SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

52. [1992, 146; 1993, 144–149] *Proposed by Dale Woods and Jin Chen, University of Central Oklahoma, Edmond, Oklahoma.*
(a) Find a closed form for the expression

$$\sum_{k=1}^{m} k\binom{2m}{m-k}.$$

(b)* Let $n \geq 2$ be an integer. Find a closed form for the expression

$$\sum_{k=1}^{m} k^n\binom{2m}{m-k}.$$

Comment by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri.

(a) The following solution may be of interest as it is shorter than any of the three solutions previously given.

$$\sum_{k=1}^{m} k\binom{2m}{m-k} = \sum_{k=1}^{m} k\binom{2m}{m+k} = \sum_{k=m+1}^{2m} (k-m)\binom{2m}{k}$$

$$= \sum_{k=m+1}^{2m} k\binom{2m}{k} - m\sum_{k=m+1}^{2m}\binom{2m}{k} = 2m\sum_{k=m+1}^{2m}\binom{2m-1}{k-1} - \frac{m}{2}\left[2^{2m} - \binom{2m}{m}\right]$$

$$= 2m\sum_{k=m}^{2m-1}\binom{2m-1}{k} - m\cdot 2^{2m-1} + \frac{m}{2}\binom{2m}{m}$$

$$= 2m\left[\frac{1}{2}\cdot 2^{2m-1}\right] - m\cdot 2^{2m-1} + \frac{m}{2}\binom{2m}{m} = \frac{m}{2}\binom{2m}{m}.$$

(b) Using the technique above, the following results are easily established, though with increasing labor:

$$n=2 \qquad \sum_{k=1}^{m} k^2 \binom{2m}{m-k} = m \cdot 2^{2m-2}$$

$$n=3 \qquad \sum_{k=1}^{m} k^3 \binom{2m}{m-k} = m \cdot \frac{m}{2} \binom{2m}{m}.$$

I have obtained corresponding formulas (by consideration of numerous special cases) for $n = 4$–8. As a conjecture, it appears that

$$S_n = \sum_{k=1}^{m} k^n \binom{2m}{m-k}$$

has the form ($n \geq 1$)

$$S_n = \begin{cases} P_n(m) \cdot \frac{m}{2}\binom{2m}{m}, & n = \text{odd}; \\ Q_n(m) \cdot m \cdot 2^{2m-\frac{1}{2}n-1}, & n = \text{even}. \end{cases}$$

where $P_n(m)$ is a polynomial in m of degree $(n-1)/2$ and $Q_n(m)$ is a polynomial in m of degree $n/2 - 1$. The following table of polynomials has been obtained so far.

n	$P_n(m)$	$Q_n(m)$
1	1	
2		1
3	m	
4		$3m-1$
5	$2m^2-m$	
6		$15m^2-15m+4$
7	$6m^3-8m^2+3m$	
8		$105m^3-210m^2+147m-34$

It may be noticed that $P_n(1) = 1$ and $Q_n(1) = 2^{\frac{1}{2}n-1}$. Let N_n be the constant term in $Q_n(m)$; it may also be noticed that $n-1$ divides each coefficient in $Q_n(m) - N_n$.

However, the overall pattern of the formulas is still obscure and I do not yet have the general closed form for

$$\sum_{k=1}^{m} k^n \binom{2m}{m-k}.$$

Solution to part (b) by Srikant Radhakrishnan, Central Missouri State University, Warrensburg, Missouri.

The following solution uses Gosper's Method. Let

$$t_k = k^n \binom{2m}{m-k}.$$

If we can find T_k such that

$$t_k = \Delta T_k = T_{k+1} - T_k,$$

then

$$\sum_{k=1}^{m} k^n \binom{2m}{m-k} = \sum_{1 \le k < m+1} t_k \delta k = T_{m+1} - T_1.$$

Then,

$$\begin{aligned}
\frac{t_{k+1}}{t_k} &= \frac{(k+1)^n \binom{2m}{m-k-1}}{k^n \binom{2m}{m-k}} \\
&= \frac{(k+1)^n (2m)!(m-k)!(m+k)!}{k^n (m+k+1)!(m-k-1)!(2m)!} \\
&= \frac{(k+1)^n (-1)(k-m)}{k^n (k+m+1)}.
\end{aligned}$$

To find T_k, we have to express

$$\frac{t_{k+1}}{t_k}$$

as a ratio

$$\frac{p(k+1)q(k)}{p(k)r(k+1)}$$

such that $p(k)$, $q(k)$, $r(k)$ are polynomials satisfying the condition

$$(k+a) \mid q(k) \text{ and } (k+b) \mid r(k) \quad \Rightarrow \quad a-b \leq 0.$$

Since

$$\frac{t_{k+1}}{t_k} = \frac{(k+1)^n(-1)(k-m)}{k^n(k+1+m)},$$

let

$$p(k) = k^n; \quad q(k) = (-1)(k-m); \quad r(k) = k+m.$$

Thus,

$$(k-m) \mid q(k); \quad (k+m) \mid r(k); \quad \text{and} \quad -m-m \leq 0,$$

since $m \geq 0$. Hence,

$$T_k = \frac{r(k)t(k)s(k)}{p(k)},$$

where $s(k)$ is a polynomial in k satisfying certain conditions. Using the fact that

$$t_k = T_{k+1} - T_k \ , \ \frac{t_{k+1}}{t_k} = \frac{p(k+1)q(k)}{p(k)r(k+1)},$$

and

$$T_k = \frac{r(k)t(k)s(k)}{p(k)}$$

we get,

$$p(k) = s(k+1)q(k) - s(k)r(k).$$

That is,

$$k^n = s(k+1)(-1)(k-m) - s(k)(k+m)$$

or

$$k^n = -k\big[s(k+1) + s(k)\big] + m\big[(s(k+1) - s(k)\big].$$

If $s(k+1)$ is a polynomial (in k) of degree d, then $s(k+1)+s(k)$ is a polynomial of degree d and $s(k+1)-s(k)$ is a polynomial of degree $d-1$. Since the left-hand side of the above equation is a polynomial of degree n, $s(k)$ has to be of degree $n-1$. Let

$$s(k) = \alpha_0 + \alpha_1 k + \alpha_2 k^2 + \cdots + \alpha_{n-1}k^{n-1}.$$

Hence,

$$T_k = \frac{(k+m)k^n\binom{2m}{m-k}s(k)}{k^n}.$$

Therefore,

$$T_{m+1} = (m+m+1)\binom{2m}{m-m-1}s(m+1) = 0,$$

$$T_1 = (m+1)\binom{2m}{m-1}s(1),$$

and our original sum is

$$T_{m+1} - T_1 = -(m+1)\binom{2m}{m-1}s(1) = -(m+1)\binom{2m}{m-1}(\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}).$$

