A DEVELOPMENT OF EULER NUMBERS

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The Euler polynomials, $\{E_n(x)\}_{n=0}^{\infty}$, are the unique polynomials with the property

$$x^n = \frac{E_n(x+1) + E_n(x)}{2}, \ x \in \mathbb{R}.$$

This definition leads to the differential equation $(E_n' - nE_{n-1})(x) = 0$, and hence, to a complete determination of the $E_n(x)$'s. It is known that when n is odd, $1/2$ is a zero of $E_n(x)$ [7]. The sequence of Euler numbers, $\{E_{2n}\}_{n=0}^{\infty}$, is then defined by

$$E_{2n} = 2^{2n} E_{2n}(1/2).$$

The first 8 Euler numbers are given in Table 1 [5]. In general, the E_n's increase continuously and rapidly in magnitude, and their algebraic signs alternate.

It is of some practical interest to have a closed form for the Euler numbers. The closely related Bernoulli numbers, for example, can each be represented by an infinite series of reciprocals of various integral powers of the positive integers

$$B_{2n} = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} 1/k^{2n}.$$

n	E_{2n}
0	1
1	-1
2	5
3	-61
4	1385
5	$-50,521$
6	$2,702,765$
7	$-199,360,981$

Table 1. The First 8 Euler Numbers.

Methods for proving this commonly employ non-elementary facts from analysis [3]; for example, one such proof begins with the infinite product formula for $\sin x$

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty}[1 - (x^2/k^2\pi^2)], \quad |x| < \infty.$$

There are similar non-elementary methods for deriving analogous closed forms for the Euler numbers. In this paper we show how only calculus can be used to obtain infinite series representations of the Euler numbers. Our treatment is motivated by Berndt's work on the Bernoulli numbers [4].

1. The Even-Indexed Euler Polynomials. We shall need one basic fact about the polynomials $E_{2n}(x)$. From the definition of the Euler polynomials, one can obtain a generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

In this, replacing t by $-t$ and setting $x = 0$, we obtain

$$\sum_{n=0}^{\infty} E_n(0)\frac{(-t)^n}{n!} = \frac{2}{e^{-t}+1} = \frac{2e^t}{e^t+1} = \sum_{n=0}^{\infty} E_n(1)\frac{t^n}{n!}.$$

Within the interval of convergence, we can subtract the two series in the last line term-by-term to give

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\left(E_n(1) - E_n(0)(-1)^n\right) = 0.$$

As t is arbitrary, we obtain for even indices $E_{2n}(1) - E_{2n}(0) = 0$. But $E_{2n}(1+x) + E_{2n}(0) = 2x^{2n}$ and for $x = 0$ this gives $E_{2n}(1) + E_{2n}(0) = 0$. Combination with the previous equation yields, finally, $E_{2n}(0) = E_{2n}(1) = 0$.

2. Two Lemmas. The general idea of the proof is to concoct a special function of $E_{2n}(x)$ and E_{2n}, integrate this function together with $\sin(\pi kx)$ ($k =$

integer), and then sum over some suitable subset of the k's. The next two lemmas are required.

Lemma 1. If j is a positive integer and $\cos\theta \neq 0$, then

$$S = \sin\theta - \sin 3\theta + \sin 5\theta - \cdots + \sin(4j+1)\theta = \frac{\sin(4j+2)\theta}{2\cos\theta}.$$

Proof.

$$\begin{aligned}
S &= \operatorname{Im}\big(e^{i\theta} - e^{3i\theta} + e^{5i\theta} - \cdots + e^{(4j+1)i\theta}\big) \\
&= \operatorname{Im}\left(\frac{e^{i\theta}(1 - \big(-e^{2i\theta})^{2j+1}\big)}{1 - (-e^{2i\theta})}\right) \\
&= \operatorname{Im}\left(\frac{e^{(4j+3)i\theta} + e^{i\theta}}{e^{2i\theta}+1} \cdot \frac{e^{-2i\theta}+1}{e^{-2i\theta}+1}\right) \\
&= \operatorname{Im}\left(\frac{e^{(4j+1)i\theta} + e^{-i\theta} + e^{(4j+3)i\theta} + e^{i\theta}}{2 + 2\cos 2\theta}\right) \\
&= \frac{\sin(4j+1)\theta - \sin\theta + \sin(4j+3)\theta + \sin\theta}{2(1+\cos 2\theta)} \\
&= \frac{2\sin(4j+2)\theta\cos\theta}{4\cos^2\theta} \\
&= \frac{\sin(4j+2)\theta}{2\cos\theta}.
\end{aligned}$$

The next lemma is a special case of a more general theorem known in the literature as the Riemann-Lebesgue Lemma [1, 2, 11].

