A CHARACTERIZATION OF THE CENTRALIZER OF A PERMUTATION

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Abstract. An explicit form is determined for those permutations commuting with a given permutation, and the number of them is also determined.

In his modern algebra textbook *Topics in Algebra* [1] I. N. Herstein, after determining the number of conjugate classes in the symmetric group S_n, states that "... we can find all the elements commuting with a given permutation" and by way of example proceeds to do this for the transposition $(1, 2)$ and for the n-cycle $(1, 2, \ldots, n)$ [1]. While this is relatively easy for permutations with such simple cycle structure, the problem is less tractable for those which are products of disjoint cycles of differing lengths, particularly if there are more than one of the same length. In this note we determine an explicit form of all permutations commuting with a given one and also determine their number.

Let σ be any permutation in S_n. The *support* of σ, denoted by $\text{supp}(\sigma)$, is the set of letters moved by σ, the *centralizer* $C(\sigma)$ of σ is the subgroup of all permutations in S_n which commute with σ, and the *conjugate class* $Cl(\sigma)$ of σ is the set of all conjugates $\theta\sigma\theta^{-1}$ where $\theta \in S_n$. The number $o(Cl(\sigma))$ of conjugates of σ is given by $o(Cl(\sigma)) = [S_n : C(\sigma)]$, the index of $C(\sigma)$ in S_n.

We first consider the case in which σ is a product of disjoint cycles, all of the *same* length, viz., $\sigma = \sigma_1\sigma_2 \cdots \sigma_r$ where $\sigma_1, \sigma_2, \ldots, \sigma_r$ are disjoint k-cycles with $k > 1$. Let $\sigma_j = (a_{1j}, a_{2j}, \ldots, a_{kj})$ for $j = 1, 2, \ldots, r$. Given a permutation $\theta \in S_r$ we define an auxiliary permutation $\theta' \in S_n$ by $\theta'(a_{ij}) = a_{i,\theta(j)}$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, r$ and $\theta'(x) = x$ for all other letters x. Note that $\text{supp}(\theta') \subseteq \text{supp}(\sigma)$.

Lemma 1. With the notation as above, a permutation in S_n commutes with σ if and only if it has the form

$$\sigma_1^{s_1}\sigma_2^{s_2} \cdots \sigma_r^{s_r}\theta'\tau$$

where $1 \leq s_i \leq k$ for $i = 1, 2, \ldots r$, where $\theta \in S_r$, and where τ is disjoint with σ. Moreover the number of such permutations is $k^r r!(n - kr)!$.

Proof. For $i = 1, 2, \ldots, r$ let $d_i = k(r-i+1)$. Since a permutation is conjugate to σ if and only if it has the same cycle structure as σ, we have

$$o(Cl(\sigma)) = \frac{\binom{n}{d_1}\binom{d_1}{k}(k-1)!\binom{d_2}{k}(k-1)!\cdots\binom{d_{r-1}}{k}(k-1)!\binom{d_r}{k}(k-1)!}{r!}$$

$$= \frac{\left[\frac{n!}{d_1!(n-d_1)!}\right]\left[\frac{d_1!(k-1)!}{k!d_2!}\right]\left[\frac{d_2!(k-1)!}{k!d_3!}\right]\cdots\left[\frac{d_{r-1}!(k-1)!}{k!d_r!}\right]\left[\frac{d_r!(k-1)!}{k!0!}\right]}{r!}$$

$$= \frac{n!}{(n-d_1)!r!k^r},$$

and hence,

$$o(C(\sigma)) = \frac{n!}{o(Cl(\sigma))} = k^r r!(n-kr)!.$$

Now let T denote the set of all permutations of the form given in the statement of the lemma. Noting that $\theta'\sigma(a_{ij}) = a_{i+1,\theta(j)} = \sigma\theta'(a_{ij})$, we have $\theta'\sigma = \sigma\theta'$ from which $T \subseteq C(\sigma)$ follows. Suppose

$$\sigma_1^{s_1}\sigma_2^{s_2}\cdots\sigma_r^{s_r}\theta'\tau = \sigma_1^{t_1}\sigma_2^{t_2}\cdots\sigma_r^{t_r}\eta'\rho$$

where $1 \le s_i, t_i \le k$ for $i = 1, 2, \ldots, r$, where θ and η are in S_r, and where τ and ρ are in S_n and disjoint with σ. Since τ and ρ are also disjoint with θ' and η', we have $\tau = \rho$. Suppose without loss of generality that $s_1 \le t_1$. Then $0 \le t_1 - s_1 < k$, whence,

$$\begin{aligned} a_{t_1-s_1+1,1} &= \sigma_1^{t_1-s_1}(a_{11}) = \sigma_1^{t_1-s_1}\sigma_2^{t_2-s_2}\cdots\sigma_r^{t_r-s_r}(a_{11}) \\ &= \theta'(\eta')^{-1}(a_{11}) = \theta'\big(a_{1,\eta^{-1}(1)}\big) = a_{1,\theta\eta^{-1}(1)}. \end{aligned}$$

Thus, $t_1 - s_1 + 1 = 1$, that is $t_1 = s_1$, and $\theta\eta^{-1}(1) = 1$. In a similar way we obtain $t_i = s_i$ and $\theta\eta^{-1}(i) = i$ for $i = 1, 2, \ldots r$. Thus, $\theta = \eta$. This establishes the unique form of the elements of T, so $o(T) = k^r r!(n-kr)!$ and $T = C(\sigma)$.