We now start trying to simplify the α_i's using the fact that

$$\begin{aligned}
k^n &= -k\big[s(k+1) + s(k)\big] + m\big[s(k+1) - s(k)\big] \\
&= -k\big[(\alpha_0 + \alpha_1(k+1) + \alpha_2(k+1)^2 + \cdots + \alpha_{n-1}(k+1)^{n-1}) \\
&\quad + (\alpha_0 + \alpha_1 k + \alpha_2 k^2 + \cdots + \alpha_{n-1}k^{n-1})\big] \\
&\quad + m\big[(\alpha_0 + \alpha_1(k+1) + \alpha_2(k+1)^2 + \cdots + \alpha_{n-1}(k+1)^{n-1}) \\
&\quad - (\alpha_0 + \alpha_1 k + \alpha_2 k^2 + \cdots + \alpha_{n-1}k^{n-1})\big].
\end{aligned}$$

We now collect like powers of k on the right to get

$$
\begin{aligned}
&k^n = m(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})\\
&+k\big[-(2\alpha_0+\alpha_1+\alpha_2+\cdots+\alpha_{n-1})+m(2\alpha_2+3\alpha_3+\cdots+(n-1)\alpha_{n-1})\big]\\
&+k^2\Bigg[-\Bigg((2\alpha_1+\binom{2}{1}\alpha_2+\binom{3}{1}\alpha_3+\cdots+\binom{n-1}{1}\alpha_{n-1}\Bigg)\\
&+m\Bigg(\binom{3}{2}\alpha_3+\binom{4}{2}\alpha_4+\cdots+\binom{n-1}{2}\alpha_{n-1}\Bigg)\Bigg]\\
&+\cdots\\
&+k^{n-1}\Bigg[-\Bigg(2\alpha_{n-2}+\binom{n-1}{n-2}\alpha_{n-1}\Bigg)\Bigg]+k^n\big[-2\alpha_{n-1}\big].
\end{aligned}
$$

Equating coefficients of like powers of k on both sides we get

$$
\begin{aligned}
&\alpha_1+\alpha_2+\cdots+\alpha_{n-1}=0\\
&-\big[2\alpha_0+\alpha_1+\alpha_2+\cdots+\alpha_{n-1}\big]\\
&+m\big[2\alpha_2+3\alpha_3+\cdots+(n-1)\alpha_{n-1}\big]=0\\
&-\Bigg[2\alpha_1+\binom{2}{1}\alpha_2+\binom{3}{1}\alpha_3+\cdots+\binom{n-1}{1}\alpha_{n-1}\Bigg]\\
&+m\Bigg[\binom{3}{2}\alpha_3+\binom{4}{2}\alpha_4+\cdots+\binom{n-1}{2}\alpha_{n-1}\Bigg]=0\\
&\quad\vdots\\
&-\Bigg[2\alpha_{n-2}+\binom{n-1}{n-2}\alpha_{n-1}\Bigg]=0\\
&-2\alpha_{n-1}=1.
\end{aligned}
$$

With some transposing, these equations become

1. $\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = 0$

2. $2\alpha_0 + (\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1})$

$$= m\left[\binom{2}{1}\alpha_2 + \binom{3}{1}\alpha_3 + \cdots + \binom{n-1}{1}\alpha_{n-1}\right]$$

3. $2\alpha_1 + \left[\binom{2}{1}\alpha_2 + \binom{3}{1}\alpha_3 + \cdots + \binom{n-1}{1}\alpha_{n-1}\right]$

$$= m\left[\binom{3}{2}\alpha_3 + \binom{4}{2}\alpha_4 + \cdots + \binom{n-1}{2}\alpha_{n-1}\right]$$

4. $2\alpha_2 + \left[\binom{3}{2}\alpha_3 + \binom{4}{2}\alpha_4 + \cdots + \binom{n-1}{2}\alpha_{n-1}\right]$

$$= m\left[\binom{4}{3}\alpha_4 + \binom{5}{3}\alpha_5 + \cdots + \binom{n-1}{3}\alpha_{n-1}\right]$$

$$\vdots$$

$n-1.$ $2\alpha_{n-3} + \left[\binom{n-2}{n-3}\alpha_{n-2} + \binom{n-1}{n-3}\alpha_{n-1}\right] = m\left[\binom{n-1}{n-2}\alpha_{n-1}\right]$

$n.$ $2\alpha_{n-2} + \left[\binom{n-1}{n-2}\alpha_{n-1}\right] = 0$

$n+1.$ $2\alpha_{n-1} = -1.$

These form $n+1$ equations in n unknowns. We now perform a series of manipulations to get a couple of results – the first being that there is no simple closed form for the sum, if n is even and m is positive; the second, a simplified recursive form for the α_i's when n is odd. For that, we number the equations starting from the topmost as 1. all the way to the one at the bottom as $(n+1)$. Equation 1. says

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = 0.$$

Therefore, Equation 2. reduces to

$$2\alpha_0 = m\left(\binom{2}{1}\alpha_2 + \binom{3}{1}\alpha_3 + \cdots + \binom{n-1}{1}\alpha_{n-1}\right).$$

Multiplying both sides of Equation 3. by m and adding it to the above equation (adding LHS to LHS and RHS to RHS) we get

$$2\alpha_0 + 2\alpha_1 m = m^2\left(\binom{3}{2}\alpha_3 + \binom{4}{2}\alpha_4 + \cdots + \binom{n-1}{2}\alpha_{n-1}\right).$$

Multiplying both sides of Equation 4. by m^2 and adding it (the same way) to the above equation we have

$$2\alpha_0 + 2\alpha_1 m + 2\alpha_2 m^2 = m^3\left(\binom{4}{3}\alpha_4 + \binom{5}{3}\alpha_5 + \cdots + \binom{n-1}{3}\alpha_{n-1}\right).$$

Continuing the process until we would have to multiply Equation n. by m^{n-2} (on both sides) and add it to the equation obtained just prior to it we would have

$$2\alpha_0 + 2\alpha_1 m + \cdots + 2\alpha_{n-2}m^{n-2} = 0$$

or

$$(A) \qquad \alpha_0 + \alpha_1 m + \alpha_2 m^2 + \cdots + \alpha_{n-2}m^{n-2} = 0.$$

We shall keep this result in abeyance until we arrive at another set of patterns later.