Lemma 2. If $F(\theta)$ is twice continuously differentiable on $[a,b]$, $F(a) = 0$, and $\cos\theta$ has no zero in $(a,b]$, then

$$\lim_{N\to\infty} \int_a^b F(\theta)\frac{\sin N\theta}{\cos\theta} d\theta = 0.$$

Proof. Let $g(\theta) = F(\theta)/\cos\theta$. Then

$$(*) \qquad \int_a^b g(\theta)\sin N\theta d\theta = -g(\theta)\frac{\cos N\theta}{N}\Bigg|_a^b + \int_a^b g'(\theta)\frac{\cos N\theta}{N}d\theta.$$

If $\cos\theta \neq 0$ at $\theta = a$, then $g(\theta)$ is finite there. By hypothesis, $g(\theta)$ is finite at $\theta = b$ also, so the integrated term in $(*)$ approaches 0 as $N \to \infty$.

If $\cos a = 0$, then by L'Hôpital's Rule

$$\lim_{\theta\to a^+} g(\theta) = \lim_{\theta\to a^+}\frac{F'(\theta)}{-\sin\theta},$$

provided the latter limit exists. But this limit does exist because $F(\theta)$ is continuously differentiable at $\theta = a$ and $\sin a \neq 0$. Hence, the integrated term in $(*)$ again approaches 0 as $N \to \infty$.

Again, if $\cos\theta \neq 0$ at $\theta = a$, then

$$g'(\theta) = \frac{\cos\theta F'(\theta) + \sin\theta F(\theta)}{\cos^2\theta}$$

is defined there and the integrand on the right-hand side of $(*)$ is continuous on $[a, b]$. Hence, the integral approaches 0 as $N \to \infty$.

On the other hand, if $\cos a = 0$, we have

$$\begin{aligned}\lim_{\theta\to a^+} g'(\theta) &= \lim_{\theta\to a^+}\frac{-\sin\theta F'(\theta) + \cos\theta F''(\theta) + \cos\theta F(\theta) + \sin\theta F'(\theta)}{-2\cos\theta\sin\theta}\\ &= \lim_{\theta\to a^+}\frac{F(\theta) + F''(\theta)}{-2\sin\theta}\\ &= -\frac{1}{2}\frac{F''(a)}{\sin a},\end{aligned}$$

because $F(\theta)$ is twice continuously differentiable on $[a, b]$ and $\sin a \neq 0$. So the integrand in $(*)$ is again continuous on $[a, b]$ and the integral approaches 0 as $N \to \infty$.

3. The Main Theorem. To begin the proof of the main result, we define the following function

$$G(x) = \frac{2^{2n}E_{2n}(x)}{E_{2n}} - 1,$$

and we choose the interval of integration to be $1/2 \le x \le 1$. We note that $G(1/2) = 0$, and also that $G(1) = -1$, in view of the discussion in Section 1. We also recall (from the opening remarks) that $E_n'(x) = nE_{n-1}(x)$.

Theorem 1. If n is a positive integer, then

$$E_{2n} = 2(-1)^n(2/\pi)^{2n+1}(2n)!\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^{2n+1}}.$$

Proof. Let K be an odd integer; integration by parts gives

$$\int_{1/2}^{1} G(x)\sin\pi Kx dx = -\frac{\cos\pi Kx}{\pi K}G(x)\Big|_{1/2}^{1} + \frac{2^{2n}}{E_{2n}}\int_{1/2}^{1}\frac{\cos\pi Kx}{\pi K}E_{2n}'(x)dx$$

$$= \frac{-1}{\pi K} + \frac{2^{2n}(2n)}{\pi K E_{2n}}\int_{1/2}^{1}E_{2n-1}(x)\cos\pi Kx dx$$

$$= \frac{-1}{\pi K} + \frac{2^{2n}(2n)}{\pi K E_{2n}}\left(\frac{\sin\pi Kx}{\pi K}E_{2n-1}(x)\Big|_{1/2}^{1} - \int_{1/2}^{1}\frac{\sin\pi Kx}{\pi K}E_{2n-1}'(x)dx\right)$$

$$= \frac{-1}{\pi K} - \frac{2^{2n}(2n)(2n-1)}{(\pi K)(\pi K)E_{2n}}\int_{1/2}^{1}E_{2n-2}(x)\sin\pi Kx dx.$$

After $2n-2$ more integrations by parts, we obtain

$$\int_{1/2}^{1} G(x)\sin\pi Kx dx = \frac{-1}{\pi K} + \frac{(-1)^n 2^{2n}(2n)!}{(\pi K)^{2n}E_{2n}}\int_{1/2}^{1}E_0(x)\sin\pi Kx dx$$

$$= \frac{-1}{\pi K} + \frac{(-1)^n 2^{2n}(2n)!}{(\pi K)^{2n+1}E_{2n}},$$

because $E_0(x) = 1$.

