REMARKS ON A FACTORIZATION OF $X^n - Y^n$

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Introduction. Early on, elementary algebra students learn to factor expressions $x^2 - y^2, x^3 - y^3, \dots$, where x and y represent real numbers. Later, they learn the generalization

$$x^n - y^n = (x-y)\sum_{m=0}^{n-1} x^m y^{n-1-m} \qquad (*)$$

for each positive integer n and all real x and y. This identity has been shown to have various applications throughout the undergraduate mathematics curriculum and beyond. For example, Johnsonbaugh, using the inequality

$$\frac{b^{n+1} - a^{n+1}}{b-a} < (n+1)b^n$$

for all positive integers n and real a and b with $0 \le a < b$, an easy consequence of $(*)$, published an old and relatively simple proof of the monotonicity and boundedness of the sequence $\{(1+1/n)^n\}$ in [2]. Evidently, the proof was discovered by Fort in 1864 (see [3]). In [1], there is a nice proof of the existence of nth roots, which is a good deal simpler than other proofs using the Fundamental Axiom of the Reals (see [3]) and proofs using the Intermediate Value Theorem (see [5]). The following characterization of $C_{n,k}$ is an interesting by-product obtained by relating $(*)$ to the Binomial Theorem: If n and k are positive integers with $k \le n$, then

$$C_{n,k} = \sum_{m=k-1}^{n-1} C_{m,k-1}.$$

We obtain this relationship by combining $(*)$ and the Binomial Theorem to get

$$(x+1)^n = 1 + x\sum_{m=0}^{n-1}(x+1)^m = 1 + \sum_{m=0}^{n-1}\sum_{k=0}^{m} C_{m,k}\ x^{k+1}$$

for each real x and then equating coefficients of x^m.

The purpose of this note is to present other interesting applications of $(*)$ and to begin an investigation of a functional inequality, which we discovered while studying $(*)$. In Section 1, we show how inequalities, which are typically found in elementary analysis courses, as well as some which seem to be absent from the literature, flow easily from $(*)$ and how those in the second category may be used to advantage in elementary courses. In [1], identity $(*)$ is used to show that for each positive integer n and all reals x and y,

$$|x^n - y^n| \le (|x-y| + |y|)^n - |y|^n. \tag{1}$$

Another known inequality, which is easily deduced from $(*)$, is

$$(n^{1/n} - 1)^2 \le 2/n$$

for each positive integer n (see [4]). We show in this article that it is possible to prove a stronger inequality, although we are unable to see how $(*)$ can be used to verify this inequality. We prove that

$$(n^{1/n} - 1)^2 \le 1/n \tag{2}$$

for each positive integer n.

Following a suggestion of W. Rudin (private communication), we give a shorter proof than that given for (2) that

$$(x^{1/x} - 1)^2 \le 1/x \tag{3}$$

for each positive real x.

In Section 2, we study real-valued functions f satisfying the functional inequality

$$|f(x) - f(y)| \le f(|x-y| + |y|) - f(|y|) \tag{4}$$

for all x and y in $D(f)$, where $D(f)$ is the domain of f. This study is motivated by the observation, from (1), that the function f, defined by $f(x) = x^n$, satisfies (4). We see also that the exponential function satisfies (4).

$$|\exp(x) - \exp(y)|$$

$$= \begin{cases} \exp(y)(\exp(x-y) - 1) \le \exp(|x-y| + |y|) - \exp(|y|), & \text{if } x \ge y \\ \exp(x)(\exp(y-x) - 1) \le \exp(|x-y| + |y|) - \exp(|y|), & \text{if } x < y. \end{cases}$$

We denote the class of functions satisfying (4) by Ω. It is obvious that constant functions are elements of Ω and fairly obvious that any $f \in \Omega$ is nondecreasing on the set of nonnegative elements of $D(f)$. We show that any $f \in \Omega$ is continuous and is convex on $[c, \infty)$ if $c \geq 0$ and $[c, \infty) \subset D(f)$. That is,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in [c, \infty)$ and $0 \leq \lambda \leq 1$. In addition, we show that if $f, g \in \Omega$ and $\lambda \geq 0$, then $f + g \in \Omega$ and $\lambda f \in \Omega$. Moreover, if $f, g \in \Omega$, $f(0) \geq 0$, and $g(0) \geq 0$, then $fg \in \Omega$.