We now claim the existence of an identity,

$$\binom{n-k}{n-k-1}\alpha_{n-k} + \binom{n-k+1}{n-k-1}\alpha_{n-k+1}$$

$$+ \binom{n-k+2}{n-k-1}\alpha_{n-k+2} + \cdots + \binom{n-1}{n-k-1}\alpha_{n-1} = 0,$$

whenever $k = 2, 4, 6, \ldots$.

The proof for this can be obtained by the brute force of induction, but it is much more pleasant to see the pattern in a similar light. For $k = 2$, the left-hand side is

$$\binom{n-2}{n-3}\alpha_{n-2} + \binom{n-1}{n-3}\alpha_{n-1}.$$

Using Equation n. we have

$$\alpha_{n-2} = \frac{-1}{2}\left[\binom{n-1}{n-2}\alpha_{n-1}\right].$$

Plugging that in the above expression we get

$$\frac{-1}{2}\binom{n-2}{n-3}\binom{n-1}{n-2}\alpha_{n-1} + \binom{n-1}{n-3}\alpha_{n-1}.$$

We now use trinomial revision which states that

$$\binom{r}{k}\binom{k}{m} = \binom{r}{m}\binom{r-m}{k-m}.$$

The expression now becomes

$$\frac{-1}{2}\binom{n-1}{n-3}\binom{2}{1}\alpha_{n-1} + \binom{n-1}{n-3}\alpha_{n-1} = -\binom{n-1}{n-3}\alpha_{n-1} + \binom{n-1}{n-3}\alpha_{n-1} = 0.$$

Hence

$$\binom{n-2}{n-3}\alpha_{n-2} + \binom{n-1}{n-3}\alpha_{n-3} = 0.$$

Using this in Equation $n-1$. would give us

$$2\alpha_{n-3} = m\binom{n-1}{n-2}\alpha_{n-1} = m(-2\alpha_{n-2}).$$

(Since equation $n.$ has

$$\binom{n-1}{n-2}\alpha_{n-1} = -2\alpha_{n-2})$$

and

$$\alpha_{n-3} = -m\alpha_{n-2}.$$

Equation $n-2.$, if it had been listed, would have read

$$2\alpha_{n-4} + \left[\binom{n-3}{n-4}\alpha_{n-3} + \binom{n-2}{n-4}\alpha_{n-2} + \binom{n-1}{n-4}\alpha_{n-1}\right]$$
$$= m\left[\binom{n-2}{n-3}\alpha_{n-2} + \binom{n-1}{n-3}\alpha_{n-1}\right].$$

The right side is zero by our latest result, hence,

$$(B) \qquad \alpha_{n-4} = \frac{-1}{2}\left[\binom{n-3}{n-4}\alpha_{n-3} + \binom{n-2}{n-4}\alpha_{n-2} + \binom{n-1}{n-4}\alpha_{n-1}\right].$$

For the induction step, we assume truth for all i from $2, 4, \dots, k-2$ and try to prove it for $i = k$ (here k is even). That is, we try and prove that

$$\binom{n-k}{n-k-1}\alpha_{n-k} + \binom{n-k+1}{n-k-1}\alpha_{n-k+1} + \cdots + \binom{n-1}{n-k-1}\alpha_{n-1} = 0.$$

Since the induction hypothesis holds for $k-2$,

$$\binom{n-k+2}{n-k+1}\alpha_{n-k+2} + \binom{n-k+3}{n-k+1}\alpha_{n-k+3} + \cdots + \binom{n-1}{n-k+1}\alpha_{n-1} = 0.$$

We know that Equation $n-(k-2)$., had it been listed, would have read

$$2\alpha_{n-k} + \left[\binom{n-k+1}{n-k}\alpha_{n-k+1} + \binom{n-k+2}{n-k}\alpha_{n-k+2} + \cdots + \binom{n-1}{n-k}\alpha_{n-1}\right]$$

$$= m\left[\binom{n-k+2}{n-k+1}\alpha_{n-k+1} + \binom{n-k+3}{n-k+1}\alpha_{n-k+1} + \cdots + \binom{n-1}{n-k+1}\alpha_{n-1}\right].$$

By our induction hypothesis, the right is equal to zero and hence,

$$\alpha_{n-k} = \frac{-1}{2}\left(\binom{n-k+1}{n-k}\alpha_{n-k+1} + \binom{n-k+2}{n-k}\alpha_{n-k+2} + \cdots + \binom{n-1}{n-k}\alpha_{n-1}\right).$$

Plugging this value of α_{n-k} in the LHS of the equation we are trying to establish, we obtain

$$\frac{-1}{2}\binom{n-k}{n-k-1}\left[\binom{n-k+1}{n-k}\alpha_{n-k+1}\right.$$

$$\left.\binom{n-k+2}{n-k}\alpha_{n-k+2} + \cdots + \binom{n-1}{n-k}\alpha_{n-1}\right]$$

$$+ \binom{n-k+1}{n-k-1}\alpha_{n-k+1} + \binom{n-k+2}{n-k+1}\alpha_{n-k+2}$$

$$+ \cdots + \binom{n-1}{n-k-1}\alpha_{n-1}.$$

We use trinomial revision again to get

$$\frac{-1}{2}\binom{n-k+1}{n-k-1}\binom{2}{1}\alpha_{n-k+1}+\binom{n-k+1}{n-k-1}\alpha_{n-k+1}$$

$$+\frac{-1}{2}\binom{n-k+2}{n-k-1}\binom{3}{1}\alpha_{n-k+2}+\binom{n-k+2}{n-k-1}\alpha_{n-k+2}$$

$$+\quad\cdots+\frac{-1}{2}\binom{n-1}{n-k-1}\binom{k}{1}\alpha_{n-1}+\binom{n-1}{n-k-1}\alpha_{n-1}$$

$$=\frac{-3}{2}\binom{n-k+2}{n-k-1}\alpha_{n-k+2}+\binom{n-k+2}{n-k-1}\alpha_{n-k+2}$$

$$+\frac{-4}{2}\binom{n-k+3}{n-k-1}\alpha_{n-k+3}+\binom{n-k+3}{n-k-1}\alpha_{n-k+2}$$

$$-\cdots+\frac{-k}{2}\binom{n-1}{n-k-1}\alpha_{n-1}+\binom{n-1}{n-k-1}\alpha_{n-1}$$

$$=\frac{-1}{2}\binom{n-k+2}{n-k-1}\alpha_{n-k+2}+\frac{-2}{2}\binom{n-k+3}{n-k-1}\alpha_{n-k+3}$$

$$-\cdots+\frac{-(k-2)}{2}\binom{n-1}{n-k-1}\alpha_{n-1}.$$

Using our induction hypothesis for α_{n-k+2} we get (this time from Equation $n-(k-4)$.)

