SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

88. [1995, 140; 1996, 160–167] *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Let m be a positive integer. Prove that

$$\prod_{i=1}^{m} \cos^2 \frac{i\pi}{2m+1} = \frac{1}{4^m}.$$

Comment by Thomas C. Leong, The City College of City University of New York, New York, New York.

The discussion of Problem 232 in D. O. Shklarsky, N. N. Chentzov and I. M. Yaglom, *The USSR Olympiad Problem Book*, Dover Publications, New York, 1993, gives several solutions to show that

$$\prod_{i=1}^{m} \sin \frac{i\pi}{2m+1} = \frac{\sqrt{2m+1}}{2^m}, \quad \prod_{i=1}^{m-1} \sin \frac{i\pi}{2m} = \frac{\sqrt{m}}{2^{m-1}},$$

$$\prod_{i=1}^{m} \cos \frac{i\pi}{2m+1} = \frac{1}{2^m}, \quad \prod_{i=1}^{m-1} \cos \frac{i\pi}{2m} = \frac{\sqrt{m}}{2^{m-1}}.$$

89. [1996, 36] *Proposed by Stanley Rabinowitz, MathPro Press, Westford, Massachusetts.*

Let ω be a primitive 49th root of unity. Prove that

$$\prod_{\substack{k=1 \\ \gcd(k,49)=1}}^{49} (1-\omega^k) = 7.$$

Solution I by Lawrence Somer, The Catholic University of America, Washington, D.C.

We note that

$$\prod_{\substack{k=1\\ \gcd(k,49)=1}}^{49} (x-\omega^k) = \frac{x^{49}-1}{x^7-1} = x^{42}+x^{35}+x^{28}+x^{21}+x^{14}+x^7+1.$$

The first equality follows since the roots of

$$\frac{x^{49}-1}{x^7-1} = 0$$

are the 49th roots of unity which are not 7th roots of unity. Letting $x = 1$, we obtain that

$$\prod_{\substack{k=1\\ \gcd(k,49)=1}}^{49} (1-\omega^k) = 7.$$

Solution II by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri.

Let

$$P = \prod_{\substack{k=1\\ \gcd(k,49)=1}}^{49} (1-\omega^k).$$

The choice of primitive root is immaterial; choose

$$\omega = e^{2\pi i/49}.$$

Then

$$\omega^k = (\omega^{49-k})^*,$$

where * means complex conjugate, so

$$(1-\omega^k)(1-\omega^{49-k}) = [1-\cos(2\pi k/49)]^2 + \sin^2(2\pi k/49) = 4\sin^2(\pi k/49).$$

Hence, by pairing off the factors in P, we obtain

$$P = \frac{4^{24}\prod_{k=1}^{24}\sin^2(\pi k/49)}{4^3\prod_{k=7,14,21}\sin^2(\pi k/49)} = 4^{21}\frac{\prod_{k=1}^{24}\sin^2(\pi k/49)}{\prod_{k=1}^{3}\sin^2(\pi k/7)}.$$

To evaluate the products we use the formula (L. B. W. Jolley, *Summation of Series*, 2nd ed., Dover, 1961, p. 190)

$$\sin^2(\pi/n)\sin^2(2\pi/n)\cdots\sin^2(((n-1)/2)\pi/n) = n/2^{n-1},$$

where n is odd. Finally, we obtain

$$P = 2^{42}\frac{49/2^{48}}{7/2^6} = 7.$$

Solution III by Frank J. Flanigan, San Jose State University, San Jose, California.

We prove that if p is a prime, m is a natural number, and ω is a primitive p^mth root of unity, then

$$\prod_{\substack{k=1 \\ \gcd(k,p^m)=1}}^{p^m} (1-\omega^k) = p.$$

First, note that the left side here is $f(1)$, where

$$f(x) = \prod_{\substack{k=1 \\ \gcd(k,p^m)=1}}^{p^m} (x-\omega^k).$$

Second, note that

$$f(x) = \Phi_{p^m}(x) = \text{ the } p^m\text{th cyclotomic polynomial over the ring of integers.}$$

This is because

$$\{\omega^k \mid 1 \le k \le p^m,\ \gcd(k, p^m) = 1\}$$

is precisely the set of primitive p^mth roots of unity in the complex numbers.

Third, recall that

$$\Phi_{p^m}(x) = \Phi_p(x^{p^{m-1}}),$$

where $\Phi_p(x)$ is the pth cyclotomic polynomial. This follows from the fact that if ζ is a primitive p^mth root of unity, then $\zeta^{p^{m-1}}$ is a primitive pth root of unity and hence, a zero of $\Phi_p(x)$.

Fourth, recall that

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1.$$

Thus, the left side of the product is

$$f(1) = \Phi_{p^m}(1) = \Phi_p(1^{p^{m-1}}) = 1 + 1 + \cdots + 1 + 1 = p,$$

which is what we asserted.

Solution IV by Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

The above result is a special case of the following proposition.

Proposition. Let $n > 1$ be an integer and let ω be a primitive nth root of unity.