1. Some Applications. As an application of (1), for any fixed y and $\epsilon > 0$, each solution to $(|x-y| + |y|)^n - |y|^n < \epsilon$ is a solution to $|x^n - y^n| < \epsilon$. Since $(|x-y| + |y|)^n - |y|^n < \epsilon$ is equivalent to $|x-y| < (\epsilon + |y|^n)^{1/n} - |y|$,

$$\text{if } \epsilon > 0 \text{ and } 0 < \delta \leq (\epsilon + |y|^n)^{1/n} - |y|, \text{ then } |x^n - y^n| < \epsilon, \text{ when } |x-y| < \delta. \quad (A_1)$$

It should be readily obtainable for the reader that if $P(x) = \sum_{m=0}^{n} a_{n-m} x^{n-m}$ is a polynomial function of degree n in x, then

$$\text{for any } \epsilon > 0 \text{ and any fixed } y, \ |P(x) - P(y)| < \epsilon, \text{ if } |x-y| < \delta, \qquad (A_2)$$

where

$$0 < \delta \leq \min\left\{ \left(\frac{\epsilon}{nK} + |y|^m \right)^{1/m} - |y| : m = 1, \dots, n \right\}$$

and

$$K = \max\{|a_{n-m}| : m = 0, \dots, n-1\}.$$

The identity $(*)$, along with the statement $(**)$, offered below without proof, leads to another useful inequality (5).

> Let x and y be real and let n be a positive integer such that $xy > 0$ and $x^{1/n}$ is real. Then $x^{k/n} y^{(n-1-k)/n} > 0$ for each integer k. $(**)$

Let x and y be real and let n be a positive integer such that $xy > 0$ and $x^{1/n}$ is real. Then the following inequality holds.

$$|x^{1/n} - y^{1/n}| \leq \frac{|x-y|}{y^{(n-1)/n}}. \tag{5}$$

The verification of (5) comes from ($*$) and ($**$) as follows.

$$|x-y| = |x^{1/n} - y^{1/n}| \sum_{m=0}^{n-1} x^{m/n} y^{(n-1-m)/n}$$

$$\geq |x^{1/n} - y^{1/n}| y^{(n-1)/n}.$$

Utilizing (5) and ($*$), we may establish that for any real x and y and any integer n for which $y^{1/n}$ is real and any $\epsilon > 0$,

$$\text{any solution to } |x-y| < \min\{\epsilon y^{(n-1)/n}, |y|\} \text{ is a solution to } |x^{1/n}-y^{1/n}| < \epsilon. \quad (A_3)$$

The exercises below may be used to obtain more experience with applying ($*$) to arrive at other elementary inequalities.

Exercise 1. If $p \geq 1$, show that $p^n \geq 1 + n(p-1)$ for each positive integer n.

Exercise 2. If $p \geq 1$, show that $p^n - 1 \geq (p-1)^2 n(n-1)/2$ for each positive integer n.

Solution. From ($*$) and the result of Exercise 1, we obtain

$$p^n - 1 \geq (p-1) \sum_{m=0}^{n-1} p^m \geq (p-1)^2 \sum_{m=1}^{n-1} m = (p-1)^2 n(n-1)/2.$$

Exercise 3. If $p > 1$, show that $n/p^n < 2p/n(p-1)^2$ for each positive integer n.

Solution. From the result of Exercise 2, for each positive integer n,

$$p^{n+1} \geq (p-1)^2 n(n+1)/2 > (p-1)^2 n^2/2.$$

Exercise 4. Show that $(n^{1/n} - 1)^2 \leq 2/n$ for each positive integer n.

Solution. From the result of Exercise 2, for $n > 1$,

$$n - 1 = (n^{1/n})^n - 1 \geq (n^{1/n} - 1)^2 n(n-1)/2.$$

Exercise 5. If $x > 0$, show that $|x^{1/n} - 1| \leq \max\{1, 1/x\}|x - 1|/n$ for each positive integer n.

Solution. For $x \geq 1$, we arrive at $|x^{1/n} - 1| \leq |x - 1|/n$ by applying the result of Exercise 1 with $p = x^{1/n}$. For $x < 1$, we apply the result of Exercise 1 with $p = (1/x)^n$ to conclude that $|x^{1/n} - 1| \leq |x - 1|/(nx)$. Hence, for all $x > 0$,

$$|x^{1/n} - 1| \leq \max\{|x - 1|/(nx),\ |x - 1|/n\} = \max\{1, 1/x\}|x - 1|/n.$$

Although we are unable to see how $(*)$ could be used to verify (2), we establish (2) below using the equivalent form (6) for (2):

$$n \leq \left(1 + \frac{1}{\sqrt{n}}\right)^n \tag{6}$$

for each positive integer n. If $x \geq 1$, there is a positive integer m such that $m \leq x < m + 1$. For such an m we have

$$(3/2)^m \leq \left(1 + \frac{1}{m+1}\right)^{m^2} < \left(1 + \frac{1}{x}\right)^{x^2},$$

since

$$\left(1 + \frac{1}{m+1}\right)^m$$

is strictly increasing, as can be seen easily by analyzing the expression obtained by multiplying and dividing

$$\left(1 + \frac{1}{m+1}\right)^m \text{ by } \left(1 + \frac{1}{m+1}\right).$$

We have by induction that $(3/2)^m \geq (m + 1)^2$ for all $m \geq 14$. We note that 14 is the smallest positive integer satisfying this property. We conclude that

$$x^2 < \left(1 + \frac{1}{x}\right)^{x^2}$$

for all $x \geq 14$. We find that (6) holds for every positive integer $m \geq 196$. We have, by direct computation, that each of the integers $1, 2, \ldots, 195$ satisfies (6). (The difference $(1 + 1/\sqrt{n})^n - n$ strictly increases from 1 at $n = 1$ to approximately 719727 at $n = 195$.)

