y_2, f_2, g_2) are objects of $\mathcal{DG}_X$ that are isomorphic to the objects (Y_1', f_1', g_1') and (Y_2', f_2', g_2') of $\mathcal{DG}_X$, respectively. Then there is a canonical isomorphism $(Y_1, f_1, g_1) \oplus (Y_2, f_2, g_2) \cong (Y_1', f_1', g_1') \oplus (Y_2', f_2', g_2')$.

Proof. Let $\phi_1: Y_1 \to Y_1'$ and $\phi_2: Y_2 \to Y_2'$ be bijections such that $f_1' \circ \phi_1 = f_1$, $g_1' \circ \phi_1 = g_1$, $f_2' \circ \phi_2 = f_2$, and $g_2' \circ \phi_2 = g_2$. Define $\phi: Y_1 \amalg Y_2 \to Y_1' \amalg Y_2'$ to be the mapping satisfying $\phi(y_1) = \phi_1(y_1)$ if $y_1 \in Y_1$ and $\phi(y_2) = \phi_2(y_2)$ if $y_2 \in Y_2$. It can be easily shown that ϕ induces an isomorphism between $(Y_1 \amalg Y_2, f_1 \amalg f_2, g_1 \amalg g_2)$ and $(Y_1' \amalg Y_2', f_1' \amalg f_2', g_1' \amalg g_2')$.

Lemma 2. Let (Y_1, f_1, g_1), (Y_2, f_2, g_2), and (Y_3, f_3, g_3) be objects of $\mathcal{DG}_X$ that are isomorphic to the objects (Y_1', f_1', g_1'), (Y_2', f_2', g_2'), and (Y_3', f_3', g_3'), respectively.

Then there are canonical isomorphisms $(Y_1, f_1, g_1) \oplus (Y_2, f_2, g_2) \cong (Y_2', f_2', g_2') \oplus (Y_1', f_1', g_1')$ and

$$\big((Y_1, f_1, g_1) \oplus (Y_2, f_2, g_2)\big) \oplus (Y_3, f_3, g_3) \cong (Y_1', f_1', g_1') \oplus \big((Y_2', f_2', g_2') \oplus (Y_3', f_3', g_3')\big).$$

<u>Proof</u>. The proof is straightforward and will be omitted.

<u>Proposition 4</u>. There are canonical isomorphisms

$$\begin{aligned} &(Y_1, f_1, g_1) \odot \big((Y_2, f_2, g_2) \oplus (Y_3, f_3, g_3)\big) \\ &\qquad \cong \big((Y_1, f_1, g_1) \odot (Y_2, f_2, g_2)\big) \oplus \big((Y_1, f_1, g_1) \odot (Y_3, f_3, g_3)\big), \\ &\big((Y_1, f_1, g_1) \odot (Y_2, f_2, g_2)\big) \oplus (Y_3, f_3, g_3) \\ &\qquad \cong \big((Y_1, f_1, g_1) \odot (Y_3, f_3, g_3)\big) \oplus \big((Y_2, f_2, g_2) \odot (Y_3, f_3, g_3)\big). \end{aligned}$$

<u>Proof</u>. We shall prove the first isomorphism. The second one can be proved similarly. Let (U, α, β) and (V, γ, δ) be the left and the right hand sides of the isomorphism, respectively. Then we have $(U, \alpha, \beta) = (Y_1, f_1, g_1) \odot (Y_2 \amalg Y_3, f_2 \amalg f_3, g_2 \amalg g_3)$, where

$$U = \{(y_1, y) \in Y_1 \times (Y_2 \amalg Y_3) \mid g_1 y_1 = (f_2 \amalg f_3) y\}.$$

