REMARK ON A GENERAL ARITHMETIC FOURIER TRANSFORM

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Abstract. In this note we prove a general result connected with arithmetic Fourier transforms from which follows the main result given by Walker [1].

1. Introduction. In the paper [1], W. J. Walker proved the following result concerning arithmetic Fourier transforms. Let f be an even function of period 2π which is normalized, so that

$$\int_0^{2\pi} f(\theta)d\theta = 0.$$

Suppose that the Fourier series

$$f(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta$$

is absolutely convergent to f. Moreover, let

$$\delta_j^q = \begin{cases} 0 & \text{if } j \text{ contains a prime factor greater than the } q\text{th prime} \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

Recall that the Möbius function is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r \\ 0 & \text{if } p^2 \mid n. \end{cases} \tag{2}$$

In addition, define $S(n)$ by

$$S(n) = \frac{1}{n} \sum_{m=0}^{n-1} f\left(\frac{2\pi m}{n}\right). \tag{3}$$

Then, it follows that

$$a_n = \lim_{q\to\infty} \sum_{j=1}^{\infty} \delta_j^q \mu(j) S(jn); \quad n \geq 1. \tag{4}$$

Moreover, Walker indicated some connections and applications to the field of signal processing and to artificial neural networks.

2. Result. In the present paper we prove a general result.

Theorem. Let the function h satisfy the following condition.

$$\frac{1}{n}\sum_{m=0}^{n-1} h\left(\frac{wmj}{n}\right) = \begin{cases} 0 & \text{if } j \neq kn \\ 1 & \text{if } j = kn, \end{cases} \tag{C}$$

where $w \neq 0$ is a fixed real constant and k is a positive integer and let the function f have the expansion on the Fourier series

$$f(x) = \sum_{n=1}^{\infty} a_n h(nx), \tag{5}$$

which is absolutely convergent to f. Then

$$a_n = \lim_{q\to\infty} \delta_j^q \mu(j) S(jn), \tag{6}$$

where

$$S(n) = \frac{1}{n}\sum_{m=0}^{n-1} f\left(\frac{wm}{n}\right) \tag{7}$$

and δ_j^q and μ are defined by (1) and (2), respectively.

Proof. First we prove that by the hypothesis it follows that

$$S(n) = \sum_{k=1}^{\infty} a_{kn}. \tag{8}$$

Rewriting the series (5) in the form

$$f(x) = \sum_{j=1}^{\infty} a_j h(jx)$$

and putting $x = (wm)/n$, where $w \neq 0$ is a real constant we obtain

$$f\left(\frac{wm}{n}\right) = \sum_{j=1}^{\infty} a_j h\left(\frac{wmj}{n}\right). \tag{9}$$

From (7) and (9) we get

$$S(n) = \sum_{j=1}^{\infty} a_j \frac{1}{n} \sum_{m=0}^{n-1} h\left(\frac{wmj}{n}\right). \tag{10}$$

Applying to (10) condition (C), we obtain

$$S(n) = \sum_{j=1}^{\infty} a_j$$

for $j = kn$, so

$$S(n) = \sum_{k=1}^{\infty} a_{kn}$$

and we see that (8) is proved. On the other hand, we know the following summation formula for the Möbius function.

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Using this formula and the Möbius inversion formula, by an easy calculation, we obtain

$$a_n = \sum_{k=1}^{\infty} \mu(k) S(kn). \tag{11}$$

Now, assuming δ_j^q is defined as in (1), consider the expression

$$T_q(n) = \sum_{j=1}^{\infty} \delta_j^q \mu(j) S(jn). \tag{12}$$

Then from (11) and (12) we can deduce that

$$T_q(n) = a_n + \sum_{k=2}^{\infty} \alpha_k^q a_{kn} \tag{13}$$

where $\alpha_k^q = 0$, if k contains one of the first q primes and $\alpha_k^q = 1$, otherwise. From (13) and by the assumption of the Theorem, it follows that

$$\lim_{q\to\infty} \sum_{k=2}^{\infty} \alpha_k^q a_{kn} = 0.$$

Hence, from (12) and (13) we obtain

$$\lim_{q\to\infty} T_q(n) = \lim_{q\to\infty} \sum_{j=1}^{\infty} \delta_j^q \mu(j) S(jn) = a_n.$$

The proof of the Theorem is complete.

3. Remark. Now, we observe that from our Theorem we can obtain Walker's result as a particular case. Indeed, let $h(x) = \cos x$; $w = 2\pi$. Then for $x = (wmj)/n$ we have

$$\sum_{m=0}^{n-1} h\left(\frac{wmj}{n}\right) = \sum_{m=0}^{n-1} \cos\frac{2\pi mj}{n}. \tag{14}$$

On the other hand, it is easy to see that

$$\begin{cases} \cos\frac{2\pi mj}{n} + i\sin\frac{2\pi mj}{n} = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{mj} = \epsilon^{mj} \\ \cos\frac{2\pi mj}{n} - i\sin\frac{2\pi mj}{n} = \left(\cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}\right)^{mj} = \epsilon^{-mj} \end{cases}. \tag{15}$$

By (15), it follows that

$$\cos\frac{2\pi mj}{n} = \frac{1}{2}\left(\epsilon^{mj} + \epsilon^{-mj}\right). \tag{16}$$

Hence, from (14) and (16) we obtain

$$\sum_{m=0}^{n-1} h\left(\frac{wmj}{n}\right) = \sum_{m=0}^{n-1} \cos\frac{2\pi mj}{n} = \frac{1}{2}\sum_{m=0}^{n-1}\left(\epsilon^{mj} + \epsilon^{-mj}\right). \tag{17}$$

But it is well-known that if ϵ is a root of unity of degree n, then

$$\sum_{m=0}^{n-1} \epsilon^{mj} = \begin{cases} 0 & \text{if } j \neq kn \\ n & \text{if } j = kn \end{cases} \tag{18}$$

for some positive integer k. Therefore, by (17) and (18) it follows that

$$\frac{1}{n}\sum_{m=0}^{n-1} h\left(\frac{wmj}{n}\right) = \begin{cases} 0 & \text{if } j \neq kn \\ 1 & \text{if } j = kn \end{cases}$$

so that condition (C) is satisfied for the function $h(x) = \cos x$. Thus, by the Theorem with $w = 2\pi$, we obtain Walker's result, since,

$$S(n) = \frac{1}{n}\sum_{m=0}^{n-1} f\left(\frac{wm}{n}\right) = \frac{1}{n}\sum_{m=0}^{n-1} f\left(\frac{2\pi m}{n}\right).$$

Reference

1. W. J. Walker, "The Arithmetic Fourier Transform and Real Neural Networks: Summability by Primes," *J. Math. Anal. Appl.*, 190 (1995), 211–219.

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