(1) If p is a prime divisor of n, then

$$\prod_{\substack{k=1 \\ \gcd(k,p)=1}}^{n} (1 - \omega^k) = p.$$

(2) If p is a prime divisor of n and n is not prime, then

$$\prod_{\substack{k=1 \\ \gcd(k,p)=p}}^{n-1} (1 - \omega^k) = n/p.$$

(3) If n is the product of two distinct primes, then

$$\prod_{\substack{k=1\\ \gcd(k,n)=1}}^{n-1} (1-\omega^k) = 1.$$

Proof. Since

$$x^n - 1 = \prod_{k=1}^{n}(x-\omega^k) = (x-1)\prod_{k=1}^{n-1}(x-\omega^k),$$

by differentiating both sides of the above equation with respect to x and by letting $x = 1$, we get

$$(*) \qquad n = \prod_{k=1}^{n-1}(1-\omega^k).$$

Let p be a prime divisor of n and let $m = (n/p)$. Since

$$(\omega^p)^k, \quad 1 \le k \le m$$

are solutions of the equation $x^m - 1 = 0$ and the solutions are distinct,

$$x^m - 1 = \prod_{k=1}^{m}(x-\omega^{pk}).$$

Hence, following the method used in the beginning of the proof, we get

$$(**) \qquad (n/p) = m = \prod_{k=1}^{m-1}(1-\omega^{pk}) = \prod_{\substack{k=1\\ \gcd(k,p)=p}}^{n-1} (1-\omega^k), \quad \text{when } m > 1.$$

Part (1) of the proposition follows from $(*)$ when $n = p$ and it follows from $(*)$ and $(**)$ when n is not prime.

Part (2) is in fact $(**)$.

To prove the third part of the proposition, let $n = pq$, where p and q are distinct primes. By $(**)$,

$$q = \prod_{k=1}^{q-1}(1-\omega^{pk}) \quad \text{and} \quad p = \prod_{k=1}^{p-1}(1-\omega^{qk}).$$

Clearly, $pk \neq qs$ for $1 \leq k \leq q-1$ and $1 \leq s \leq p-1$. Hence,

$$pq = \prod_{\substack{k=1 \\ p|k \ \text{ or } \ q|k}}^{n-1} (1-\omega^k)$$

and by $(*)$ we get

$$pq = \prod_{k=1}^{n-1}(1-\omega^k).$$

These two equations imply that

$$1 = \prod_{\substack{k=1 \\ \gcd(k,n)=1}}^{n-1} (1-\omega^k).$$

Solution V by Thomas C. Leong, The City College of City University of New York, New York, New York.

In general, we show that if ω is a primitive nth root of unity, $n \geq 2$, then

$$\prod_{\substack{k=1 \\ (k,n)=1}}^{n} (1-\omega^k) = \begin{cases} p, & \text{if } n = p^\alpha \text{ is the power of a prime } p,\ \alpha \geq 1 \\ 1, & \text{otherwise.} \end{cases}$$

The primitive nth roots of unity are

$$\{\exp(2\pi i r_j/n)\}_{j=1}^{\phi(n)},$$

where

$$\{r_j\}_{j=1}^{\phi(n)}$$

is a reduced residue system modulo n. Now

$$\{k \ : \ 1 \leq k \leq n,\ (k,n)=1\}$$

is a reduced residue system modulo n, and for any integer a relatively prime to n,

$$\{ak \ : \ 1 \leq k \leq n,\ (k,n)=1\}$$

is also a reduced residue system modulo n. Thus, as k runs through the positive integers relatively prime to and not exceeding n, ω^k runs through the primitive nth roots of unity. Hence,

$$\prod_{\substack{k=1 \\ (k,n)=1}}^{n} (x-\omega^k)$$

is the nth cyclotomic polynomial

$$\Phi_n(x).$$

Thus, we wish to show that for $n \geq 2$,

$$(*) \qquad \Phi_n(1) = \begin{cases} p, & \text{if } n = p^\alpha \text{ is the power of a prime } p,\ \alpha \geq 1 \\ 1, & \text{otherwise.} \end{cases}$$

We induct on n. The case $n = 2$ is simple to verify, so suppose that $(*)$ holds for integers less than n. Now ζ is a root of $x^n - 1$ if and only if it is a primitive dth root of unity for some d which divides n; thus,

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

If $n = p^\alpha$ is a prime power, then

$$\Phi_{p^\alpha}(x) = \frac{x^{p^\alpha} - 1}{\prod_{\substack{d|p^\alpha \\ d \neq p^\alpha}} \Phi_d(x)} = \frac{1 + x + x^2 + \cdots + x^{p^\alpha - 1}}{\prod_{\substack{d|p^\alpha \\ d \neq 1, p^\alpha}} \Phi_d(x)} = \frac{1 + x + x^2 + \cdots + x^{p^\alpha - 1}}{\Phi_p(x) \cdot \Phi_{p^2}(x) \cdots \Phi_{p^{\alpha-1}}(x)}$$

and so by the inductive hypothesis

$$\Phi_{p^\alpha}(1) = \frac{p^\alpha}{\Phi_p(1) \cdot \Phi_{p^2}(1) \cdots \Phi_{p^{\alpha-1}}(1)} = \frac{p^\alpha}{p^{\alpha-1}} = p.$$

If n is not a prime power, say,

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$$

with $r \geq 2$, $\alpha_i \geq 1$, then

$$\Phi_n(x) = \frac{x^n - 1}{\prod_{\substack{d|n \\ d \neq n}} \Phi_d(x)} = \frac{1 + x + x^2 + \cdots + x^{n-1}}{\prod_{\substack{d|n \\ d \neq 1, n}} \Phi_d(x)}.$$

By the inductive hypothesis, $\Phi_d(1) = 1$ if d is not a prime power. Thus,

$$\Phi_n(1) = \frac{n}{\Phi_{p_1}(1) \cdot \Phi_{p_1^2}(1) \cdots \Phi_{p_1^{\alpha_1}}(1) \cdots \Phi_{p_r}(1) \cdot \Phi_{p_r^2}(1) \cdots \Phi_{p_r^{\alpha_r}}(1)} = \frac{n}{p_1^{\alpha_1} p_2^{\alpha_2} \cdot p_r^{\alpha_r}} = 1$$

and we are done.