We should point out that we have exhibited

$$x < \left(1 + \frac{1}{\sqrt{x}}\right)^x$$

for all real $x \geq 196$.

Following a suggestion of Rudin, we note that if f is the function defined by $f(x) = \ln(1 + x) - x + x^2/2$, then

(a) f is increasing on $[0, \infty)$ and

(b) $f(0) = 0$

(observe that $x - x^2/2$ is the sum of the second and third terms of the Taylor expansion of $\ln(1 + x)$ about 0). It follows that

$$x \ln\left(1 + \frac{1}{\sqrt{x}}\right) - \ln x > \sqrt{x} - \frac{1}{2} - \ln x.$$

Since the minimum value of the expression on the right hand side of the last inequality is $3/2 - \ln 4$ which is positive, (3) holds.

2. Some Properties of Ω. The first item in this section is a verification that functions in Ω are continuous. This will be accomplished by establishing (1°)–(3°) in succession:

(1°) If $f \in \Omega$, then $|y| + n|x - y| \in D(f)$ for all $x, y \in D(f)$ and $n \geq 0$.

(2°) If $f \in \Omega$ and $x_0, x_1, \ldots$ is an increasing sequence of equally spaced nonnegative reals in $D(f)$, i.e., $\{x_n - x_{n-1}\}$ is a constant sequence, then

(a) $f(x_n) - f(x_{n-1}) \geq f(x_1) - f(x_0)$ for all n.

(b) $f(x_n) - f(x_0) \geq n(f(x_1) - f(x_0))$ for all n.

(3°) If $f \in \Omega$, then f is continuous.

To show that (1°) and (2°) are valid, we use induction on n. As for (1°), let $x, y \in D(f)$. Since $f \in \Omega$, $|y|, |y| + |x - y| \in D(f)$. If $|y| + m|x - y| \in D(f)$ for each nonnegative $m \leq n$, then $|y| + (n + 1)|x - y| \in D(f)$, since

$$|y| + (n + 1)|x - y| = ||y| + n|x - y|| + ||y| + n|x - y| - (|y| + (n - 1)|x - y|)|.$$

Parts (a) and (b) of (2°) clearly hold for $n = 1$. Suppose n is an integer for which (a) is true, i.e., for which $f(x_n) - f(x_{n-1}) \geq f(x_1) - f(x_0)$. Then

$$f(x_{n+1}) - f(x_n) = f(x_n + |x_n - x_{n-1}|) - f(x_n) \geq f(x_n) - f(x_{n-1}) \geq f(x_1) - f(x_0).$$

The proof of (a) is complete. If (b) holds for n, then

$$f(x_{n+1}) - f(x_0) = (f(x_{n+1}) - f(x_n)) + (f(x_n) - f(x_0)) \geq (n+1)(f(x_1) - f(x_0))$$

from part (a). This completes the demonstration that (1°) and (2°) are valid. Now for (3°), suppose that f is not constant and that y is a non-isolated point of $D(f)$. We will show that f is continuous at y. We show first that for each $\epsilon > 0$, there is a $z \in D(f)$ satisfying $z > |y|$ and $f(z) < f(|y|) + \epsilon$. Assume $\epsilon > 0$ and there is no such z. Since f is not constant, there is a $v > |y|$ such that $f(v) > f(|y|)$. Choose a positive integer m such that $m\epsilon > f(v) - f(|y|)$ and an $x \in D(f)$ such that $x \neq y$ and $m|x - y| < v - |y|$. Then $|y| + m|x - y| \in D(f)$ from (1°) and (2°) yields

$$f(v) - f(|y|) \geq f(|y| + m|x - y|) - f(|y|) \geq m(f(|y| + |x - y|) - f(|y|)),$$

so $m\epsilon > f(v) - f(|y|) \geq m\epsilon$, a contradiction. Now, if $\epsilon > 0$, choose $z \in D(f)$ such that $z > |y|$, $f(z) < f(|y|) + \epsilon$ and let $\delta = z - |y|$. Then, if $x \in D(f)$ and $|x - y| < \delta$, it follows that

$$|f(x) - f(y)| \leq f(|x - y| + |y|) - f(|y|) \leq f(\delta + |y|) - f(|y|) = f(z) - f(|y|) < \epsilon.$$

Hence, f is continuous at y.