$$-\frac{1}{2}\binom{n-k+2}{n-k-1}\left(\frac{-1}{2}\right)\left[\binom{n-k+3}{n-k+2}\alpha_{n-k+3}+\binom{n-k+4}{n-k+2}\alpha_{n-k+4}\right.$$

$$\left.+\cdots+\binom{n-1}{n-k+2}\alpha_{n-1}\right]-\frac{2}{2}\binom{n-k+3}{n-k-1}\alpha_{n-k+3}$$

$$-\frac{3}{2}\binom{n-k+4}{n-k-1}\alpha_{n-k+4}-\cdots-\frac{k-2}{2}\binom{n-1}{n-k-1}\alpha_{n-1}.$$

Now trinomial revision yields

$$\frac{1}{4}\binom{n-k+3}{n-k-1}\binom{4}{3}\alpha_{n-k+3}-\binom{n-k+3}{n-k-1}\alpha_{n-k+3}$$

$$+\frac{1}{4}\binom{n-k+4}{n-k-1}\binom{5}{3}\alpha_{n-k+4}-\frac{3}{2}\binom{n-k+4}{n-k-1}\alpha_{n-k+4}$$

$$+\cdots+\frac{1}{4}\binom{n-1}{n-k-1}\binom{k}{3}\alpha_{n-1}+\frac{-(k-2)}{2}\binom{n-1}{n-k-1}\alpha_{n-1}.$$

It is obvious that if we continue this process, the α_i's keep cancelling out until we get down to zero. This gives us the identity

$$\binom{n-k}{n-k-1}\alpha_{n-k}+\binom{n-k+1}{n-k-1}\alpha_{n-k-1}+\cdots+\binom{n-1}{n-k-1}\alpha_{n-1}=0$$

for all $k=2,4,6,\cdots$ and

$$\alpha_{n-k}=\frac{-1}{2}\left[\binom{n-k+1}{n-k}\alpha_{n-k+1}+\binom{n-k+2}{n-k}\alpha_{n-k+2}+\cdots+\binom{n-1}{n-k}\alpha_{n-1}\right]$$

(just as we saw in (B)) for all $k=2,4,6,\cdots$. This follows by using the identity on Equation $n-(k-2)$. for each even k. The two identities, when plugged in each of our $n+1$ equations for the appropriate k's, would yield

$$\alpha_{n-k}=-m\alpha_{n-k+1}$$

for $k=3,5,7,\cdots$. (Just as we had

$$\alpha_{n-3}=-m\alpha_{n-2}$$

at the beginning of our induction proof.) Summarizing what we have obtained, the recurrence solution is

$$\alpha_{n-1} = \frac{-1}{2} \text{ from Equation } n+1.$$

$$\alpha_{n-k} = \frac{-1}{2}\left[\binom{n-k+1}{n-k}\alpha_{n-k+1} + \binom{n-k+2}{n-k}\alpha_{n-k+2} + \cdots + \binom{n-1}{n-k}\alpha_{n-1}\right]$$

$$\text{for } k = 2, 4, 6, \cdots$$

$$\alpha_{n-k} = -m\alpha_{n-k+1} \quad \text{for } k = 3, 5, 7, \cdots .$$

This defines a recurrence for all the α_i's. Now let us see why there is no simple form for even values of n.

Recalling our result (A) left in abeyance we had,

$$(A) \qquad \alpha_0 + \alpha_1 m + \alpha_2 m^2 + \cdots + \alpha_{n-2}m^{n-2} = 0.$$

If n is even, by the third of our recurrence definitions

$$\alpha_{n-1} = \frac{-1}{2}; \quad \alpha_{n-3} = -m\alpha_{n-2};$$

$$\alpha_{n-5} = -m\alpha_{n-4}; \cdots ; \alpha_{n-(n-1)} = -m\alpha_{n-(n-2)},$$

since $n-1$ is odd. The last in this string of equalities says that

$$\alpha_1 = -m\alpha_2.$$

Plugging each of these in (A),

$$\alpha_0 + (-m\alpha_2 m + \alpha_2 m^2) + (-m\alpha_4 m^3 + \alpha_4 m^4) + \cdots + (-m\alpha_{n-2}m^{n-3} + \alpha_{n-2}m^{n-2}) = 0.$$

That is,

$$\alpha_0 + (-m^2\alpha_2 + m^2\alpha_2) + (-m^4\alpha_4 + m^4\alpha_4) + \cdots + (-m^{n-2}\alpha_{n-2} + m^{n-2}\alpha_{n-2}) = 0.$$

Thus, $\alpha_0 = 0$. Equation 1. says

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = 0.$$

Hence,

$$\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1} = 0,$$

since $\alpha_0 = 0$. The original sum which was

$$\sum_{k=1}^{m} k^n \binom{2m}{m-k} = -(m+1)\binom{2m}{m-1}(\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1})$$

would hence reduce to zero (meaning that m would have to be nonpositive). The $n+1$ equations constitute a non-consistent system when n is even, hence there is no closed form for these values of n. When n is odd we have (since

$$\alpha_1 + \cdots + \alpha_{n-1} = 0$$

from Equation 1.)

$$\begin{aligned}\sum_{k=1}^{m} k^n \binom{2m}{m-k} &= -(m+1)\binom{2m}{m-1}(\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}) \\ &= -(m+1)\binom{2m}{m-1}\alpha_0,\end{aligned}$$

where α_0 is obtained from the recursion defined above as

$$\begin{aligned}\alpha_0 &= -m\alpha_1 \\ &= \frac{m}{2}\left(\binom{2}{1}\alpha_2 + \binom{3}{1}\alpha_3 + \cdots + \binom{n-1}{1}\alpha_{n-1}\right) \\ &= \frac{m}{2}(2\alpha_2 + 3\alpha_3 + \cdots + (n-1)\alpha_{n-1}).\end{aligned}$$

That is, for all $n = 3, 5, 7, \ldots$

$$\sum_{k=1}^{m} k^n \binom{2m}{m-k} = \frac{-m(m+1)}{2}\binom{2m}{m-1}(2\alpha_2 + 3\alpha_3 + \cdots + (n-1)\alpha_{n-1}),$$

where $\alpha_2, \alpha_3, \dots, \alpha_{n-1}$ are calculated using the above recurrence solution.