90. [1996, 36–37] *Proposed by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri.*

The Fibonacci polynomials, $\{U_n(x)\}$, are defined by $U_1(x) = 1$, $U_2(x) = x$, and $U_n(x) = xU_{n-1}(x) + U_{n-2}(x)$, for $n \geq 3$.

(a) Derive a Binet-like formula for $U_n(x)$.

(b) Prove that

$$(U_n(x))^2 - U_{n-1}(x)U_{n+1}(x) = (-1)^{n-1}, \quad n \geq 2.$$

(c) Find a formula for the sum

$$\sum_{k=1}^{n} (U_k(x))^2.$$

(d) Let $\{L_n\}$ be the Lucas numbers: $L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$ $(n \geq 3)$. Prove that

$$U_n'(1) = \frac{nL_n - F_n}{5},$$

where F_n denotes the nth Fibonacci number and $U_n'(x)$ denotes the derivative of $U_n(x)$.

(e) Find a generating function for the $U_n(x)$'s, that is, a function $f(x,y)$ such that, formally,

$$f(x,y) = \sum_{n=1}^{\infty} U_n(x)y^n.$$

(f) Prove that for $n > 1$ all the zeroes of $U_n(x)$ lie along the imaginary axis.

Solution I by the proposer.

(a) Suppose

$$U_1(x) = \frac{a(x) - b(x)}{d(x)}$$

and

$$U_2(x) = \frac{[a(x)]^2 - [b(x)]^2}{d(x)},$$

so

$$U_2(x)/U_1(x) = x = a(x) + b(x).$$

Next,

$$U_3(x)/U_1(x) = x^2 + 1 = [a(x)]^2 + a(x)b(x) + [b(x)]^2.$$

Comparison then gives

$$[a(x)]^2 + a(x)b(x) + [b(x)]^2 = [a(x) + b(x)]^2 + 1,$$

so $a(x)b(x) = -1$. It follows that

$$x = a(x) - \frac{1}{a(x)},$$

or

$$[a(x)]^2 - xa(x) - 1 = 0,$$

and

$$a(x) = \frac{1}{2}\left(x \pm \sqrt{x^2 + 4}\right).$$

Since, in the equation above, $a(1)$ cannot be negative, we must take the $(+)$ sign in the brackets. Thus, we conjecture that

$$U_n(x) = \frac{[\frac{1}{2}(x + \sqrt{x^2 + 4})]^n - [\frac{1}{2}(x - \sqrt{x^2 + 4})]^n}{\sqrt{x^2 + 4}}.$$

This is automatically true for $n = 1, 2$; assume it to be true for $n = k-2, k-1$. Then,

$$U_k(x) = x\left(\frac{[a(x)]^{k-1} - [b(x)]^{k-1}}{d(x)}\right) + \left(\frac{[a(x)]^{k-2} - [b(x)]^{k-2}}{d(x)}\right)$$

$$= \frac{[a(x)]^{k-2}(1 + xa(x)) - [b(x)]^{k-2}(1 + xb(x))}{d(x)}$$

$$= \frac{[a(x)]^{k-2}[a(x)]^2 - [b(x)]^{k-2}[b(x)]^2}{d(x)}$$

identically. The conjecture is therefore true for all n by induction.

(b) This follows by straightforward induction, upon making use of $a(x)b(x) = -1$ and $a(x) - b(x) = d(x)$. If we define $U_0(x) = 0$, then the relation is also true for $n = 1$.

(c) We have the identity

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

The analogous equation for $U_n(x)$'s would not be correct since the terms on the right-hand side would all be one power of x too great. Hence, we conjecture

$$\sum_{j=1}^{n} [U_j(x)]^2 = \frac{U_n(x)U_{n+1}(x)}{x}.$$

This holds for $n = 1$; assume it holds for $n = k$. Then

$$\sum_{j=1}^{k+1} [U_j(x)]^2 = \frac{U_k(x)U_{k+1}(x)}{x} + [U_{k+1}(x)]^2$$

$$= \frac{[U_{k+2}(x) - xU_{k+1}(x)]U_{k+1}(x)}{x} + [U_{k+1}(x)]^2$$

$$= \frac{U_{k+1}(x)U_{k+2}(x)}{x},$$

so the conjecture is true for all n by induction.