For the general case suppose $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ where $\sigma_1, \sigma_2, \ldots, \sigma_m$ are disjoint permutations such that for $1 \le i \le m$, σ_i is a product of r_i disjoint k_i-cycles, and $k_1, k_2, \ldots, k_m$ are distinct integers greater than 1. For $i = 1, 2, \ldots, m$ let $\sigma_i = \sigma_1^{(i)}\sigma_2^{(i)}\cdots\sigma_{r_i}^{(i)}$ where $\sigma_1^{(i)}, \sigma_2^{(i)}, \ldots, \sigma_{r_i}^{(i)}$ are disjoint k_i-cycles.

Theorem 2. With the notation as above, a permutation in S_n commutes with σ if and only if it has the form

$$\prod_{i=1}^{m}\left(\sigma_1^{(i)}\right)^{s_1^{(i)}}\left(\sigma_2^{(i)}\right)^{s_2^{(i)}}\cdots\left(\sigma_{r_i}^{(i)}\right)^{s_{r_i}^{(i)}}\theta_i'\delta$$

where $1 \le s_j^{(i)} \le k_i$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, r_i$, where $\theta_i \in S_{r_i}$ for $i = 1, 2, \ldots, m$, and where δ is a permutation in S_n disjoint with σ. Moreover the number of such permutations is

$$\left(\prod_{i=1}^{m} k_i^{r_i} r_i!\right)\left(n - \sum_{i=1}^{m} k_i r_i\right)!.$$

Proof. Let T denote the set of all permutations of the form given in the statement of the theorem. By using the lemma it is easy to see that $T \subseteq C(\sigma)$. Now suppose $\alpha \in C(\sigma)$. Then

$$\sigma = \alpha\sigma\alpha^{-1} = \prod_{i=1}^{m}\prod_{j=1}^{r_i}\alpha\sigma_j^{(i)}\alpha^{-1}$$

yields a decomposition of σ into a product of disjoint cycles. Since

$$\sigma = \prod_{i=1}^{m}\prod_{j=1}^{r_i}\sigma_j^{(i)}$$

is also a decomposition of σ into a product of disjoint cycles and $k_1, k_2, \ldots, k_m$ are distinct, by the uniqueness of decomposition we must have

$$\prod_{j=1}^{r_i} \alpha\sigma_j^{(i)}\alpha^{-1} = \prod_{j=1}^{r_i} \sigma_j^{(i)}$$

for $i = 1, 2, \ldots, m$, so we have $\alpha\sigma_i\alpha^{-1} = \sigma_i$ for $i = 1, 2, \ldots, m$, that is, α commutes with σ_i for $i = 1, 2, \ldots, m$. Thus, by the lemma, for $i = 1, 2, \ldots, m$, there must exist integers $s_1^{(i)}, s_2^{(i)}, \ldots, s_{r_i}^{(i)}$ and permutations $\theta_i \in S_{r_i}$ and $\tau_i \in S_n$ such that $1 \leq s_j^{(i)} \leq k_i$ for $j = 1, 2, \ldots, r_i$, τ_i is disjoint with σ_i, and

$$\alpha = \left(\sigma_1^{(i)}\right)^{s_1^{(i)}} \left(\sigma_2^{(i)}\right)^{s_2^{(i)}} \cdots \left(\sigma_{r_i}^{(i)}\right)^{s_{r_i}^{(i)}} \theta_i'\tau_i.$$

For brevity we let

$$\beta_i = \left(\sigma_1^{(i)}\right)^{s_1^{(i)}} \left(\sigma_2^{(i)}\right)^{s_2^{(i)}} \cdots \left(\sigma_{r_i}^{(i)}\right)^{s_{r_i}^{(i)}} \theta_i'$$

for $i = 1, 2, \ldots, m$; thus,

$$\alpha = \beta_1\tau_1 = \beta_2\tau_2 = \cdots = \beta_m\tau_m.$$

Consider the two permutations τ_1 and $\prod_{i=2}^m \beta_i$. Suppose $x \in \operatorname{supp}(\sigma)$, the latter being the disjoint union of $\operatorname{supp}(\sigma_i)$ for $i = 1, 2, \ldots, m$. If $x \in \operatorname{supp}(\sigma_1)$, then $\tau_1(x) = x = \left(\prod_{i=2}^m \beta_i\right)(x)$; if $x \in \operatorname{supp}(\sigma_j)$ with $j \neq 1$, then