85. [1995, 139] *Proposed by Robert E. Kennedy and Curtis Cooper, Central Missouri State University, Warrensburg, Missouri.*

A positive integer n is said to be a Niven number if it is divisible by its digital sum. Let A be the set of Niven numbers and $A(x)$ be the number of Niven numbers not exceeding x. Prove or disprove the following statement.

For every integer $m \geq 1$, there exists a positive integer n such that

$$\frac{n}{A(n)} = m.$$

Solution by the proposers. The statement is true.

Theorem. For $n > 1$, the ratio $n/A(n)$ takes on every integer value $m \geq 1$.

Proof. In [1], we showed that

$$\lim_{x\to\infty} \frac{A(x)}{x} = 0.$$

Hence $A(x)/x$ ultimately becomes and remains less than any assigned $\epsilon > 0$, as $x \to \infty$. It starts at $A(1)/1 = \frac{1}{1}$.

For any $m \geq 1$, there is a unique largest Niven number $n_k = n_{k(m)}$ for which $A(n_k) = k \geq n_k/m$. Thus, $mA(n_k) = mk \geq n_k$. Either $mk < n_{k+1}$ or $mk \geq n_{k+1}$.

If $mk < n_{k+1}$, and since $n_k \leq mk$, $A(n_k) \leq A(mk) < A(n_{k+1})$, from which $A(mk) = k$, and $mk/A(mk) = m$, so that $n = mk$ is an integer for which $n/A(n) = m$.

If $mk \geq n_{k+1}$, then $A(n_{k+1}) = k+1 > k = mk/m \geq n_{k+1}/m$ which contradicts the choice of n_k as the largest Niven number for which $A(n) \geq n/m$.

Reference

1. R. E. Kennedy and C. Cooper, "On the Natural Density of the Niven Numbers," *College Mathematics Journal*, 15 (1984), 309–312.

86. [1995, 139] *Proposed by Alan H. Rapoport, M.D., Ashford Medical Center, Santurce, Puerto Rico.*

An urn contains 20,000 balls consisting of 500 balls of each of 40 different colors. 100 balls consisting of exactly $1 \leq d \leq 40$ colors are selected at random without replacement from the urn. Let $P(d)$ denote the probability that the 100 balls consist of exactly d colors. Find d_0 such that

$$P(d_0) = \max_{1 \leq d \leq 40} P(d).$$

Solution by the proposer. We begin with a more specific problem.

An urn contains 20,000 balls consisting of 500 balls of each of 40 different colors. If 100 balls are selected at random without replacement, what is the probability of every color being represented at least once?

The number of ways to pick 100 balls without replacement from 20,000 balls is

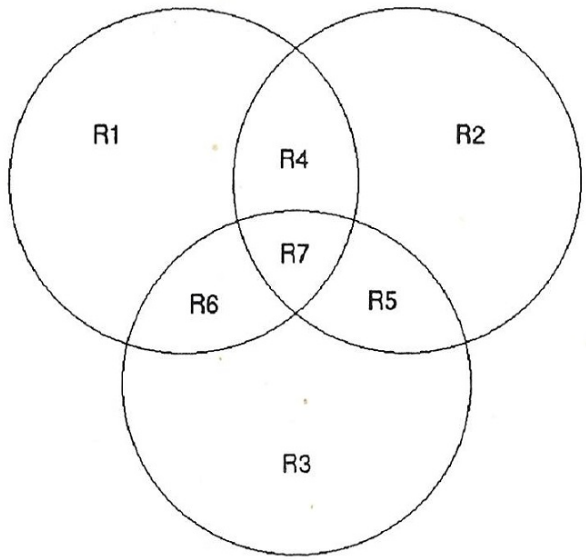
$$\binom{20000}{100}.$$

Using the Principle of Inclusion-Exclusion, the number of ways to pick 40 different colored balls among the 100 balls chosen is

$$\sum_{k=1}^{40} (-1)^{40-k} \binom{500k}{100} \binom{40}{k}.$$

Therefore, the probability of every color being represented at least once is the ratio of these two quantities.

To better explain the last summation, let us use 1500 balls consisting of 500 balls of each of 3 different colors. We have the following Venn diagram.



The elements in the circled sets are all the possible selections of 100 balls. The 3 sets represent a selection with one of the colors present. An element in any one of the 7 regions represents a selection of 100 balls where certain colors are present. An element in only one of the sets, outside the other 2 sets, represents a selection of 100 balls with only one color present. We want to count the number of elements in the intersection of all 3 sets. To do this we will label the regions and show what each quantity in the Principle of Inclusion-Exclusion represents.

First, the number of ways to select 100 balls from the 1500 is

$$\binom{1500}{100}.$$

This is the number of elements in

$$R1 \cup R2 \cup R3 \cup R4 \cup R5 \cup R6 \cup R7.$$

Second, the number of ways to select 100 balls from the 1000 (we exclude selections which include a particular color) is

$$\binom{1000}{100}.$$

This is the number of elements in

$$R1 \cup R2 \cup R4.$$

It is also the number of elements in

$$R2 \cup R3 \cup R5 \quad \text{and} \quad R1 \cup R3 \cup R6.$$

Third, the number of ways to select 100 balls from the 500 (we exclude selections which include 2 specific colors) is

$$\binom{500}{100}.$$

This is the number of elements in $R1$. It is also the number of elements in $R2$ and the number of elements in $R3$.

Finally, the number of elements in $R7$ is (by the Principle of Inclusion-Exclusion)

$$\binom{1500}{100} - 3\binom{1000}{100} + 3\binom{500}{100}.$$

This follows from the identity below (vertical bars represent the number of elements in the set).

$$\begin{aligned}|R7| = {} & |R1| + |R2| + |R3| + |R4| + |R5| + |R6| + |R7| \\ & - |R1| - |R2| - |R4| - |R2| - |R3| - |R5| - |R1| - |R3| - |R6| \\ & + |R1| + |R2| + |R3|.\end{aligned}$$

Finally, regarding the original problem, again we use the Principle of Inclusion-Exclusion. The denominator of $P(d)$ is

$$\binom{20000}{100}$$

and the numerator of $P(d)$, the probability that d different colors are represented among the 100 balls is

$$\binom{40}{d}\sum_{k=1}^{d}(-1)^{d-k}\binom{500k}{100}\binom{d}{k}.$$

Computing these quantities for $0 \le d \le 40$, $d_0 = 37$.