But we have

$$(f_2 \amalg f_3) y = \begin{cases} f_2 y_2, & \text{if } y = i_2(y_2),\ y_2 \in Y_2 \\ f_3 y_3, & \text{if } y = i_3(y_3),\ y_3 \in Y_3, \end{cases}$$

where $i_2\colon Y_2 \to Y_2 \amalg Y_3$ and $i_3\colon Y_3 \to Y_2 \amalg Y_3$ are the natural embeddings. Thus, it follows that U is equal to the set

$$\{(y_1, i_2(y_2)) \in Y_1 \times i_2(Y_2) \mid g_1 y_1 = f_2 y_2\} \amalg \{(y_1, i_3(y_3)) \in Y_1 \times i_3(Y_3) \mid g_1 y_1 = f_3 y_3\}.$$

On the other hand, $V = V_1 \amalg V_2$, where

$$V_1 = \{(y_1, y_2) \in Y_1 \times Y_2 \mid g_1 y_1 = f_2 y_2\}, \quad V_2 = \{(y_1, y_3) \in Y_1 \times Y_3 \mid g_1 y_1 = f_3 y_3\}.$$

We denote by $j_1: V_1 \to V_1 \amalg V_2$ and $j_2: V_2 \to V_1 \amalg V_2$ the natural embeddings and define a mapping $\phi: V \to U$ by $\phi\big(j_1(y_1, y_2)\big) = (y_1, i_2(y_2))$ and $\phi\big(j_2(y_1, y_3)\big) = (y_1, i_3(y_3))$ for $(y_1, y_2) \in V_1$ and $(y_1, y_2) \in V_2$. Then ϕ is a bijection and it remains to show that $\gamma = \alpha \circ \phi$ and $\delta = \beta \circ \phi$. Indeed, we have

$$\alpha \circ \phi\big(j_1(y_1, y_2)\big) = \alpha(y_1, i_2(y_2)) = f_1 \circ \mathrm{pr}_1(y_1, i_2(y_2)) = f_1(y_1),$$

where pr_1 is the projection onto Y_1. On the other hand, we have

$$\gamma\big(j_1(y_1, y_2)\big) = (f_1 \circ p_1 \amalg g_3 \circ p_3)\big(j_1(y_1, y_2)\big) = f_1 \circ p_1(y_1, y_2) = f_1(y_1),$$

where p_1 and p_3 are natural projections onto Y_1 and Y_3, respectively. It follows that $\gamma = \alpha \circ \phi$. Similarly, we can show that $\delta = \beta \circ \phi$.

By Lemma 1, the operation $\oplus$ is well-defined on M_X and by Lemma 2, it is commutative and associative. The operation $\oplus$ is also distributive over $\odot$ on M_X by Proposition 4.

5. The Ring Structure. From the results of Sections 3 and 4, it follows that the operations $\odot$ and $\oplus$ on the set M_X of isomorphism classes of the objects of $\mathcal{DG}_X$ satisfy most of the axioms necessary to make $(M_X, \odot, \oplus)$ a ring, except the existence of an identity and an inverse for the operation $\oplus$. In order to remedy this problem, we shall extend the operation $\oplus$ to the addition operation in the monoid algebra $\mathbb{Z}[M_X]$ of M_X over $\mathbb{Z}$. More precisely, we define the ring $\mathcal{R}_X$ by $\mathcal{R}_X = \mathbb{Z}[M_X]/\mathcal{I}$, where $\mathcal{I}$ is the ideal of $\mathbb{Z}[M_X]$ generated by the elements of the form $x \oplus y - x - y$, with $x, y \in M_X$.

Theorem 1. If X is a finite set with m elements, then $\mathcal{R}_X$ is isomorphic to the ring $M_m(\mathbb{Z})$ of $m \times m$ matrices of integers.

