A CHARACTERIZATION OF ULTRAFILTERS IN COMPLEMENTARY TOPOLOGIES

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Abstract. A topology τ on a non-empty set X is called a complementary topology if each open set U in τ, its complement $X - U$ is also in τ. These topologies, maximal ideals and their relation to ultrafilters were characterized by this author. In this paper, the structure of filters in τ are investigated. Finally, the ultrafilters are characterized.

1. Introduction. Let τ be a topology on a set X. Then τ is called a complementary topology (comp-topology) if for each $U \in \tau$, its complementary $X - U$ is also in τ. Since complementary topologies are the only topologies that form a Boolean ring under the usual operations, these spaces along with their maximal ideals have been characterized in [4]. This may be a result of some interest due to the many applications of Boolean rings to such fields as logic and switching circuits. In [5], the relation between maximal ideals in Boolean rings and ultrafilters in τ have been characterized. In this article, the structure of filters in τ are investigated. Finally, the ultrafilters in these topologies independent of ideals are characterized.

2. Complementary Topology. The following lemmas and theorem have been proved in [4] and [5].

Lemma 1. In a comp-topology, the intersection of an arbitrary collection of open sets is an open set.

Note that the converse of this lemma is not true. For example, see [5].

Lemma 2. If τ is a non-trivial comp-topology on a set X, then τ admits a unique basis $B(\tau)$ which forms a partition for the space X. This partition is called a *disjoint basis*.

Lemma 3. If the topology τ on a set X admits a basis which forms a partition for X, then τ is a comp-topology.

Lemmas 2 and 3 have been employed to prove the following theorem, which characterizes comp-topological spaces.

Theorem 1. Let τ be a non-trivial topology on a set X, then τ is a comp-topology if and only if τ admits a unique basis that forms a partition for the set X.

Example. Let X be the set of real numbers $\mathbb{R}$ and

$$B(\tau) = \{\big[n, n+1\big) \ : \ n \text{ is an integer } \}.$$

It is easy to see that X is a complementary topology with $B(\tau)$ as its unique basis.

3. Filters and Ultrafilters. Let τ be a non-trivial comp-topology on a set X with a disjoint basis

$$B(\tau) = \{U_\alpha \ : \ \alpha \in A\}.$$

We define a filter in τ as follows.

Definition 1. A non-empty subset F of the topology τ is called a comp-filter or simply filter if i) $U \subseteq V$ where $U, V \in \tau$ and $U \in F$, then $V \in F$; and ii) For any $U, V \in F$, $U \cap V \in F$. A filter is said to be an ultrafilter or a maximal filter if $F \subset \tau$ and F is not contained in any other filter in τ.

To characterize the filters in τ, we first observe that by Theorem 1, there is a unique disjoint basis

$$B(\tau) = \{U_\alpha \ : \ \alpha \in A\}$$

for τ, where A is an index set for the elements in the partition. Now, for each subcollection of indices $A_0 \subseteq A$, let $F(A_0)$ be the set consisting of all those $X - U_\alpha$'s with $\alpha \in A_0$ together with all the finite intersections of those $X - U_\alpha$'s.

Theorem 2. Let τ be a comp-topology on a set X and

$$B(\tau) = \{U_\alpha \ : \ \alpha \in A\}$$

be a disjoint base for τ. For each filter F in τ, there exists a unique subcollection A_0 of the index set A such that $F(A_0) \subseteq F$. The only sets that are possibly in F, but not in $F(A_0)$, are those that are intersections of the $X - U_\alpha$'s, for infinitely many α's in A_0. $F(A_0)$ will be called the associated set of F.

Proof. Let F be a filter and a subset of τ. Let

$$A_0 = \{\alpha \in A \ : \ U \cap U_\alpha = \emptyset \text{ for some } U \in F\}.$$

Since $U \cap U_\alpha = \emptyset$ for each $\alpha \in A_0$ and $U \subseteq X - U_\alpha$, we see that F contains all those $X - U_\alpha$'s with $\alpha \in A_0$. Therefore, $F(A_0) \subseteq F$. It is clear that the only sets that can possibly be in F, but not in $F(A_0)$, must be an intersection of $X - U_\alpha$'s, for infinitely many α in A. The following theorem will enable us to characterize the ultrafilters in τ.

Theorem 3. Let τ be a comp-topology on a set X and

$$B(\tau) = \{U_\alpha \ : \ \alpha \in A\}$$

be a disjoint basis for τ. Suppose α_0 is a fixed element of A. Then the set

$$F = \{X - U \ : \ U \in \tau \text{ and } U \cap U_{\alpha_0} = \emptyset\}$$

is an ultrafilter in τ.