87*. [1995, 140] *Proposed by James H. Taylor, Central Missouri State University, Warrensburg, Missouri.*

Let n be a nonnegative integer and $0 \le k \le n$. Let $J(\lambda)$ be the $(n+1)\times(n+1)$ matrix

$$\begin{pmatrix}
-\lambda & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\\
n & -\lambda & 2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\\
\lambda & n-2 & -\lambda & 3 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\\
-n & \lambda & n-4 & -\lambda & 4 & \cdots & 0 & 0 & 0 & 0 & 0\\
0 & -n+1 & \lambda & n-6 & -\lambda & \cdots & 0 & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & \cdots & -\lambda & n-3 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 8-n & -\lambda & n-2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & \lambda & 6-n & -\lambda & n-1 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & 4-n & -\lambda & n\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -3 & \lambda & 2-n & -\lambda
\end{pmatrix}.$$

Also, let v be the coefficient vector of the polynomial

$$(x+1)^{n-k}(x-1)^k.$$

Show that $J(n-2k)v = 0$.

Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri.

Proof. It is a routine matter to show that the theorem is true for $n = 1, 2, 3, 4, 5$. So assume for some $n = N \geq 5$, the theorem is true for all k, $0 \leq k \leq N$.

Now let $n = N + 1$. Let k be an integer, $0 \leq k \leq N$. Assume

$$(x+1)^{N-k}(x-1)^k = x^N + a_1 x^{N-1} + a_2 x^{N-2} + \cdots + a_{N-1}x + (-1)^k.$$

Then

$$\begin{aligned}
&(x+1)^{N+1-k}(x-1)^k \\
&= (x^N + a_1 x^{N-1} + a_2 x^{N-2} + \cdots + a_{N-1}x + (-1)^k)(x+1) \\
&= x^{N+1} + (1+a_1)x^N + (a_1+a_2)x^{N-1} + (a_2+a_3)x^{N-2} + \cdots \\
&\quad + (a_{N-1} + (-1)^k)x + (-1)^k.
\end{aligned}$$

Let

$$v_N = \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ a_{N-1} \\ (-1)^k \end{pmatrix} \text{ and } v_{N+1} = \begin{pmatrix} 1 \\ 1+a_1 \\ a_1+a_2 \\ . \\ . \\ . \\ a_{N-1}+(-1)^k \\ (-1)^k \end{pmatrix}.$$

v_N is $(N+1) \times 1$ and v_{N+1} is $(N+2) \times 1$.

$J(N-2k) =$

$$\begin{pmatrix}
2k-N & 1 & 0 & . & ..0 & 0 & 0 & 0 & 0 \\
N & 2k-N & 2 & 0 & ..0 & 0 & 0 & 0 & 0 \\
N-2k & N-2 & 2k-N & 3 & ..0 & 0 & 0 & 0 & 0 \\
-N & N-2k & N-4 & 2k-N & ..0 & 0 & 0 & 0 & 0 \\
0 & -N+1 & N-2k & N-6 & ..0 & 0 & 0 & 0 & 0 \\
. & . & . & . & ... & . & . & . & . \\
. & . & . & . & ... & . & . & . & . \\
. & . & . & . & ... & . & . & . & . \\
0 & 0 & 0 & 0 & ..0 & -3 & N-2k & 2-N & 2k-N
\end{pmatrix}$$

and

$J(N+1-2k) =$

$$\begin{pmatrix} 2k-N-1 & 1 & 0 & ..0 & 0 & 0 & 0 \\ N+1 & 2k-N-1 & 2 & ..0 & 0 & 0 & 0 \\ N+1-2k & N-1 & 2k-N-1 & ..0 & 0 & 0 & 0 \\ -N-1 & N+1-2k & N-3 & ..0 & 0 & 0 & 0 \\ 0 & -N & N+1-2k & ..0 & 0 & 0 & 0 \\ . & . & . & ... & . & . & . \\ . & . & . & ... & . & . & . \\ . & . & . & ... & . & . & . \\ 0 & 0 & 0 & ..0 & N+1-2k & 1-N & 2k-N-1 \end{pmatrix}$$

$J(N-2k)v_N = \mathbf{0}$ by the induction hypothesis, so now it is necessary to show that $J(N+1-2k)v_{N+1} = \mathbf{0}$.

The first row of $J(N+1-2k)$ times v_{N+1} is

$$\begin{aligned}(2k-N-1)\cdot 1 + 1\cdot(1+a_1) &= (2k-N)\cdot 1 - 1 + 1 + a_1 \\ &= (2k-N)\cdot 1 + 1\cdot a_1.\end{aligned}$$

This last expression is the first row of $J(N-2k)$ times v_N, which is 0, so the first row of $J(N+1-2k)$ times v_{N+1} is 0.

The second row of $J(N+1-2k)$ times v_{N+1} is

$$\begin{aligned}&(N+1)\cdot 1 + (2k-N-1)\cdot(1+a_1) + 2\cdot(a_1+a_2) \\ &= N+1+(2k-N)\cdot a_1 - a_1 + 2k - N - 1 + 2a_1 + 2a_2 \\ &= N\cdot 1 + (2k-N)\cdot a_1 + 2\cdot a_2 + (2k-N)\cdot 1 + 1\cdot a_1 \\ &= 0.\end{aligned}$$

This zero comes from the second row of $J(N-2k)$ times v_N plus the first row of $J(N-2k)$ times v_N.

Similarly, the third and fourth rows of $J(N+1-2k)$ times v_{N+1} are zero.

Let r be a row index of $J(N+1-2k)$, $5 \leq r \leq N+1$. The rth row of $J(N+1-2k)$ times v_{N+1} is

$$\begin{aligned}
&(-N+r-5)\cdot(a_{r-5}+a_{r-4})+(N+1-2k)\cdot(a_{r-4}+a_{r-3})\\
&+(N+1-2(r-2))\cdot(a_{r-3}+a_{r-2})+(2k-N-1)\cdot(a_{r-2}+a_{r-1})\\
&+r\cdot(a_{r-1}+a_r)\\
&=(-N+r-5)\cdot a_{r-5}+(N-2k)\cdot a_{r-4}+(N-2(r-3))\cdot a_{r-3}\\
&+(2k-N)\cdot a_{r-2}+(r-1)\cdot a_{r-1}+(-N+r-4)\cdot a_{r-4}+(N-2k)\cdot a_{r-3}\\
&+(N-2(r-2))\cdot a_{r-2}+(2k-N)\cdot a_{r-1}+r\cdot a_r.
\end{aligned}$$

This last expression is the sum of the rth and $(r-1)$st rows of $J(N-2k)$ times v_N. Both of these are zero, so the rth row of $J(N+1-2k)$ times v_{N+1} is zero.