(d) This follows by straightforward induction, upon making use of the identity

$$L_j = F_{j-1} + F_{j+1}.$$

(e) We have

$$yf(x,y) = \sum_{n=1}^{\infty} U_n(x)y^{n+1} = \sum_{m=2}^{\infty} U_{m-1}(x)y^m$$

$$xf(x,y) = \sum_{n=1}^{\infty} xU_n(x)y^n = \sum_{n=2}^{\infty} xU_n(x)y^n + xU_1(x)y$$

$$= \sum_{m=2}^{\infty} [U_{m+1}(x) - U_{m-1}(x)]y^m + xyU_1(x)$$

$$\frac{1}{y}f(x,y) = \sum_{n=1}^{\infty} U_n(x)y^{n-1} = \sum_{m=0}^{\infty} U_{m+1}(x)y^m$$

$$= \sum_{m=2}^{\infty} U_{m+1}(x)y^m + U_1(x) + U_2(x)y.$$

Combination of these equations yields

$$xf(x,y) - xyU_1(x) = \frac{1}{y}f(x,y) - U_1(x) - U_2(x)y - yf(x,y),$$

or equivalently,

$$f(x,y) = \frac{y}{1 - xy - y^2}.$$

(f) In the result of part (a) let $x = 2i\cos\theta$; then from deMoivre's theorem we obtain

$$U_n(2i\cos\theta) = -i^{n+1}\left(\frac{\sin n\theta}{\sin\theta}\right).$$

For θ satisfying $0 \leq \theta < 2\pi$, the right-hand side is 0 only when $\theta = j\pi/n$, $j = 1, 2, \ldots, 2n-1$. But of these zeroes, only $n-1$ are distinct because $\cos(j\pi/n) = \cos[2\pi - (j\pi/n)]$. If θ lies outside of $[0, 2\pi)$ and is a zero of U_n, then θ is congruent modulo 2π to $j\pi/n$, for some integer $j \in [1, 2n-1]$. Hence, there are just $n-1$ incongruent values of θ modulo 2π, and since $U_n(x)$ is of degree $n-1$, it can only have $n-1$ zeroes. It follows that all the zeroes of $U_n(x)$ are the pure imaginaries

$$\{2i\cos(j\pi/n) \ : \ j = 1, 2, \ldots, 2n-1\}.$$

We note that when n is even, but not odd, $x = 0$ is a zero (choose $j = \frac{1}{2}n$).

Solution II by Russell Euler, Northwest Missouri State University, Maryville, Missouri.

(a) The characteristic equation for $U_n(x)$ is $\lambda^2 - x\lambda - 1 = 0$. The roots of this equation are $u = u(x) = (x + \sqrt{x^2+4})/2$ and $v = v(x) = (x - \sqrt{x^2+4})/2$. So,

$$U_n(x) = c_1 u^n + c_2 v^n,$$

where c_1 and c_2 are constants. Using the initial conditions, one obtains the system of equations

$$U_1(x) = c_1 u + c_2 v = 1,$$

$$U_2(x) = c_1 u^2 + c_2 v^2 = x.$$

The solution of this system is $c_1 = 1/\sqrt{x^2+4}$, $c_2 = -1/\sqrt{x^2+4}$ if $x \neq \pm 2i$. Hence, for $x \neq \pm 2i$.

$$U_n(x) = (u^n - v^n)/\sqrt{x^2+4}. \tag{1}$$

(b) From (1),

$$
\begin{aligned}
&[U_n(x)]^2 - U_{n-1}(x)U_{n+1}(x) \\
&= \frac{u^{2n} - 2(uv)^n + v^{2n} - [u^{2n} - u^{n-1}v^{n+1} - u^{n+1}v^{n-1} + v^{2n}]}{x^2+4} \\
&= \frac{2(-1)^{n-1} + (uv)^{n-1}(u^2+v^2)}{x^2+4} \\
&= (-1)^{n-1}\frac{u^2+2+v^2}{x^2+4} \\
&= (-1)^{n-1}\frac{(u-v)^2}{x^2+4} \\
&= (-1)^{n-1}.
\end{aligned}
$$

(c) The sequence of Fibonacci polynomials can be extended backwards by defining $U_0(x) = 0$. Then

$$
\begin{aligned}
&\sum_{k=1}^{n}[U_k(x)]^2 \\
&= \sum_{k=1}^{n}\Big(U_k(x)[U_{k+1}(x) - U_{k-1}(x)]/x\Big) \\
&= \frac{1}{x}\left(\sum_{k=1}^{n}U_k(x)U_{k+1}(x) - \sum_{k=1}^{n}U_k(x)U_{k-1}(x)\right) \\
&= \frac{1}{x}\Big([U_1(x)U_2(x) - U_1(x)U_0(x)] + [U_2(x)U_3(x) - U_2(x)U_1(x)] \\
&+ \cdots + [U_n(x)U_{n+1}(x) - U_n(x)U_{n-1}(x)]\Big) \\
&= \frac{U_n(x)U_{n+1}(x)}{x}.
\end{aligned}
$$

(d) Using (1) it is straightforward to show that

$$U_n'(x) = \frac{\sqrt{x^2+4}[nu^{n-1}u' - nv^{n-1}v'] - xU_n(x)}{x^2+4}$$

$$= \frac{n\sqrt{x^2+4}[u^n/\sqrt{x^2+4} + v^n/\sqrt{x^2+4}] - xU_n(x)}{x^2+4}$$

$$= \frac{n(u^n+v^n) - xU_n(x)}{x^2+4}.$$

In particular, if $x = 1$, then

$$U_n'(1) = \frac{n[u^n(1)+v^n(1)] - U_n(1)}{5}$$

$$= \frac{nL_n - F_n}{5}.$$

(e)

$$f(x,y) = \sum_{n=1}^{\infty} U_n(x)y^n$$

$$= y + xy^2 + \sum_{n=3}^{\infty}[xU_{n-1}(x) + U_{n-2}(x)]y^n$$

$$= y + xy^2 + xy\sum_{n=3}^{\infty} U_{n-1}(x)y^{n-1} + y^2\sum_{n=3}^{\infty} U_{n-2}(x)y^{n-2}.$$

Thus,

$$f(x,y) = y + xy^2 + xy[f(x,y) - y] + y^2 f(x,y).$$

So, $(1 - xy - y^2)f(x,y) = y$ and therefore, $f(x,y) = y/(1 - xy - y^2)$.

