SOME IMPROVED RESULTS ON B-RINGS

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Abstract. In this work, all rings are commutative rings with identity. Let R be a ring and $J(R)$ denote the Jacobson radical of R. A sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq 1$, of elements in R is said to be unimodular if 1 is in the ideal $(a_1, a_2, \ldots, a_s, a_{s+1})$. A ring R is said to be a B-ring, whenever for any unimodular sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq 2$, of elements in R with $(a_1, a_2, \ldots, a_{s-1}) \not\subset J(R)$, there exists an element b in R such that $(a_1, a_2, \ldots, a_s + ba_{s+1}) = R$. Besides some other results, two basic facts about B-rings will be proved and they are: 1) R is a B-ring if and only if for any unimodular sequence (a_1, a_2, a_3) of elements in R with a_1 not in $J(R)$, there exists an element b in R such that $(a_1, a_2 + ba_3) = R$, and 2) A homomorphic image of a B-ring is again a B-ring. At the end, some applications of these results will be discussed.

1. Introduction. The main purpose of this paper which will be discussed in section 2 of this work, is to provide a simpler alternative proof to some results of section 2 in [1] and also to generalize some results of that section in [1].

Throughout this paper all rings are commutative rings with identity. For a ring R, let $J(R)$ denote the Jacobson radical of R and define a sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq 1$, of elements in R to be unimodular, whenever 1 is in the ideal $(a_1, a_2, \ldots, a_s, a_{s+1})$. Notice that without having any confusion in the context, parentheses will be used to show both the sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$ of elements in R and the ideal $(a_1, a_2, \ldots, a_s, a_{s+1})$ generated by $a_1, a_2, \ldots, a_s, a_{s+1} \in R$. A ring R is said to be a B-ring, whenever for any unimodular sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq 2$, of elements in R with $(a_1, a_2, \ldots, a_{s-1}) \not\subset J(R)$, there exists an element b in R such that $(a_1, a_2, \ldots, a_s + ba_{s+1}) = R$. For more information on B-rings see [1] and also for a generalization of B-rings (B-type-rings) see the dissertation of the author [2].

2. Main Results.

Theorem 1. Let $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq 1$, be a unimodular sequence in R, then $(a_1, a_2, \ldots, a_s + a_{s+1}) = R$, whenever a_s or a_{s+1} is in $J(R)$.

Proof. Here we just give a proof to the case when $a_s \in J(R)$ and the proof of the other case which is similar, is left to the reader. Assume $A = (a_1, a_2, \dots, a_s + a_{s+1}) \neq R$. Then there exists a maximal ideal M of R such that $A \subset M$. Now $a_s \in J(R) \subset M$ and $a_s + a_{s+1} \in M$ imply $a_{s+1} \in M$ and this makes $R = (a_1, a_2, \dots, a_{s+1}) \subset M$, which is a contradiction.

Theorem 2. A ring R is a B-ring if and only if for any unimodular sequence (a_1, a_2, a_3) in R with $a_1 \notin J(R)$, there exists an element $b \in R$ such that $(a_1, a_2 + ba_3) = R$.

Proof. The necessary part is quite clear. To prove the sufficient part, let $(a_1, a_2, \dots, a_s, a_{s+1})$, $s \geq 2$, be a unimodular sequence in R with the condition $(a_1, a_2, \dots, a_{s-1}) \not\subset J(R)$. Without loss of generality, assume $a_1 \notin J(R)$. Now, $1 \in (a_1, a_2, \dots, a_s, a_{s+1})$ implies $1 = \sum_{i=1}^{s+1} a_i x_i$ for some $x_1, x_2, \dots, x_s, x_{s+1} \in R$. Thus, $1 \in (a_1, a_s, l)$, where $l = a_2x_2 + a_3x_3 + \cdots + a_{s-1}x_{s-1} + a_{s+1}x_{s+1}$. Now by the hypothesis there exists $b \in R$ such that $1 \in (a_1, a_s + bl) \subset (a_1, a_2, \dots, a_{s-1}, a_s + bx_{s+1}a_{s+1})$.

Theorem 3. A homomorphic image of a B-ring is a B-ring.

Proof. Let A be a nonzero proper ideal of R and $\phi\colon R \to R/A$ be the canonical epimorphism of rings. Assume $(a_1 + A, a_2 + A, \dots, a_s + A, a_{s+1} + A)$ is a unimodular sequence in R/A with $(a_1 + A, a_2 + A, \dots, a_{s-1} + A) \not\subset J(R/A)$, where $s \geq 2$. $1 + A \in (a_1 + A, a_2 + A, \dots, a_s + A, a_{s+1} + A)$ implies $1 = \sum_{i=1}^{s+1} a_i x_i + a$ for some $a \in A$ and $x_1, x_2, \dots, x_s, x_{s+1} \in R$. Now $1 \in (a_1, a_2, \dots, a_s, a_{s+1}x_{s+1} + a)$ and since $\phi(J(R))$ is contained in $J(R/A)$, then by the hypothesis there exists $b \in R$ such that $(a_1, a_2, \dots, a_s + bx_{s+1}a_{s+1} + ba) = R$, which implies $(a_1 + A, a_2 + A, \dots, a_s + bx_{s+1}a_{s+1} + A) = R/A$.

Remark. By applying Theorem 2 in the proof of the above theorem, it suffices to make the argument only for the unimodular sequences of the form $(a_1 + A, a_2 + A, a_3 + A)$ with $a_1 + A \notin J(R/A)$.

3. Some Applications of the Above Results. For a ring R and any element $a \in R$, let $Z(a)$ denote the set of all maximal ideals of R, where each contains the element a. A ring R is said to be Bezoutian or an F-ring, if every finitely generated ideal in R is a principal ideal in R.

Remark. R in Lemma 2.2 of [1] does not need to be an F-ring and actually, as a stronger alternative of this lemma, we can apply Theorem 2 above. Consequently, in Theorem 2.4 of [1] there is no need to show or mention that R is an F-ring. Also, we exclude the Bezoutian condition from the hypothesis of Theorem 2.3 in [1] and state it in the following general form.

Theorem 4. If R is a ring which satisfies the condition that for every $a, c \in R$ with $a \notin J(R)$, there exists an $r \in R$ such that $Z(R) = Z(a) - Z(c)$, then R is a B-ring.

Proof. The proof follows from Theorem 2 above and the argument in the proof of Theorem 2.3 in [1].

Next, by applying Theorem 3 above, a simpler alternative proof can be provided to Theorem 2.6 and to the necessity part of Lemma 2.1 in [1].

Theorem 5. Let D be an integral domain, K its quotient field. Let $R = \{(a_1, a_2, \ldots, a_k, a, a, \ldots) \mid a_i \in K, a \in D\}$, where k is a nonnegative integer (k may be different for distinct elements in R). The operations in R are component-wise addition and multiplication. If R is a B-ring, then D is a B-domain.

Proof. Since D is a homomorphic image of R under the mapping given by $(a_1, a_2, \ldots, a_k, a, a, \ldots) \mapsto a$, the proof is immediate by Theorem 3.

Remark. By Theorem 2 above, the illustration of the proof of Theorem 2.6 in [1] is actually a complete proof of that theorem.

Finally, as an application of Theorem 3 to the necessity part of Lemma 2.1 in [1], which states "for any nonzero proper ideal $A \subset J(R)$, R is a B-ring if and only if R/A is a B-ring," we can conclude that R/A is a B-ring for any nonzero proper ideal A of R, whenever R is a B-ring.

References

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