Finally, the $(N+2)$nd row of $J(N+1-2k)$ times v_{N+1} is

$$\begin{aligned}
&-3\cdot(a_{N-3}+a_{N-2})+(N+1-2k)\cdot(a_{N-2}+a_{N-1})+(1-n)\cdot(a_{N-1}+(-1)^k)\\
&+(2k-N-1)\cdot(-1)^k\\
&=-3\cdot a_{N-3}+(N-2k)\cdot a_{N-2}+(2-N)\cdot a_{N-1}+(2k-N)\cdot(-1)^k\\
&+(-2)\cdot a_{N-2}+(N-2k)\cdot a_{N-1}+(-N)\cdot(-1)^k.
\end{aligned}$$

The sum of the first four terms is the product of the $(N+1)$st row of $J(N-2k)$ times v_N, so this sum is zero. Checking the expansion of $(x+1)^{N-k}(x-1)^k$ shows that $a_{N-2}=(-1)^k\cdot a_2$ and $a_{N-1}=(-1)^k\cdot a_1$. This implies the sum of the last three terms is $(-1)^{k+1}\cdot[N+(2k-N)\cdot a_1+2a_2]$, and the bracketed quantity is the second row of $J(N-2k)$ times v_N which again equals zero. This concludes the proof.

88. [1995, 140] *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Let m be a positive integer. Prove that

$$\prod_{i=1}^{m} \cos^2 \frac{i\pi}{2m+1} = \frac{1}{4^m}.$$

Solution I by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin. We begin by establishing the following preliminary results.

Lemma. Let t be a non-negative integer. Then

$$\binom{t}{0} + \binom{t}{2} + \binom{t}{4} + \cdots = \binom{t}{1} + \binom{t}{3} + \binom{t}{5} + \cdots .$$

Proof. By the binomial theorem,

$$\binom{t}{0} - \binom{t}{1} + \binom{t}{2} - \cdots + (-1)^r \binom{t}{r} + \cdots + (-1)^t \binom{t}{t}$$

$$= \sum_{k=0}^{t} \binom{t}{k} 1^{t-k}(-1)^k$$

$$= (1 + (-1))^t = 0.$$

Thus,

$$\binom{t}{0} + \binom{t}{2} + \binom{t}{4} + \cdots = \binom{t}{1} + \binom{t}{3} + \binom{t}{5} + \cdots .$$

Corollary 1. Let t be a non-negative integer. Then

$$\binom{t}{1} + \binom{t}{3} + \binom{t}{5} + \cdots = 2^{t-1}.$$

Proof. By the binomial theorem,

$$2^t = (1+1)^t = \sum_{k=0}^{t} \binom{t}{k} 1^{t-k} 1^k = \sum_{k=0}^{t} \binom{t}{k}$$

$$= \left[\binom{t}{0} + \binom{t}{2} + \binom{t}{4} + \cdots\right] + \left[\binom{t}{1} + \binom{t}{3} + \binom{t}{5} + \cdots\right]$$

$$= 2\left[\binom{t}{1} + \binom{t}{3} + \binom{t}{5} + \cdots\right],$$

by the preceding Lemma. Thus,

$$\binom{t}{1} + \binom{t}{3} + \binom{t}{5} + \cdots = \frac{2^t}{2} = 2^{t-1}.$$

Corollary 2. Let m be a non-negative integer. Then

$$\binom{2m+1}{1} + \binom{2m+1}{3} + \binom{2m+1}{5} + \cdots + \binom{2m+1}{2m+1} = 2^{2m}.$$

Proof. Let $t = 2m+1$ in the preceding corollary.
By DeMoivre's theorem

$$\cos ja + i \sin ja = (\cos a + i \sin a)^j.$$

Expanding the right-hand side of the above and equating imaginary parts gives

$$\sin ja = \binom{j}{1} \cos^{j-1} a \sin a - \binom{j}{3} \cos^{j-3} a \sin^3 a + \cdots .$$

Letting $j = 2m+1$ yields

$$\begin{aligned}
&\sin(2m+1)a \\
&= \binom{2m+1}{1}\cos^{2m} a \sin a - \binom{2m+1}{3}\cos^{2(m-1)}\sin^3 a + \cdots \\
&= \sin a\Bigg[\binom{2m+1}{1}\cos^{2m} a - \binom{2m+1}{3}\cos^{2(m-1)} a(1-\cos^2 a) \\
&\quad + \cdots + (-1)^m(1-\cos^2 a)^m\Bigg].
\end{aligned}$$

Observe that

$$\sin(2m+1)a = 0 \quad \text{when} \quad a = \frac{\pi}{2m+1}, \frac{2\pi}{2m+1}, \dots, \frac{m\pi}{2m+1},$$

but $\sin a \neq 0$ for any of these values of a. So the expression in the brackets above must be equal to zero for all these m values of a. Therefore for these values of a, $\cos^2 a$ are the zeros of

$$\binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1}(1-x) + \cdots + (-1)^m(1-x)^m.$$

This is a polynomial of degree m with zeros

$$\cos^2\frac{\pi}{2m+1}, \cos^2\frac{2\pi}{2m+1}, \dots, \cos^2\frac{m\pi}{2m+1}.$$

The constant term of the polynomial is $(-1)^m$. The coefficient of x^m is

$$\binom{2m+1}{1} + \binom{2m+1}{3} + \binom{2m+1}{5} + \cdots + \binom{2m+1}{2m+1}$$

which equals 2^{2m} by Corollary 2. Hence the product of the zeros of the polynomial is

$$\cos^2\frac{\pi}{2m+1}\cos^2\frac{2\pi}{2m+1}\cdots\cos^2\frac{m\pi}{2m+1} = (-1)^m\cdot\frac{(-1)^m}{2^{2m}} = \frac{1}{2^{2m}} = \frac{1}{4^m}.$$

Solution II by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri. By induction one can show that $\cos n\theta$ is expressible as a polynomial of degree n in $\cos\theta$, where the coefficient of the term in $\cos^n\theta$ is 2^{n-1}. Hence, if ψ denotes another real variable (besides θ), we can write

$$\cos n\theta - \cos n\psi = \sum_{k=0}^{n} A_k \cos^k\theta,$$

where $A_k = A_k(n,\psi)$ and $A_n = 2^{n-1}$. The left-hand side vanishes when $n\theta = n\psi$, $n\psi = 2\pi$, $n\psi = 4\pi$, $\ldots$, $n\psi = 2(n-1)\pi$, so we may factor the right-hand side as

$$\cos n\theta - \cos n\psi = 2^{n-1}\prod_{k=0}^{n-1}\left[\cos\theta - \cos\bigl(\psi + \frac{2k\pi}{n}\bigr)\right].$$