(f) Using (1), $U_n(x) = 0$ is equivalent to $u^n = v^n$ and so

$$\frac{u}{v} = 1^{1/n} = e^{2k\pi i/n}$$

for $k = 0, 1, \ldots, n-1$. Since

$$\frac{u}{v} = (x + \sqrt{x^2+4})^2/(-4),$$

$$x\sqrt{x^2+4} = \pm 2ie^{k\pi i/n}.$$

Solving this equation for x gives

$$x = \pm 2i\cos\left(\frac{k\pi}{n}\right).$$

91. [1996, 37] *Proposed by Herta T. Freitag, Roanoke, Virginia.*

Pythagoras did not have our computational facilities for trigonometric functions (calculators or tables) at his disposal, but he may have had a "feeling" for the aesthetic beauty of the golden ratio in his soul, as he is said to have chosen the pentagram as the design for the fraternity pin of his academy.

Let R, the radius of a circle be given. How could one obtain the area of the inscribed pentagram on this basis? (Leave your answer in terms of $G = (\sqrt{5}+1)/2$, the golden ratio.)

Solution I by the proposer.

Referring to Figure 1, the area of the pentagram is given by

(1) $$K = 5(\text{Area of } \triangle AME - \text{ Area of } \triangle APE).$$

But,

$$\text{Area of } \triangle AME = \frac{R^2}{2}\sin 72^\circ$$

and

$$\text{Area of } \triangle APE = \frac{a^2}{2} \sin 72^\circ,$$

and thus,

$$K = \frac{5(R^2 - a^2)}{2} \sin 72^\circ. \tag{2}$$

For determining a, we consider $\triangle AMP$ (see Figure 2), and, using the law of sines

$$a = \frac{\sin 36^\circ}{\sin 54^\circ} R. \tag{3}$$

Now, we wish to obtain all these trigonometric functions. Referring to a regular decagon inscribed into our circle (see Figure 3) and letting its side be s_{10}, from similar triangles, we have

$$\frac{s_{10}}{R} = \frac{R - s_{10}}{s_{10}},$$

from which

$$s_{10} = R(\sqrt{5} - 1)/2 = \frac{R}{G}.$$

Furthermore, since

$$\sin 18^\circ = \frac{R}{2s_{10}},$$

$$\sin 18^\circ = \frac{1}{2G}. \tag{4}$$

Next, since $\sin 36^\circ = 2 \sin 18^\circ \cos 18^\circ$,

$$\sin 36^\circ = \frac{\sqrt{4G+3}}{2G^2}. \tag{5}$$

To obtain $\sin 54^\circ = \sin(3(18))^\circ$, we resort to $\sin 3x = \sin x(3 - 4\sin^2 x)$ to have