$$\tau_1(x) = \beta_1\tau_1(x) = \beta_j\tau_j(x) = \beta_j(x) = \left(\prod_{i=2}^m \beta_i\right)(x),$$

the last equation holding since $x \notin \operatorname{supp}(\sigma_i)$ with $i \neq j$. Thus, τ_1 and $\prod_{i=2}^m \beta_i$ agree on $\operatorname{supp}(\sigma)$. Hence, $\delta = \left(\prod_{i=2}^m \beta_i\right)^{-1}\tau_1$ is a permutation of the remaining $n - \sum_{i=1}^m k_i r_i$ letters not in $\operatorname{supp}(\sigma)$. We now have

$$\alpha = \beta_1\tau_1 = \beta_1\left(\prod_{i=2}^m \beta_i\right)\delta = \prod_{i=1}^m \left(\sigma_1^{(i)}\right)^{s_1^{(i)}} \left(\sigma_2^{(i)}\right)^{s_2^{(i)}} \cdots \left(\sigma_{r_i}^{(i)}\right)^{s_{r_i}^{(i)}} \theta_i'\delta,$$

showing that $\alpha \in T$. So we have established $T = C(\sigma)$.

To see that the representation of α in this form is unique, suppose we also have

$$\alpha = \prod_{i=1}^{m} \big(\sigma_1^{(i)}\big)^{t_1^{(i)}} \big(\sigma_2^{(i)}\big)^{t_2^{(i)}} \cdots \big(\sigma_{r_i}^{(i)}\big)^{t_{r_i}^{(i)}} \eta_i' \rho.$$

Considering supports, it is easy to see that $\delta = \rho$ and that

$$\big(\sigma_1^{(i)}\big)^{s_1^{(i)}} \big(\sigma_2^{(i)}\big)^{s_2^{(i)}} \cdots \big(\sigma_{r_i}^{(i)}\big)^{s_{r_i}^{(i)}} \theta_i' = \big(\sigma_1^{(i)}\big)^{t_1^{(i)}} \big(\sigma_2^{(i)}\big)^{t_2^{(i)}} \cdots \big(\sigma_{r_i}^{(i)}\big)^{t_{r_i}^{(i)}} \eta_i'$$

for $i = 1, 2, \dots, m$. But this is the case considered in the proof of the lemma, so for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, r_i$ we have $s_j^{(i)} = t_j^{(i)}$ and $\theta_i = \eta_i$. Hence, the representation of α is unique and counting elements of this form, we obtain $o(C(\sigma)) = \big(\prod_{i=1}^{m} k_i^{r_i} r_i!\big)\big(n - \sum_{i=1}^{m} k_i r_i\big)!$, which completes the proof of the theorem.

Example 3. In S_{25} let

$$\sigma = (1,2)(3,4)(5,6)(7,8,9)(10,11,12)(13,14,15,16)(17,18,19,20).$$

Then

$$\begin{aligned} \sigma_1 &= (1,2)(3,4)(5,6), \quad k_1 = 2, \quad r_1 = 3, \\ \sigma_2 &= (7,8,9)(10,11,12), \quad k_2 = 3, \quad r_2 = 2, \\ \sigma_3 &= (13,14,15,16)(17,18,19,20), \quad k_3 = 4, \quad r_3 = 2. \end{aligned}$$

The number of permutations in S_{25} commuting with σ is

$$\big(2^3 \cdot 3!\big)\big(3^2 \cdot 2!\big)\big(4^2 \cdot 2!\big)(25 - 20)! = 3,317,760,$$

and they all have the form

$$(1,2)^{c_1}(3,4)^{c_2}(5,6)^{c_3}(7,8,9)^{d_1}(10,11,12)^{d_2}(13,14,15,16)^{e_1}(17,18,19,20)^{e_2}\theta_1'\theta_2'\theta_3'\delta$$

where $c_i = 1$ or 2, $d_i = 1, 2,$ or 3, $e_i = 1, 2, 3,$ or 4 for each i, $\theta_1 \in S_3$, $\theta_2 \in S_2$, $\theta_3 \in S_2$, and δ is a permutation of $\{21, 22, 23, 24, 25\}$. We also note that, for example, if $\theta_1 = (1,3,2)$, $\theta_2 = (1,2)$, and $\theta_3 = 1$, then

$$\theta_1' = (1,5,3)(2,6,4), \quad \theta_2' = (7,10)(8,11)(9,12), \quad \theta_3' = 1.$$

Reference

1. I. N. Herstein, *Topics in Algebra*, 2nd ed., Xerox Corporation, Lexington, Massachusetts, 1975.

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