Now let $\theta = 0$, $\psi = \pi$, $n = 2m+1$; the equation reduces to

$$\begin{aligned} 1 = \sin^2\left(\frac{2m+1}{2}\pi\right) &= 2^{4m}\prod_{k=0}^{2m}\sin^2\left(\frac{\pi}{2} + \frac{k\pi}{2m+1}\right) \\ &= 2^{4m}\prod_{k=1}^{2m}\cos^2\left(\frac{k\pi}{2m+1}\right). \end{aligned}$$

Since

$$\cos^2\left(\frac{k\pi}{2m+1}\right) = \cos^2\left(\frac{(2m+1-k)\pi}{2m+1}\right)$$

for any integer k, then upon taking square roots we obtain

$$1 = 2^{2m} \prod_{k=1}^{m} \cos^2\left(\frac{k\pi}{2m+1}\right).$$

Solution III by F. J. Flanigan, San Jose State University, San Jose, California.

Outline of Proof.

Step 1. Reduction to the "standard" product.

$$\prod_{i=1}^{m} \cos^2 \frac{i\pi}{2m+1} = \left| \prod_{k=1}^{2m} \cos \frac{k\pi}{2m+1} \right|.$$

Step 2. For positive integers N,

$$\prod_{k=1}^{N-1} \cos \frac{k\pi}{N} = \begin{cases} 0, & N \text{ even} \\ (-1)^{(1/2)(N-1)}/2^{N-1}, & N \text{ odd.} \end{cases}$$

Step 3. In the above, let $N = 2m+1$ and take absolute value to get $1/4^m$, as required.

Proofs.

Step 1. Note

$$\cos^2 \frac{i\pi}{2m+1} = \cos \frac{i\pi}{2m+1} \cdot \left| \cos \frac{(2m+1-i)\pi}{2m+1} \right|$$

for $i = 1, \dots, m$. Now re-index using $k = 1, \dots, m, m+1, \dots, 2m$.

Step 2. The key is the identity (see R. C. Mullin, *American Mathematical Monthly*, 69 (1962), 217).

$$(*) \qquad \prod_{k=1}^{N-1} (z^2 - 2z\cos(k\pi/N) + 1) = \sum_{k=0}^{N-1} z^{2k}$$

which holds for all positive integers N, because

$$\sum_{k=0}^{N-1} z^{2k} = (z^{2N} - 1)/(z^2 - 1)$$

and this polynomial has the $2N-2$ roots, $z = e^{ik\pi/N}$ with $k = \pm 1, \pm 2, \ldots, \pm(N-1)$. Now put $z = -i$ in $(*)$ and obtain

$$\prod_{k=1}^{N-1} 2i\cos(k\pi/N) = (1/2)(1 - (-1)^N)$$

which readily gives the assertions of Step 2.

Step 3. This is immediate.

Comments and Queries.

(a) Are there other references for the cosine product in Step 2?

(b) By letting $z = \pm 1$, $(*)$ also yields

$$\prod_{k=1}^{N-1} \sin\frac{k\pi}{N} = N/2^{N-1}.$$

Generalized Solution by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri.

We will prove the stronger result

$$\prod_{i=1}^{m} \cos^n\left(\frac{\pi i}{2m+1}\right) = \frac{1}{2^{nm}}, \tag{1}$$

for all real numbers n.

Let $m = 2j$ for some positive integer j. Then

$$\prod_{i=1}^{m} \sin\left(\frac{\pi i}{2m+1}\right) = \prod_{i=1}^{j} \sin\left(\frac{2\pi i}{4j+1}\right) \prod_{i=1}^{j} \sin\left(\frac{\pi(2i-1)}{4j+1}\right).$$

Since $\sin\theta = \sin(\pi - \theta)$, the second product on the right hand side of the above identity can be rewritten to give

$$\prod_{i=1}^{m} \sin\left(\frac{\pi i}{2m+1}\right) = \prod_{i=1}^{j} \sin\left(\frac{2\pi i}{4j+1}\right) \prod_{i=1}^{j} \sin\left(\frac{2\pi(2j-i+1)}{4j+1}\right)$$

and so

$$\begin{aligned} \prod_{i=1}^{m} \sin\left(\frac{\pi i}{2m+1}\right) &= \prod_{i=1}^{j} \sin\left(\frac{2\pi i}{4j+1}\right) \prod_{i=j+1}^{2j} \sin\left(\frac{2\pi i}{4j+1}\right) \\ &= \prod_{i=1}^{2j} \sin\left(\frac{2\pi i}{4j+1}\right) \\ &= \prod_{i=1}^{m} \sin\left(\frac{2\pi i}{2m+1}\right). \end{aligned} \tag{2}$$

Similarly, identity (2) holds if m is odd. Now

$$\begin{aligned} \prod_{i=1}^{m} \cos\left(\frac{\pi i}{2m+1}\right) \prod_{i=1}^{m} \sin\left(\frac{\pi i}{2m+1}\right) &= \prod_{i=1}^{m} \left[\cos\left(\frac{\pi i}{2m+1}\right) \sin\left(\frac{\pi i}{2m+1}\right)\right] \\ &= \prod_{i=1}^{m} \left[\frac{1}{2} \sin\left(\frac{2\pi i}{2m+1}\right)\right] \\ &= \frac{1}{2^m} \prod_{i=1}^{m} \sin\left(\frac{2\pi i}{2m+1}\right). \end{aligned}$$

Using (2) gives

$$\prod_{i=1}^{m} \cos\left(\frac{\pi i}{2m+1}\right) = \frac{1}{2^m}.$$

Since

$$\cos\left(\frac{\pi i}{2m+1}\right) > 0$$

for $1 \leq i \leq m$, raising both sides of the latter identity to the power n yields (1) and the desired result follows when $n = 2$.

Comment by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri. The result in Problem 88 is tabulated in the equivalent form

$$2^{(n-1)/2} \cos\left(\frac{2\pi}{2n}\right) \cos\left(\frac{4\pi}{2n}\right) \cdots \cos\left(\frac{(n-1)\pi}{2n}\right) = 1,$$

if n is odd, as Formula No. 1022 in L. B. W. Jolley, *Summation of Series*, 2nd ed. Dover Publications, New York, 1961, p. 190.

Also solved by the proposers.