CONCURRENCY THEOREMS FOR PENTAGRAMS

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Although concurrency theorems for cevians of triangles are very well known, there appear to be few comparable results for either simple polygons or for star polygons. A short history of *regular* star polygons, or regular polygrams, is given in [1, pp. 44–45]. Most of this history is prior to 1678 when the Italian mathematician, Giovanni Ceva (1647–1736), published his famous Ceva's theorem. In this note we give what may be a new proof of Ceva's theorem and use the same techniques to prove several results about pentagrams.

By definition a *cevian of a triangle* is a line segment from one vertex to a point on the opposite side which is not an endpoint. In the same manner we define a *cevian of a pentagon* as a line segment from one vertex to a point on the opposite side which is not an endpoint. The most ubiquitous examples of cevians of triangles are medians, angle-bisectors, and altitudes; there does not, however, appear to be any special named cevians for pentagons. We make frequent use of the following.

Lemma 1. If $\overline{CD}$ is a cevian of $\triangle ABC$ (see Figure 1), then

$$\frac{AD}{DB} = \frac{AC \sin ACD}{BC \sin DCB}.$$

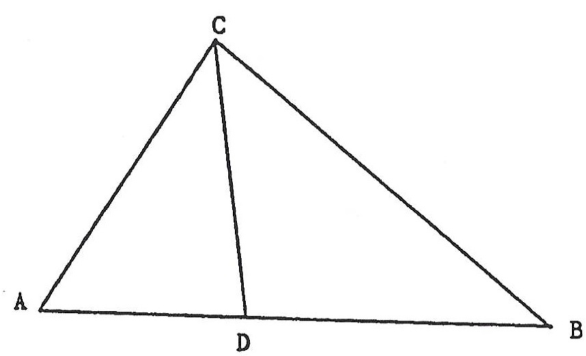

Figure 1.

Proof. By the law of sines in $\triangle ADC$ and $\triangle BDC$, respectively, we have $AD/AC = \sin ACD/\sin ADC$ and $DB/BC = \sin DCB/\sin BDC$. Dividing the first equation by the second, noting that $\sin ADC = \sin BDC$, and simplifying yields the result.

As an application of this lemma, we prove the following:

Theorem 1. (Ceva's Theorem): If cevians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ of $\triangle ABC$ are concurrent at a point P, then

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1,$$

and conversely.

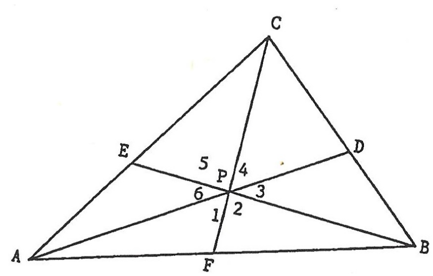

Figure 2.

Proof. With notation as in Figure 2 and by Lemma 1, we have

$$\frac{AF}{FB} = \frac{PA \sin \angle 1}{PB \sin \angle 2},$$

$$\frac{BD}{DC} = \frac{PB \sin \angle 3}{PC \sin \angle 4},$$

$$\text{and} \quad \frac{CE}{EA} = \frac{PC \sin \angle 5}{PA \sin \angle 6}.$$

By multiplying these three equations, noting that vertical angles are congruent, and simplifying we obtain the result. With minor variations this proof also holds when P is outside $\triangle ABC$. The converse, which is sometimes considered a part of Ceva's theorem, can be proved in the usual way.

From an examination of approximately thirty available "college geometry" textbooks, it was found that the methods of proof of Ceva's theorem involved either areas of triangles, similar triangles, or else Menelaus' theorem. None used the above approach which appears (to this author) to be a much simpler method.

Next we use this approach to prove a generalization of Ceva's theorem for pentagons. We note that this result is also given as a problem in [1, p. 67].

Theorem 2. Let $ABCDE$ be a convex pentagon with a point P inside the pentagon such that there exist cevians from each vertex through P, to points A', B', C', D', and E' on the opposite sides (see Figure 3), then

$$\frac{AD'}{D'B} \cdot \frac{BE'}{E'C} \cdot \frac{CA'}{A'D} \cdot \frac{DB'}{B'E} \cdot \frac{EC'}{C'A} = 1,$$

and conversely, if four cevians are concurrent at P and the formula holds, then the fifth cevian is concurrent at P.