$$\sin 54^\circ = \frac{1}{2G}\left(3 - \frac{1}{G^2}\right),$$

or

$$\sin 54^\circ = \frac{G}{2}. \tag{6}$$

To compute $\sin 72^\circ = \sin(4(18))^\circ$, we use

$$\sin 4x = \sin 2(2x) = 2\sin 2x(1 - 2\sin^2 x).$$

Thus,

$$\sin 72^\circ = 2\sin 36^\circ(1 - 2\sin^2 18^\circ)$$

or

$$\sin 72^\circ = \frac{\sqrt{4G+3}}{G^2}\left(1 - \frac{1}{2G^2}\right),$$

from which

$$\sin 72^\circ = \frac{\sqrt{4G+3}}{2G}. \tag{7}$$

Finally, from (3), (5), and (6),

$$a = \frac{\sqrt{4G+3}}{G^3 R}$$

and

$$R^2 - a^2 = R^2\left(1 - \frac{4G+3}{G^6}\right) = \frac{2R^2}{G^3}.$$

Hence, by (2),

$$K = \frac{5}{2}\frac{2R^2}{G^3}\frac{\sqrt{4G+3}}{2G},$$

and the area of our pentagram becomes

$$K = \frac{5\sqrt{4G+3}}{2G^4}R^2.$$

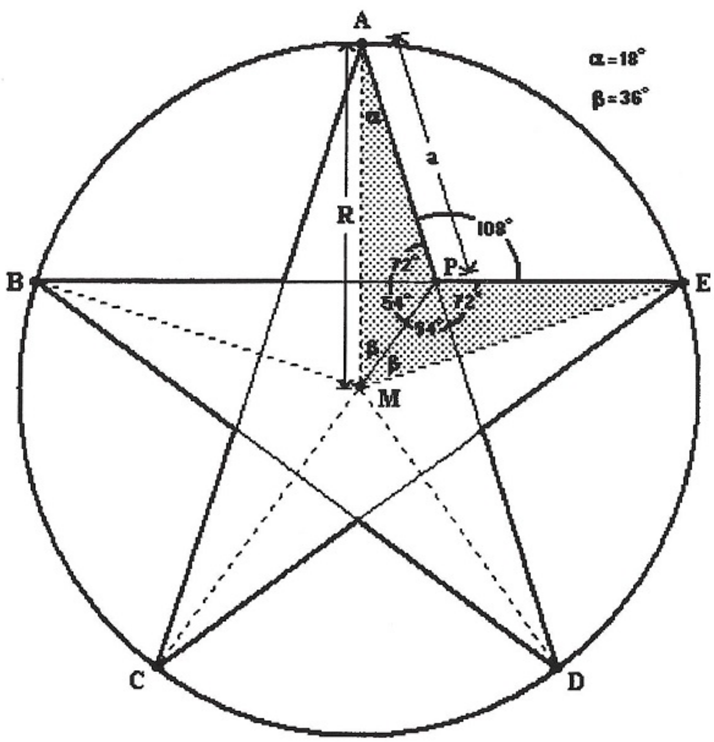


Figure 1.

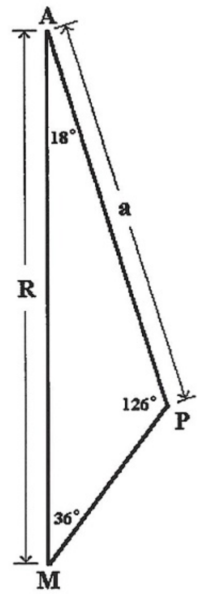


Figure 2.

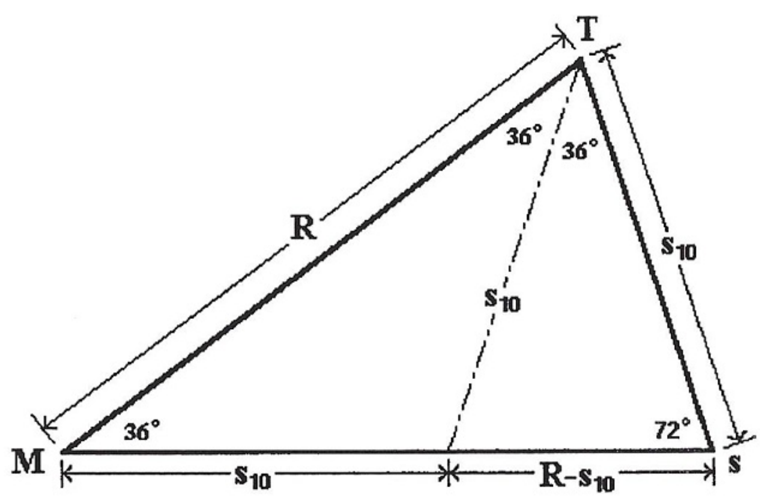


Figure 3.

Solution II by Russell Euler, Northwest Missouri State University, Maryville, Missouri.

Let x be the length of a side of the pentagon inscribed in the given circle. By the Law of Cosines,

$$x^2 = R^2 + R^2 - 2R^2\cos 72^\circ$$

$$= 2R^2(1 - \sin 18^\circ).$$

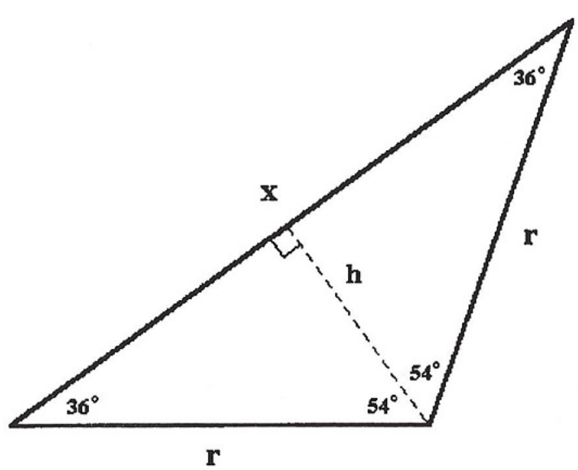
Since it is known that $\sin 18^\circ = (G-1)/2$, $x^2 = R^2(3-G)$ and so $x = R\sqrt{3-G}$. Therefore, the area of the pentagon is given by

$$A_1 = \frac{5}{4}x^2 \cot 36^\circ.$$

Since $\cot 36^\circ = G/\sqrt{3-G}$,

$$A_1 = \frac{5}{4}R^2 G\sqrt{3-G}.$$

Then the area, A, of the pentagram is $A = A_1 - 5A_2$, where A_2 is the area of the triangle in the figure below.



Then, $A_2 = xh/2$. From the Law of Sines,

$$\frac{r}{\sin 36^\circ} = \frac{x}{\sin 108^\circ}$$

and so

$$r = \frac{x \sin 36^\circ}{\sin 72^\circ} = \frac{x \sin 36^\circ}{2 \sin 36^\circ \cos 36^\circ} = \frac{x}{2 \sin 54^\circ} = \frac{x}{G}.$$