Proof. First we show that F is a filter. It is clear that $\emptyset \notin F$. Let B and C be in τ such that $B \subseteq C$ and $B \in F$. Then, there exists a $U \in \tau$ such that $B = X - U$ and $U \cap U_{\alpha_0} = \emptyset$. We need to show that there exists a $V \in \tau$ such that $C = X - V$ for some V in τ and $V \cap U_{\alpha_0} = \emptyset$. Let $V = U - C = U \cap (X - C)$. By using De Morgan's Law it is easy to see that $X - V = B \cup C = C$. To show $V \cap U_{\alpha_0} = \emptyset$, we consider the fact that $V \cap U_{\alpha_0} = U \cap (X - C) \cap U_{\alpha_0} = \emptyset$. Let $B, C \in F$. Then there exists a $V, W \in \tau$ such that $B = X - V$, $C = X - W$. Then $B \cap C = X - (V \cup W)$, where $V \cup W \in \tau$ and $(V \cup W) \cap U_{\alpha_0} = \emptyset$. Therefore, F is a filter. To show that F is an ultrafilter, let F^* be a filter containing F. Let V be an arbitrary element in F^*. Either $V \cap U_{\alpha_0} = \emptyset$ or $V \cap U_{\alpha_0} \neq \emptyset$. If $V \cap U_{\alpha_0} = \emptyset$, then $X - V \in F \subset F^*$; which implies $\emptyset \in F^*$, a contradiction to the fact that F^* is a filter. So $V \cap U_{\alpha_0} \neq \emptyset$. But since U_{α_0} is a basis element from the partition, it follows that $U_{\alpha_0} \subseteq V$. Considering that $U_{\alpha_0} = X - (X - U_{\alpha_0})$ and $(X - U_{\alpha_0}) \cap U_{\alpha_0} = \emptyset$, it follows that $U_{\alpha_0} \in F$. Since F is a filter and $U_{\alpha_0} \subseteq V$, we can conclude that $V \in F$ and $F^* = F$. This shows that F is an ultrafilter.

We will now characterize all ultrafilters of τ whose associated sets are generated by the index sets A_0 that are proper subsets of A.

Theorem 4. Let τ be a comp-topology on a set X and

$$B(\tau) = \{U_\alpha \ : \ \alpha \in A\}$$

be a disjoint basis for τ. Suppose $F \subset \tau$ is a filter whose associated set corresponds to a proper subset A_0 of A. Then F is an ultrafilter of τ if and only if

i. A_0 misses exactly one element α_0 of A, and

ii. F contains all the possible unions of elements of the set

$$\{X - U_\alpha \ : \ \alpha \in A_0\}.$$

Condition (ii) may be replaced by the equivalent condition.

iii. F contains all the elements of τ that have a non-empty intersection with U_{α_0}.

Proof. Since τ is a comp-topology, it is easy to see that Conditions ii and iii are equivalent. Let F be a filter that satisfies the following two conditions.

I. Let $F(A_0)$ be the associated set of F, where the index set A_0 misses exactly one element α_0 of A.

II. F contains all the possible elements of the topology τ that are non-disjoint from U_{α_0}.

We need to show that F is an ultrafilter. If F^* is any filter containing F as a proper subset, then F^* must contain a set U such that $U \notin F$ and $U \cap U_{\alpha_0} = \emptyset$. But this implies $U \subset X - U_{\alpha_0}$. Since F^* is a filter, then $X - U_{\alpha_0} \in F^*$. By (II), $U_{\alpha_0} \in F \subset F^*$. This contradicts the fact that $\emptyset \notin F^*$. Therefore, F is an ultrafilter.

Conversely, let F be an ultrafilter with the associated set $F(A_0)$ such that $A - A_0$ contains more than one element or F does not contain all possible $X - U_\alpha$'s. Fix an element α_0 in $A - A_0$ and let $A_1 = A - \{\alpha_0\}$. Suppose F^* is the subset of τ containing all possible unions of elements of the set

$$\{X - U_\alpha \ : \ \alpha \in A_1\}.$$

It is obvious that F^* contains F as a proper subset. Now it suffices to show that F^* is a filter. Replacing Condition II by the equivalent Condition iii, then F^* contains all the possible elements of the topology τ that are non-disjoint from U_{α_0}. It is straight forward to verify that F^* is a filter in τ by using the fact that for any $U \in \tau$, if $U \cap U_{\alpha_0} \neq \emptyset$, then $U_{\alpha_0} \subseteq U$. Therefore, if A_0 is a proper subset of A and if either condition I or II is not satisfied, then F is not an ultrafilter of τ.

Corollary. Let τ be a comp-topology with a disjoint basis

$$B(\tau) = \{U_\alpha \ : \ \alpha \in A\}.$$

If the index set A is finite consisting of n elements, then τ has exactly 2^n distinct filters (including trivial ones) and among these, exactly n of them are ultrafilters.

Proof. If the index set A is finite, it is easy to show that for each subcollection of indices $A_0 \subset A$, $F(A_0) \cup \{X\}$ is a filter. Moreover if F is an arbitrary filter in τ, Theorem 2 guarantees a unique subcollection A_0 of the index set A such that $F(A_0) \subset F$. Thus, there is a one-to-one correspondence between the filters of τ and the subsets of A with a filter F corresponding to a subset $A_0 \subset A$. As for ultrafilters, we merely observe that each ultrafilter F of τ must be equal to its associated set $F(A_0)$ where A_0 is a proper subset of A. Thus, by Theorem 4, there is a one-to-one correspondence between the set of all ultrafilters of τ and the index set A.