Now let $K = K(k) = (-1)^{(k-1)/2}k$, and sum both sides of the last equation over all the *odd* k's from $k = 1$ to $k = 4j + 1$

$$\int_{1/2}^{1} G(x)\Bigg(\sum_{\substack{k=1\\k=odd}}^{4j+1} \sin \pi K x\Bigg) dx = -\frac{1}{\pi}\sum_{\substack{k=1\\k=odd}}^{4j+1} \frac{1}{K} + \frac{(-1)^n 2^{2n}(2n)!}{E_{2n}} \sum_{\substack{k=1\\k=odd}}^{4j+1} \frac{1}{(\pi K)^{2n+1}}.$$

Application of Lemma 1 to this yields

$$\frac{1}{2}\int_{1/2}^{1} G(x)\frac{\sin(4j+2)\pi x}{\cos \pi x} dx = -\frac{1}{\pi}\sum_{\substack{k=1\\k=odd}}^{4j+1} \frac{1}{K} + \frac{(-1)^n 2^{2n}(2n)!}{E_{2n}} \sum_{\substack{k=1\\k=odd}}^{4j+1} \frac{1}{(\pi K)^{2n+1}}.$$

On the left-hand side, let $\theta = \pi x$, $F(\theta) = G(x)$, $N = 4j + 2$; the integral becomes

$$\frac{1}{2\pi}\int_{\pi/2}^{\pi} F(\theta)\frac{\sin N\theta}{\cos\theta} d\theta.$$

The integrand satisfies all of the hypotheses in Lemma 2, so upon passage to the limit $j \to \infty$, we obtain

$$0 = -\frac{1}{\pi}\sum_{\substack{k=1\\k=odd}}^{\infty} \frac{1}{K} + \frac{(-1)^n 2^{2n}(2n)!}{E_{2n}} \sum_{\substack{k=1\\k=odd}}^{\infty} \frac{1}{(\pi K)^{2n+1}}.$$

The first series is just Gregory's series for $\mathrm{Tan}^{-1} 1$ [10]. In the second series, let $k = 2j + 1$ and replace K by $(-1)^j(2j + 1)$. We obtain

$$0 = -\frac{1}{4} + \frac{(-1)^n 2^{2n}(2n)!}{\pi^{2n+1} E_{2n}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^{2n+1}}$$

and the theorem follows by a simple rearrangement.

The theorem permits an asymptotic estimate of the E_{2n}'s and also of their rate of growth.

Corollary. As $n \to \infty$, we have $|E_{2n}| \sim 8\sqrt{(n/\pi)}(4n/(\pi e))^{2n}$ and

$$|E_{2n+2}/E_{2n}| \sim (4(n+1)/\pi)^2.$$

Proof. Let $Z_n = \sum_{j=0}^{\infty}(-1)^j/(2j+1)^{2n+1}$. Clearly $1 - 1/3^{2n+1} < Z_n < 1$, so $Z_n \sim 1$ as $n \to \infty$. From Stirling's approximation [6, 8, 9] we have $(2n)! \sim \sqrt{2\pi(2n)}(2n)^{2n}e^{-2n}$ as $n \to \infty$. Hence, using these asymptotic results in the series given in Theorem 1, we obtain

$$\begin{aligned} |E_{2n}| &\sim 2\sqrt{2\pi(2n)}(2n)^{2n}e^{-2n}(2/\pi)^{2n+1} \\ &\sim 8(n/\pi)^{1/2}(4n/(\pi e))^{2n}. \end{aligned}$$

Finally, making use of the result $((n+1)/n)^{2n} \sim e^2$, we have

$$\begin{aligned} |E_{2n+2}/E_{2n}| &\sim \sqrt{(n+1)/n}(4(n+1)/(\pi e))^{2n+2}(\pi e/4n)^{2n} \\ &\sim \frac{16(n+1)^2}{\pi^2} \\ &= \left(\frac{4(n+1)}{\pi}\right)^2. \end{aligned}$$

n	$\lvert E_{2n+2}/E_{2n}\rvert$	$(4(n+1)/\pi)^2$
1	5	6.48
3	22.71	25.94
5	53.50	58.36
7	97.27	103.75

Table 2. Asymptotic Estimation of the Growth of the Euler Numbers.

Table 2 shows how well the asymptotic formula agrees with the exact values for the ratios $|E_{2n+2}/E_{2n}|$; at $n = 7$, for example, the percent error is about 6.7%. At $n = 29$, the percent error drops to about 1.7%.

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