Also,

$$r^2 = \left(\frac{x}{2}\right)^2 + h^2$$

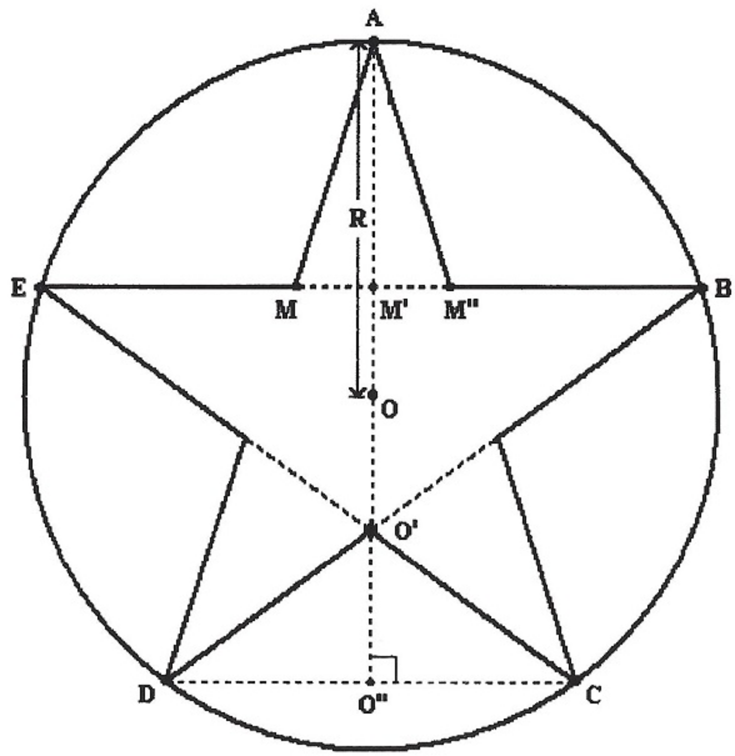
and hence, $h = \sqrt{4 - G^2}\,x/(2G)$. Then

$$A_2 = \sqrt{4 - G^2}\,x^2/(4G) = \sqrt{4 - G^2}\,R^2(3 - G)/(4G).$$

Therefore,

$$\begin{aligned} A &= \frac{5}{4} R^2 G \sqrt{3 - G} - \frac{5R^2(3 - G)\sqrt{4 - G^2}}{4G} \\ &= \frac{5}{4} R^2 \left(G\sqrt{3 - G} - \frac{(3 - G)\sqrt{4 - G^2}}{G} \right). \end{aligned}$$

Solution III by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri.



Pythagoras knew that intersecting diagonals cut each other in the golden ratio γ, that is,

$$\frac{\overline{MB}}{\overline{EM}} = \gamma.$$

Let $\overline{AO} = R$, $\overline{OO''} = h$, $\overline{DC} = s$; as $\triangle DO'C \sim \triangle EO'B$, we obtain

$$\frac{\overline{EB}}{\overline{DC}} = \frac{\overline{AD}}{\overline{DC}} = \gamma.$$

Then $\triangle OO''D$ and $\triangle AO''D$ furnish the Pythagorean equations $R^2 = (\frac{1}{2}s)^2 + h^2$ and $\gamma^2 s^2 = (\frac{1}{2}s)^2 + (R+h)^2$, respectively. These yield $h = \frac{1}{2}\gamma R$, $s = R\sqrt{3-\gamma}$, upon making use of $\frac{1}{\gamma} = \gamma + 1$. Additionally, $\overline{AO''} = R + h = R(1 + \frac{1}{2}\gamma) = 2\overline{AM'} + \overline{O'O''} = 2\overline{AM'} + \frac{\overline{AM'}}{\gamma}$, so

$$\overline{AM'} = \frac{1}{2}R\left(\frac{3\gamma+1}{2\gamma+1}\right).$$

Finally, we have

$$\begin{aligned}
\text{Area of pentagram } &= 2\left(\text{ Area of } \triangle AO'D + \text{ Area of } \triangle AMM''\right) \\
&= 2\left(\frac{1}{2}(\overline{AO'})(\overline{DO''}) + \frac{1}{2}(\overline{MM''})(\overline{AM'})\right) \\
&= (2\overline{AM'})\left(\frac{1}{2}s\right) + \left(\frac{s}{1+r}\right)(\overline{AM'}) \\
&= s(\overline{AM'})\left(1 + \frac{1}{1+\gamma}\right) \\
&= R\sqrt{3-\gamma}\left(\frac{R}{2}\frac{3\gamma+1}{2\gamma+1}\right)\left(\frac{\gamma+2}{\gamma+1}\right) \\
&= \frac{s}{2}R^2\sqrt{3-\gamma}\left(\frac{2\gamma+1}{5\gamma+3}\right) \\
&= \frac{5}{2}R^2\sqrt{3-\gamma}(2-\gamma) \\
&= \frac{5}{2}R^2\sqrt{18-11\gamma}.
\end{aligned}$$

Equivalent formulations of the final answer are possible.

92. [1996, 37; 1996, 89] *Proposed by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri.*

It is easy to show that the homogeneous quadratic expression $A_n^2 - 2B_n^2$ is invariant for all members of the sequence $\{\gamma_n\}_{n=1}^{\infty}$, defined by $\gamma_n = (3+2\sqrt{2})^n = A_n + B_n\sqrt{2}$. Find a homogeneous cubic expression that is invariant for all members of the sequence $\{I_n\}_{n=1}^{\infty}$, defined by $I_n = (1+\sqrt[3]{2}+\sqrt[3]{4})^n = A_n + B_n\sqrt[3]{2} + C_n\sqrt[3]{4}$.

Solution I by Lamarr Widmer, Messiah College, Grantham, Pennsylvania.

