PAIRWISE LOCALLY COMPACT BICONVERGENCE SPACES

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A generalization of a topological space into a convergence space [1] and a generalization of a topological space into a bitopological space [4] serve as a forerunner to a biconvergence space [1]. In this paper we show that the existence of a continuous, open mapping from a pairwise locally compact biconvergence space X onto a pairwise weakly Hausdorff biconvergence space Y assures the pairwise local compactness of Y.

Definition 1. (Convergence Space.) We consider a set S of arbitrary elements and associate with each sequence $\{p_n\}$ of elements $p_1, p_2, p_3, \cdots, p_n, \cdots$ of S of a family of sequences (called the convergent sequences) an element p of S and say that the sequence $\{p_n\}$ converges to p, in symbols,

$$\lim_{n\to\infty} p_n = p \quad \text{or} \quad \{p_n\} \to p$$

if and only if the following conditions are satisfied.

(C_1) If $\lim_{n\to\infty} p_n = p$ and $k_1 < k_2 < k_3 < \cdots < k_n < \cdots$ then $\lim_{n\to\infty} p_{k_n} = p$, i.e., if $\{p_n\} \to p$ then its every subsequence must converge to the same limit, $\{p_{k_n}\} \to p$.

(C_2) If for every n from some point on $p_n = p$, then $\lim_{n\to\infty} p_n = p$.

(C_3) If the sequence $\{p_n\}$ does not converge to a point (element) p then there exists a subsequence, such that no subsequence of it converges to p.

The set S together with this definition of convergence constitute a *convergence space*, which we denote by (S, C^*). Here S is a set over which the convergence C^* is defined. If $\{p_n\} \to p$, then p will be called a limit of $\{p_n\}$.

Definition 2. (Limiting Point.) Limiting point of a set A is a point x, such that there exists a sequence $\{x_n\}$ of distinct points of A, such that $\lim_{n\to\infty} x_n = x$.

Definition 3. (Closure.) The closure of a set A is denoted by $\overline{A}$. A point x of the convergence space (S, C^*) is said to belong to the closure of a subset A if and only if there exists a sequence $\{x_n\}$ in A such that $\{x_n\} \to x$.

Remark 4. $\overline{\overline{A}} \neq \overline{A}$. Refer to Theorem 1.8 (iv) [1].

Remark 5. The above remark points out that every topological space is a convergence space but one can find a convergence space which fails to be a topological space when an open set is defined as follows.

Definition 6. (Open Set.) A set G in a convergence space (S, C^*) is said to be open if and only if no sequence outside G under C^* converges to a point in G.

Definition 7. (Closed Set.) A set F in a convergence space (S, C^*) is said to be closed if and only if F contains the limit points of all convergent sequences contained in F. Equivalently, F is closed if and only if $S - F$ is open.

Definition 8. (Neighborhood.) An open set U in a convergence space (S, C^*) such that $x \in U$ will be called a neighborhood of x and in this context will be written U_x.

Definition 9. (Compact Set.) A set in convergence space (S, C^*) is said to be compact if and only if every open cover has a finite subcover. Using neighborhood, we say that for a set K in the space (S, C^*) and any neighborhood system $\{U_k : k \in K\}$ if K can be covered by a finite number of these neighborhoods, for definiteness

$$K \subseteq \cup_i \{U_{k_j} : j = 1, 2, \cdots, n_0\}$$

for a fixed integer $n_0 \geq 1$ then K is compact.

Definition 10. (Continuous Mapping.) Let (A, C^*) and (B, C^*) be convergence spaces. A function $f\colon A \to B$ is said to be continuous at $x \in A$ under the scheme C^* if for every sequence $\{x_n\}$ in A converging to x, the corresponding sequence $\{f(x_n)\}$ in B converges to $f(x)$.

If f is continuous at each and every point of A, then f is called C^*-continuous over A.

Definition 11. (Closed Mapping.) Let (A, C^*) and (B, C^*) be convergence spaces. A function $f\colon A \to B$ is said to be closed mapping if given any closed subset $F \subseteq A$, in (A, C^*) the set $f(F)$ is closed in (B, C^*).

This definition is in conformity with the corresponding one in a topological setting, but still there is a fine generalization dormant as stipulated by the remark in section 4, which points out towards the existence of a set H for which $\overline{\overline{H}} \neq \overline{H}$.