It is fairly obvious, from the arguments made to establish (3°) that, if f is strictly increasing and $[f(0), \infty)$ is a subset of the range of f, for $\epsilon > 0$, we may take $0 < \delta \leq f^{-1}[f(|y|) + \epsilon] - |y|$; in particular, for any $\epsilon > 0$ and any fixed y, we have

$$|\exp(x) - \exp(y)| < \epsilon, \quad \text{if} \quad |x - y| < \ln[\exp(|y|) + \epsilon] - |y|.$$

Interestingly, any continuous function f satisfying the condition in (a) under 2° for any increasing sequence x_0, x_1, … of equally spaced nonnegative reals in $D(f)$ also satisfies

(4°) If w, x, y, and z are nonnegative reals satisfying $w < x < z$, $x - w = z - y$, and $[w, x] \subset D(f)$. Then $f(z) - f(y) \geq f(x) - f(w)$.

We show first that this assertion is true when w, x, y, and z are rationals. To this end, let the function f and rationals w, x, y, and z satisfy the hypothesis of (4°) and let m be a positive integer such that mx, my, and mz are integers; let P be a partition of $[w, z]$ into intervals of length $1/m$. If $P = \{w + (j-1)/m : j = 1, 2, \ldots, m(z - w) + 1\}$, then for each integer $1 \leq k \leq m(x - w)$, the sequence in P

(in order of magnitude) beginning at $w + (k-1)/m$ and terminating at $y + k/m$ is an increasing sequence of equally spaced terms in $D(f)$;

$$f(y + k/m) - f(y + (k-1)/m) \geq f(w + k/m) - f(w + (k-1)/m),$$

since f satisfies part (a) of (2°). Thus,

$$\sum_{k=1}^{m(x-w)} (f(y+k/m) - f(y+(k-1)m)) \geq \sum_{k=1}^{m(x-w)} (f(w+k/m) - f(w+(k-1)/m)).$$

Since $x-w = z-y$, we get $f(z)-f(y) \geq f(x)-f(w)$. Now, let the real numbers w, x, y, z and the function f satisfy the hypothesis of (4°) and let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative rationals with $w \leq a_n < b_n \leq x$, $y \leq c_n \leq y + (a_n - w)$, $a_n \to w$, $b_n \to x$. Let $d_n = c_n + b_n - a_n$. Then $a_n < b_n < d_n$, $b_n - a_n = d_n - c_n$, $[a_n, d_n] \subset D(f)$, so a_n, b_n, c_n, d_n and f satisfy the hypothesis of (4°) and $f(d_n) - f(c_n) \geq f(b_n) - f(a_n)$. By continuity, we have $f(z) - f(y) \geq f(x) - f(w)$.

From part (b) of (2°), we observe that if $f \in \Omega$, $a < b$ and $a, (a+b)/2, b \in D(f)$, then $f((a+b)/2) \leq (f(a)+f(b))/2$. This is equivalent to f being convex on $[c, \infty)$, if $c \geq 0$ and $[c, \infty) \subset D(f)$ (see [5]).

It is not difficult to show that $f + g, \lambda f \in \Omega$, whenever $f, g \in \Omega$ and $\lambda \geq 0$. We will now establish that if $f, g \in \Omega$, $f(0) \geq 0$, and $g(0) \geq 0$, then the product $fg \in \Omega$. We observe first that $f(0) \geq 0$ if and only if $|f(x)| \leq f(|x|)$ for each $x \in D(f)$. Now if $x, y \in D(fg)$, then

$$\begin{aligned}
&|fg(x) - fg(y)| \leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\
&\leq f(|x|)(g(|x-y| + |y|)) - g(|y|)) + g(|y|)(f(|x-y| + |y|) - f(|y|)) \\
&\leq (f(|x-y| + |y|)(g(|x-y| + |y|) - g(|y|)) + g(|y|)(f(|x-y| + |y|) - f(|y|)) \\
&\leq fg(|x-y| + |y|) - fg(|y|).
\end{aligned}$$

We close with two examples and an observation for complex-valued functions of a complex variable.

Example 1. The function f defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 0 \\ x^3, & \text{if } x \geq 0 \end{cases}$$

is differentiable, increasing, and convex on $[0,\infty)$; however, $f \notin \Omega$, since if $x = -1/2$ and $y = -1/4$, we have $|f(x) - f(y)| = 12/64$ and $f(|x-y| + |y|) - f(|y|) = 7/64$.

Example 2. If f and g are defined by $f(x) = x^3 - 1$ and $g(x) = x^2$, then $f, g \in \Omega$, while $fg \notin \Omega$.

We point out that $(*)$ and (1) are also valid if x and y are complex numbers and that any complex-valued function of a complex variable satisfying (4) is continuous.

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