For the case in which the index set is infinite and ultrafilters are those as described in Theorem 3, it is easy to see that there are infinitely many ultrafilters in this topology.

The following theorem is true in the case that the index set is infinite. Before we state and prove the theorem, we need a definition.

Definition. Let τ be a comp-topology with a disjoint basis

$$B(\tau) = \{U_\alpha \ : \ \alpha \in A\}.$$

For each subset P of the index set A, we will let

$$U(P) = \bigcup \{U_\alpha \ : \ \alpha \in P\}$$

and call it the open set determined by P.

Lemma 4. Let τ be a comp-topology with a disjoint basis

$$B(\tau) = \{U_\alpha \ : \ \alpha \in A\}.$$

Let F be an ultrafilter in τ. Suppose that $U(P)$ is an open set determined by an infinite subset P of the index set A such that the set $U(P)$ belongs to F. Then, for any partition $P = R \cup S$ of P into infinite subsets R and S, the filter F contains exactly one of the open sets $U(R)$ and $U(S)$.

Proof. It is easy to see that F does contain both $U(R)$ and $U(S)$, for otherwise, being a filter, F would have contained $U(R) \cap U(S) = \emptyset$. We show that if F contains neither $U(R)$ nor $U(S)$, F would not have been an ultrafilter. Suppose F contains neither $U(R)$ nor $U(S)$. Let F^* be a filter generated by F and $U(R)$, i.e.,

$$F^* = \{U \ : \ U = B \cup C, \ B \in F \text{ and } C \subseteq U(R)\}.$$

It is easy to check that F^* is a filter. For example, if $U \subseteq V$ and $U \in F^*$, then there is a $B \in F$ and $C \subseteq U(R)$ such that $U = B \cup C \subseteq V$. But $B \cup C \subseteq V$ implies that $B \subseteq V$ and $C \subseteq V$. Since F is also a filter, it follows that $V \in F$ and $V = V \cup C \in F^*$. By taking $C = \emptyset$, we see that $F^* \supset F$. Thus, F^* is a filter containing F as a proper subset. Thus, F cannot be an ultrafilter in τ.

Theorem 5. Let τ be a comp-topology on a set X with a disjoint basis

$$B(\tau) = \{U_\alpha \ : \ \alpha \in A\}.$$

If the index set A is infinite, then there are infinitely many distinct ultrafilters of τ that contain $F(A)$.

Proof. Since τ is a comp-topology, it is not difficult to show that the set F consisting of all those $X - U_\alpha$'s with $\alpha \in A$ together with all the finite intersections of those $X - U_\alpha$'s and the set X is a filter containing $F(A)$.

By using Zorn's Lemma, there is at least one ultrafilter F that contains $F(A)$. Furthermore, since F is also a filter, it does contain the set

$$X = \bigcup \{U_\alpha \ : \ \alpha \in A\} = U(A).$$

We now show that by the following argument, there are at least two ultrafilters that contain $F(A)$. Partition the index set A as a union of two disjoint, infinite subsets A_0 and A_1. By the preceding lemma, the ultrafilter F must contain exactly one of the two sets $U(A_0)$ and $U(A_1)$. Without loss of generality, assume that $U(A_0) \notin F$. We now relabel F as F_0 and show that there is a second ultrafilter F_1 that contains $F(A)$ and the set $U(A_0)$, but not $U(A_1)$. First, let F_1^* be the filter generated by $F(A)$ and the set $U(A_0)$. Then apply Zorn's Lemma to obtain an ultrafilter F_1 containing the filter F_1^*. Since the ultrafilter F_1 already contains the set $U(A_0)$, by the preceding lemma, F_1 does not contain the set $U(A_1)$. Thus, F_0 and F_1 are two distinct ultrafilters. Applying the same argument on the sets $U(A_0)$ and $U(A_1)$, we now show that there are at least four ultrafilters containing $F(A)$.

First partition the set A_0 into the union of two disjoint infinite subsets A_{00} and A_{01}. As before, the ultrafilter F_0 must contain exactly one of the sets $U(A_{00})$ and $U(A_{01})$, say, F_0 does not contain the set $U(A_{00})$. Relabeling F_0 as F_{00} and using Zorn's Lemma again, we can construct another ultrafilter F_{01} containing $F(A)$, but not the sets $U(A_0)$ and $U(A_{01})$. Likewise, applying the same argument on a partition of $A_1 = A_{10} \cup A_{11}$, we can show that there exist two ultrafilters F_{10}, which does not contain $U(A_{10})$ and F_{11}, which does not contain $U(A_{11})$ (and F_1 is one of these two ultrafilters).

Since this argument can be repeated as many times as we wish, there must be infinitely many such ultrafilters.

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