Definition 12. (Biconvergence Space.) A biconvergence space (X, A, B) is a set X on which two convergence schemes A and B are defined.

Definition 13. (Weak Hausdorff Biconvergence Space T_2'.) We define this in the setting of biconvergence space (X, A, B) based on the corresponding definition for bitopological space introduced by Bose [2].

Let (X, A, B) be a biconvergence space. It is said to be a T_2' space or pairwise weak Hausdorff space if given $K_1 \neq \emptyset$, $K_2 \neq \emptyset$ where $K_1 \cap K_2 = \emptyset$ such that K_1 is an A-compact and K_2 is a B-compact set, there exists U_1 an A-open and U_2 a B-open set such that $K_1 \subseteq U_1$, $K_2 \subseteq U_2$ and $K_i \cap U_j = \emptyset$, $i \neq j$, $i, j \in \{1, 2\}$.

Theorem 14. (Characterization Theorem.) A biconvergence space (X, A, B) is pairwise T_2' if and only if every A-compact set is B-closed and every B-compact set is A-closed.

Proof. Let (X, A, B) be pairwise T_2'. Suppose K is an A-compact set. $x \in X \Rightarrow \{x\}$ is B-compact (since every finite set is compact). By Definition 13, pairwise T_2'-ness implies that there exist U and V respectively A-open and B-open sets with $K \subseteq U$ and $\{x\} \subseteq V$ and $U \cap \{x\} = \emptyset \Rightarrow K \cap \{x\} = \emptyset \Rightarrow x \notin K$. Also, $K \cap V = \emptyset$. Thus, there exists a B-neighborhood of x which has empty intersection with K. Therefore, $x \notin B$-closure of K. Hence, K is B-closed. Likewise a B-compact set can be shown to be A-closed.

Conversely, suppose that each A-compact set K is B-closed. Thus, $X \setminus K$ is B-open. Suppose $\emptyset \neq K$ is A-compact and $\emptyset \neq J$ is B-compact such that $K \cap J = \emptyset$ (being disjoint). Hence, $J \subseteq X \setminus K = V$. Thus, J is contained in a B-open set V. Similarly K is contained in an A-open set U. Now, $K \cap V = K \cap (X \setminus K) = \emptyset$. Also, $J \cap U = J \cap (X - J) = \emptyset$. Hence, the space is weakly pairwise Hausdorff (T_2').

Definition 15. (Pairwise Compact Biconvergence Space.) Swart [3] has defined pairwise compactness in bitopological spaces. If (X, A) and (X, B) are compact convergence spaces, then we say that the biconvergence space (X, A, B) is pairwise compact. $K \subseteq X$ is pairwise compact if it is pairwise compact in the subspace K.

Theorem 16. If (X, A, B) is pairwise compact and T_2'-biconvergence space then $A = B$.

Proof. Let F be an A-closed set. Then F is A-compact since X is A-compact. By T_2'-ness, F is B-closed. Every A-closed set is also B-closed. Similarly, we can show that every B-closed set is also A-closed. Hence, $A = B$.

Definition 17. (a) (Weakly Continuous.) Let $f\colon (X, C_X) \to (Y, C_Y)$ be a function from one convergence space to another convergence space. Then f is said to be *weakly continuous* at $x \in X$ if and only if whenever $\{x_n\} \to x$; $\{f(x_{n_k})\} \to f(x)$ for some subsequence $\{f(x_{n_k})\}$ of the sequence $\{f(x_n)\}$. The function f is *weakly continuous on* X if and only if f is weakly continuous at each $x \in X$.

(b) (Weak Homeomorphism.) The function f is called a *weak homeomorphism* if and only if f is one-to-one, onto, and weakly continuous, and f maps closed sets to closed sets.

(c) (Cartesian Product Convergence Structure). Let $(X_i, C_{X_i}) i = 1, 2$ be convergence spaces and $\prod_{i=1}^{2} X_i$ be the Cartesian product of these convergence spaces. Let $\{s_n\} n \in \mathbb{N}$ be a sequence in $\prod_{i=1}^{2} X_i$. Define $\{s_n\} \to x$ if and only if $\{\pi_i(s_n)\} \to \pi_i(x)$ with $i = 1, 2, \pi_i$ is the i-th projection map. It is easy to check that this defines a convergence structure on $\prod_{i=1}^{2} X_i$ and we denote it by

$$C_{\prod_{i=1}^{2} X_i}.$$

Theorem 17. Let (X, A, B) be T_2' and pairwise compact. Then every closed one-to-one map from X onto any other pairwise compact biconvergence space is a weak homeomorphism.