We have

$$A_{n+1} + B_{n+1}\sqrt[3]{2} + C_{n+1}\sqrt[3]{4} = (1+\sqrt[3]{2}+\sqrt[3]{4})(A_n + B_n\sqrt[3]{2} + C_n\sqrt[3]{4}).$$

It is routine to expand the right hand side and equate coefficients of like terms to obtain

$$A_{n+1} = A_n + 2B_n + 2C_n, \quad B_{n+1} = A_n + B_n + 2C_n, \quad \text{and} \quad C_{n+1} = A_n + B_n + C_n.$$

We will denote the general homogeneous cubic expression in three variables x, y, z as follows.

$$Q(x,y,z) = ax^3 + by^3 + cz^3 + dx^2y + exy^2 + fx^2z + gxz^2 + hy^2z + jyz^2 + kxyz.$$

Now the desired invariance condition can be stated as follows.

$$\begin{aligned}
0 =& Q(A_{n+1}, B_{n+1}, C_{n+1}) - Q(A_n, B_n, C_n) \\
=& Q(A_n + 2B_n + 2C_n, A_n + B_n + 2C_n, A_n + B_n + C_n) - Q(A_n, B_n, C_n) \\
=& A_n^3(b + c + d + e + f + g + h + j + k) \\
&+ B_n^3(8a + c + 4d + 2e + 4f + 2g + h + j + 2k) \\
&+ C_n^3(8a + 8b + 8d + 8e + 4f + 2g + 4h + 2j + 4k) \\
&+ A_n^2B_n(6a + 3b + 3c + 4d + 4e + 5f + 4g + 3h + 3j + 4k) \\
&+ A_nB_n^2(12a + 3b + 3c + 8d + 4e + 8f + 5g + 3h + 3j + 5k)
\end{aligned}$$

$$+ A_n^2 C_n(6a + 6b + 3c + 6d + 6e + 4f + 4g + 5h + 4j + 5k)$$

$$+ A_n C_n^2(12a + 12b + 3c + 12d + 12e + 8f + 4g + 8h + 5j + 8k)$$

$$+ B_n^2 C_n(24a + 6b + 3c + 16d + 10e + 12f + 6g + 4h + 4j + 8k)$$

$$+ B_n C_n^2(24a + 12b + 3c + 20d + 16e + 12f + 6g + 8h + 4j + 10k)$$
$$+ A_n B_n C_n(24a + 12b + 6c + 20d + 16e + 16f + 10g + 10h + 8j + 12k).$$

So we need to solve the system

$$\begin{aligned}
b + c + d + e + f + g + h + j + k &= 0\\
8a + c + 4d + 2e + 4f + 2g + h + j + 2k &= 0\\
8a + 8b + 8d + 8e + 4f + 2g + 4h + 2j + 4k &= 0\\
6a + 3b + 3c + 4d + 4e + 5f + 4g + 3h + 3j + 4k &= 0\\
12a + 3b + 3c + 8d + 4e + 8f + 5g + 3h + 3j + 5k &= 0\\
6a + 6b + 3c + 6d + 6e + 4f + 4g + 5h + 4j + 5k &= 0\\
12a + 12b + 3c + 12d + 12e + 8f + 4g + 8h + 5j + 8k &= 0\\
24a + 6b + 3c + 16d + 10e + 12f + 6g + 4h + 4j + 8k &= 0\\
24a + 12b + 3c + 20d + 16e + 12f + 6g + 8h + 4j + 10k &= 0\\
24a + 12b + 6c + 20d + 16e + 16f + 10g + 10h + 8j + 12k &= 0.
\end{aligned}$$

This is a straightforward but sizable exercise (I used the Derive® CAS) in row reduction. The solution is

$$a = 1,\ b = 2,\ c = 4,\ d = e = f = g = h = j = 0,\ k = -6.$$

The desired cubic expression is

$$A_n^3 + 2B_n^3 + 4C_n^3 - 6A_n B_n C_n.$$

Solution II by the proposer.

Consider the number field $\mathbb{Q}(\sqrt[3]{2})$; viewed as a vector space over $\mathbb{Q}$, this has as a basis $B = \{\alpha_1, \alpha_2, \alpha_3\} = \{1, \sqrt[3]{2}, \sqrt[3]{4}\}$. Let

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix};$$

then for any element $\gamma \in \mathbb{Q}(\sqrt[3]{2})$, define the matrix A by

$$\gamma\alpha = A\alpha.$$

The norm of γ (in the field $\mathbb{Q}(\sqrt[3]{2})$ is then given by (J. S. Chahal, *Topics in Number Theory*, Plenum Press, 1988, p. 75)

$$N(\gamma) = \det A.$$

For $\gamma = A_n + B_n\sqrt[3]{2} + C_n\sqrt[3]{4}$, we obtain

$$N(\gamma) = \begin{vmatrix} A_n & B_n & C_n \\ 2C_n & A_n & B_n \\ 2B_n & 2C_n & A_n \end{vmatrix} = A_n^3 + 2B_n^3 + 4C_n^3 - 6A_nB_nC_n.$$

This immediately gives $N(I_1) = 1 + 2 + 4 - 6 = 1$, so I_1 is a unit in $\mathbb{Q}(\sqrt[3]{2})$. Since norms in a number field obey $N(x)N(y) = N(xy)$, it follows that $N(I_n) = 1$ for all $n \in \mathbb{Z}^+$, and the cubic above is the desired homogeneous invariant expression.