Proof. We shall first construct a mapping $\psi: \mathbb{Z}[M_X] \to M_m(\mathbb{Z})$ from the monoid algebra $\mathbb{Z}[M_X]$ to the set $M_m(\mathbb{Z})$ of $m \times m$ matrices of integers. Suppose $X = \{x_1, \ldots, x_m\}$. If $(\{y\}, f, g) \in M_X$ with $f(y) = x_i$ and $g(y) = x_j$, then we set $\psi(\{y\}, f, g) = e_{i,j}$, where $\{e_{i,j} \mid 1 \le i, j \le m\}$ is the standard basis for $M_m(\mathbb{Z})$. Consider an element

$$V = \sum_{j=1}^{k} n_j(Y_j, f_j, g_j) \in \mathbb{Z}[M_X].$$

For $1 \leq j \leq k$, we assume that $Y_j = \{y_{j,i} \mid 1 \leq i \leq l_j\}$ and denote by $f_{j,i}, g_{j,i}$, the restrictions of f_j, g_j to the set $\{y_{j,i}\}$. Then we have $(Y_j, f_j, g_j) = \oplus_{i=1}^{l_j}(\{y_{j,i}, f_{j,i}, g_{j,i}\})$. We define $\psi(V)$ by

$$\psi(V) = \sum_{j=1}^{k}\sum_{i=1}^{l_j} n_j \psi(\{y_{j,i}, f_{j,i}, g_{j,i}\}).$$

Now, using the fact that two elements $(\{x\}, f, g)$ and $(\{y\}, f', g')$ in $\mathbb{Z}[M_X]$ are isomorphic if and only if $f(x) = f'(y)$ and $g(x) = g'(y)$, it follows that ψ induces a homomorphism of abelian groups from $\mathcal{R}_X$ to $M_m(\mathbb{Z})$. In order to consider the product operations, suppose $\psi(\{y_1\}, f_1, g_1) = e_{i,j}$, $\psi(\{y_2\}, f_2, g_2) = e_{k,l}$, so that $f_1(y_1) = x_i$, $g_1(y_1) = x_j$, $f_2(y_2) = x_k$, and $g_2(y_2) = x_l$. Let $(\{y_1\}, f_1, g_1) \odot (\{y_2\}, f_2, g_2) = (U, \lambda, \mu)$ with

$$U = \{(y_1, y_2) \in \{y_1\} \times \{y_2\} \mid g_1(y_1) = f_2(y_2)\} = \begin{cases} \emptyset, & \text{if } x_j \neq x_k \\ \{(y_1, y_2)\}, & \text{if } x_j = x_k. \end{cases}$$

If $x_j = x_k$, then

$$\lambda(y_1, y_2) = f_1 \circ \mathrm{pr}_1(y_1, y_2) = f_1(y_1) = x_i, \quad \mu(y_1, y_2) = g_2 \circ \mathrm{pr}_2(y_1, y_2) = g_2(y_2) = x_l.$$

Hence, we have

$$\psi(U, \lambda, \mu) = \begin{cases} 0, & \text{if } j \neq k \\ e_{i,l}, & \text{if } j = k. \end{cases}$$

Thus, ψ transfers the operation $\odot$ to the multiplication operation of matrices. It follows that ψ induces a ring homomorphism from $\mathcal{R}_X$ to $M_m(\mathbb{Z})$.

6. Magic Squares. In this section we slightly generalize the usual definition of magic squares and show that these generalized magic squares correspond to regular digraphs under the isomorphism described in Theorem 1.

Definition 3. An element $A = (a_{i,j})$ of $M_m(\mathbb{Z})$ is a generalized magic square if

$$\sum_{i=1}^{m} a_{i,j} = \sum_{l=1}^{m} a_{k,l}$$

for all $j, k \in \{1, \ldots, m\}$.

The set of generalized magic squares in $M_m(\mathbb{Z})$ is a subring of $M_m(\mathbb{Z})$, which we denote by $M_m^0(\mathbb{Z})$.

Definition 4. An object (Y, f, g) of $\mathcal{DG}_X$ is regular if f and g are surjective and

$$|f^{-1}(x)| = |g^{-1}(x)| = |Y|/|X|$$

for all $x \in X$, where $|\cdot|$ denotes the cardinality of a set.

Remark 4. The directed graph represented by a regular object (Y, f, g) is a $(|Y|/|X|)$ – regular digraph in the usual sense (see e.g. [1]), that is, a directed graph such that the in-degree and the out-degree of each vertex is equal to $|Y|/|X|$.