Proof. Suppose $(X, A, B,)$ be T_2'. By Theorem 16, $A = B$. Let (Y, C, D) be an arbitrary pairwise compact space. Suppose $f \colon X \to Y$ is a one-to-one, onto, closed map. Then it suffices to show that f is weakly continuous. Let $\{x_n\} \to x$. Without loss of generality, assume that $x \notin \cup_{n \in \mathbb{N}} \{x_n\}$.

Suppose $\{f(x_n)\}$ has convergent subsequence, $\{f(x_{n_k})\}$. Then $\{f(x_{n_k})\} \to f(x^*)$ for some x^* in X. Since $f^{-1}\{f(x_{n_k})\} = \{f^{-1}(f(x_{n_k})\} = \{x_{n_k}\} \to x^*$, and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and $\{x_n\} \to x$, we have $x = x^*$. Thus, $\{f(x_n)\}$ has a convergent subsequence which converges to $f(x)$ and therefore f is weakly continuous. Now, suppose $\{f(x_n)\}$ has no convergent subsequence. Then, [5], the set $H = \{(f(x_n), n) : n \in \mathbb{N}\}$ is closed in the product space $Y \times (\mathbb{N} \cup \{w\})$. We shall show that $K_1 = \cup_{n \in \mathbb{N}} \{f(x_n)\}$ is closed. Let $\{y_n\}$ be a sequence in K_1 such that $\{y_n\} \to y$. Suppose $y \notin K_1$. Let i_0 be a fixed element in $\mathbb{N}$. Then $\{(y_n, i_0)\}$ is a sequence in H. Since

$$\overline{\wedge}_1\{(y_n, i_0)\} = \{\overline{\wedge}_1(y_n, i_0)\} = \{y_n\} \to y$$

and

$$\overline{\wedge}_2\{(y_n, i_0)\} = \{\overline{\wedge}_2(y_n, i_0)\} = \{i_0\} \to i_0,$$

by the definition of the convergence structure on the product spaces and the fact that H is closed, we have $y \in K_1$; which is a contradiction. Therefore, K_1 is closed. Let $K_2 = \{f(x)\}$. Since K_1 and K_2 are closed and Y is compact, K_1 and K_2 are compact. Since f is a closed map, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are compact and hence, by Theorem 14, they are also closed. Since $\{x_n\} \to x$ and $\cup_{n\in\mathbb{N}}\{x_n\} = f^{-1}(K_1)$, $x \in f^{-1}(K_1)$ because $f^{-1}(K_1)$ is closed. This contradicts the assumption $x \notin \cup_{n\in\mathbb{N}}\{x_n\}$.

Definition 18. (Pairwise Locally Compact.) A biconvergence space (X, A, B) is said to be pairwise locally compact at $x \in X$, if there exists an A-open neighborhood U_x of x and a B-open neighborhood V_x of x such that A-cl. U_x is B-compact and B-cl. V_x is A-compact. A biconvergence space (X, A, B) is pairwise locally compact if and only if it is pairwise locally compact at each point $x \in X$.

Theorem 19. If $f\colon (X, A, B) \to (Y, C, D)$ is continuous open mapping of a pairwise locally compact space (X, A, B) onto a pairwise T_2'-space (Y, C, D); then Y is pairwise locally compact.

Proof. Let $y \in Y$. Then there exists $x \in X$ such that $y = f(x)$. By pairwise local compactness of X, we have that there exists an A-open neighborhood U_x of x and a B-open neighborhood V_x of x such that A-cl. U_x is B-compact and B-cl. V_x is A-compact. Since f is continuous, $f(A\text{-cl.}U_X)$ is D-compact in Y, a pairwise T_2'-space $\Rightarrow f(A\text{-cl.}U_x)$ is C-closed in Y, so $f(A\text{-cl.}U_x)$ is C- cl. $f(U_x)$. Now f being open, $f(U_x)$ is C-open and its C-closure is D-compact. Similarly $f(V_x)$ is D-open and it can be shown that its D-closure is C-compact. Thus, Y is pairwise locally compact at y. Since y was arbitrarily chosen in Y, the space Y is pairwise locally compact.

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