Let $\mathcal{DG}_X^0$ be the subcategory of $\mathcal{DG}_X$ consisting of the objects of $\mathcal{DG}_X$ that are regular. If (Y_1, f_1, g_1) and (Y_2, f_2, g_2) are objects of $\mathcal{DG}_X^0$, then both $(Y_1, f_1, g_1) \odot (Y_2, f_2, g_2)$ and $(Y_1, f_1, g_1) \oplus (Y_2, f_2, g_2)$ are objects of $\mathcal{DG}_X^0$. Thus, if we set

$$M_X^0 = \mathcal{DG}_X^0 / \cong, \quad \mathcal{R}_X^0 = \mathbb{Z}[M_X^0]/\mathcal{I}^0,$$

where $\mathcal{I}^0$ is the ideal of $\mathbb{Z}[M_X^0]$ generated by elements of the form $x \oplus y - x - y$, with $x, y \in M_X^0$, then $\mathcal{R}_X^0$ becomes a subring of $\mathcal{R}_X$.

Theorem 2. The isomorphism described in Theorem 1 induces an isomorphism between $\mathcal{R}_X^0$ and $M_m^0(\mathbb{Z})$.

Proof. Let $\tilde{\psi}\colon \mathcal{R}_X \to M_m(\mathbb{Z})$ be the isomorphism constructed in the proof of Theorem 1 and let $V = \sum_{j=1}^{k} n_j(Y_j, f_j, g_j) \in \mathbb{Z}[M_X]$ be a representative of $\tilde{V} \in \mathcal{R}_X$. For $1 \le j \le k$, we assume that $Y_j = \{y_{j,i} \mid 1 \le i \le l_j\}$ and denote by $f_{j,i}, g_{j,i}$, the restrictions of f_j, g_j to the set $\{y_{j,i}\}$. If $\tilde{\psi}(\tilde{V}) = (\nu_{a,b}) \in M_m(\mathbb{Z})$, with $1 < a, b < m$, then we have

$$\nu_{a,b} = \sum_{j=1}^{K} n_j |f_j^{-1}(x_a) \cap g_j^{-1}(x_b)|;$$

hence, we have

$$\sum_{a=1}^{m} \nu_{a,b} = \sum_{j=1}^{k} n_j \left| \left(\bigcup_{a=1}^{m} f_j^{-1}(x_a) \right) \cap g_j^{-1}(x_b) \right| = \sum_{j=1}^{k} n_j |g_j^{-1}(x_b)| = \sum_{j=1}^{k} \frac{n_j|Y|}{|X|}.$$

Similarly, we have

$$\sum_{b=1}^{m} \nu_{a,b} = \sum_{j=1}^{k} n_j \left| f_j^{-1}(x_a) \cap \left(\bigcup_{b=1}^{m} g_j^{-1}(x_b) \right) \right| = \sum_{j=1}^{k} n_j |f_j^{-1}(x_a)| = \sum_{j=1}^{k} \frac{n_j|Y|}{|X|}.$$

Thus, it follows that $(\nu_{a,b}) \in M_m^0(\mathbb{Z})$.

7. Grothendieck Topologies. In the previous sections we regarded directed graphs as the objects in the category of pairs of mappings of finite sets. In this section, we consider the category of single mappings of finite sets, which is essentially a Grothendieck topology on the category of finite sets.

Definition 5. We define $\mathcal{S}$ to be the category whose objects are mappings of finite sets $\psi: A \to B$ and whose morphisms from $\psi: A \to B$ to $\psi': A' \to B'$ are pairs of mappings $\alpha: A \to A'$ and $\beta: B \to B'$, such that $\beta \circ \psi = \psi' \circ \alpha$.

Definition 6. Given a finite set X, we define $\mathcal{S}_X$ to be the subcategory of $\mathcal{S}$ whose objects are the mappings $\phi: Z \to X$ in $\mathcal{S}$ and whose morphisms from $\phi: Z \to X$ to $\phi': Z' \to X$ are maps $\nu: Z \to Z'$ such that $\phi = \phi' \circ \nu$.

Remark 5. The category $\mathcal{S}$ can be regarded as a Grothendieck topology on the category of finite sets by using the objects of $\mathcal{S}_X$ as a covering of X, and therefore the category of finite sets equipped with this topology is a site (see [2, 3]).

Given objects (Y, f, g) and (Z, ϕ) of $\mathcal{DG}_X$ and $\mathcal{S}_X$, respectively, let W be the pullback of $g: Y \to X$ and $\phi: Z \to X$, that is, $W = \{(y, z) \in Y \times Z \mid g(y) = \phi(z)\}$. Then we define the action of (Y, f, g) on (Z, ϕ) by $(Y, f, g) \cdot (Z, \phi) = (W, f \circ \pi_Y)$, where $\pi_Y: W \to Y$ is the natural projection. If $(M_X, \odot)$ denotes the monoid considered in Proposition 3, then it follows from the next theorem that M_X acts on the set $\mathrm{Obj}(\mathcal{S}_X)$ of objects of $\mathcal{S}_X$.

Theorem 3. Given a pair of objects (Y_1, f_1, g_1) and (Y_2, f_2, g_2) of $\mathcal{DG}_X$ and an object (Z, ϕ) of $\mathcal{S}_X$, we have

$$(Y_1, f_1, g_1) \cdot \big((Y_2, f_2, g_2) \cdot (Z, \phi)\big) = \big((Y_1, f_1, g_1) \odot (Y_2, f_2, g_2)\big) \cdot (Z, \phi).$$

Proof. We have $(Y_2, f_2, g_2) \cdot (Z, \phi) = (W, f_2 \circ \pi_{Y_2})$, where $\pi_{Y_2}\colon W \to Y_2$ is the natural projection and $W = \{(y_2, z) \in Y_2 \times Z \mid g_2(y_2) = \phi(z)\}$. Thus, we obtain

$$(Y_1, f_1, g_1) \cdot \big((Y_2, f_2, g_2) \cdot (Z, \phi)\big) = (W', f_1 \circ \pi_{Y_1}),$$

with

$$W' = \{(y_1, y_2, z) \in Y_1 \times Y_2 \times Z \mid g_1(y_1) = (f_2 \circ \pi_{Y_2})(y_2, z) = f_2(y_2)\},$$

$\pi_{Y_1}\colon W' \to Y_1$, $\pi_{Y_2}\colon W' \to Y_2$, and $(f_1 \circ \pi_{Y_1})(y_1, y_2, z) = f_1(y_1)$. On the other hand, we have

$$(Y_1, f_1, g_1) \odot (Y_2, f_2, g_2) = (U, f_1 \circ \pi'_{Y_1}, g_2 \circ \pi'_{Y_2}),$$

where $\pi'_{Y_1}\colon U \to Y_1$ and $\pi'_{Y_2}\colon U \to Y_2$ are natural projections and

$$U = \{(y_1, y_2) \in Y_1 \times Y_2 \mid g_1(y_1) = f_2(y_2)\};$$

hence, it follows that

$$\big((Y_1, f_1, g_1) \odot (Y_2, f_2, g_2)\big) \cdot (Z, \phi) = (U', (f_1 \circ \pi'_{Y_1}) \circ \pi'_U),$$

with

$$U' = \{(y_1, y_2, z) \in Y_1 \times Y_2 \times Z \mid g_2 \circ \pi'_{Y_2}(y_1, y_2) = g_2(y_2) = \phi(z)\},$$

$\pi'_U\colon U' \to U$ and $((f_1 \circ \pi'_{Y_1}) \circ \pi'_U)(y_1, y_2, z) = (f_1 \circ \pi'_{Y_1})(y_1, y_2) = f_1(y_1)$. Therefore, we have $W' = U$ and $f_1 \circ \pi_{Y_1} = (f_1 \circ \pi'_{Y_1}) \circ \pi'_U$; hence, the theorem follows.

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