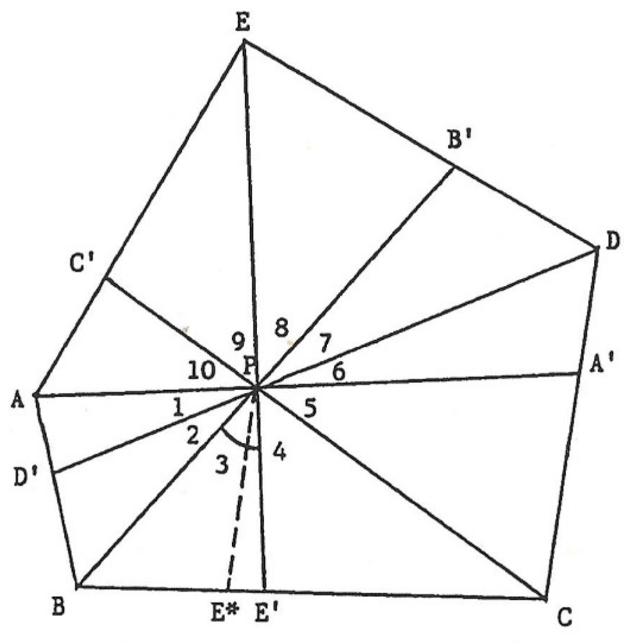

Figure 3.

Proof. Proceeding as in the proof of Ceva's theorem and using Lemma 1 five times we have

$$\frac{AD'}{D'B}\cdot\frac{BE'}{E'C}\cdot\frac{CA'}{A'D}\cdot\frac{DB'}{B'E}\cdot\frac{EC'}{C'A}$$

$$=\frac{PA\sin\angle 1}{PB\sin\angle 2}\cdot\frac{PB\sin\angle 3}{PC\sin\angle 4}\cdot\frac{PC\sin\angle 5}{PD\sin\angle 6}\cdot\frac{PD\sin\angle 7}{PE\sin\angle 8}\cdot\frac{PE\sin\angle 9}{PA\sin\angle 10}$$

$$=1.$$

Conversely, suppose that cevians $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$, and $\overline{DD'}$ are concurrent at P. We must show that $\overline{EE'}$ also passes through P. Let $\overline{EP}$ be extended through P to meet $\overline{BC}$ at E^*. Then by the first part of this theorem we have

$$\frac{AD'}{D'B}\cdot\frac{BE^*}{E^*C}\cdot\frac{CA'}{A'D}\cdot\frac{DB'}{B'E}\cdot\frac{EC'}{C'A}=1.$$

By equating this formula with the formula of our hypothesis, and simplifying we obtain $BE^*/E^*C = BE'/E'C$. By adding one to each side of this equation we obtain $BC/E^*C = BC/E'C$ which implies $E^* = E'$. Hence, $\overline{EE'}$ is concurrent with the other cevians.

As an aside we note that we can obtain a trigonometric form of Ceva's theorem for pentagons by using the law of sines in $\triangle APB$, $\triangle BPC$, $\triangle CPD$, $\triangle DPE$, and $\triangle EPA$, respectively (Figure 3) to obtain

$$\frac{\sin PAB}{\sin PBA}=\frac{PB}{PA},\quad \frac{\sin PBC}{\sin PCB}=\frac{PC}{PB},\quad \frac{\sin PCD}{\sin PDC}=\frac{PD}{PC},$$

$$\frac{\sin PDE}{\sin PED}=\frac{PE}{PD},\quad \text{and}\quad \frac{\sin PEA}{\sin PAE}=\frac{PA}{PE}.$$

By multiplying these five equations and simplifying, we have

$$\frac{\sin PAB}{\sin PBA}\cdot\frac{\sin PBC}{\sin PCB}\cdot\frac{\sin PCD}{\sin PDC}\cdot\frac{\sin PDE}{\sin PED}\cdot\frac{\sin PEA}{\sin PAE}=1.$$

Next we give a new proof of a result that appeared in [2]. Not only is the new proof shorter and easier to follow, but it also shows that the result can be easily generalized.

<u>Theorem 3</u>. Let $ABCDEFGHIJ$ be a pentagram (Figure 4), then

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FG} \cdot \frac{GH}{HI} \cdot \frac{IJ}{JA} = 1.$$

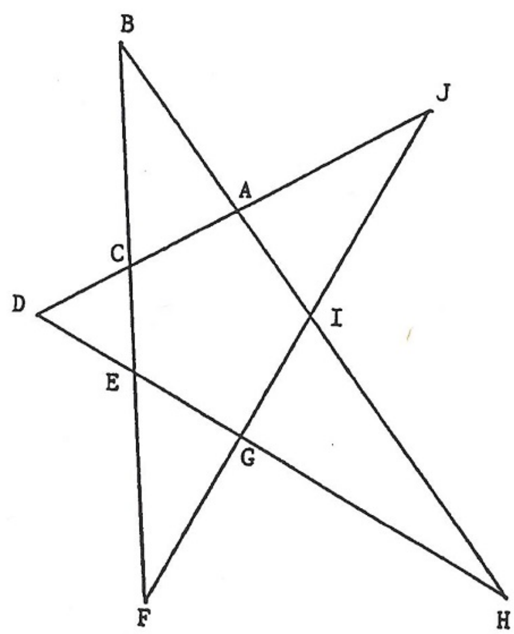

Figure 4.

<u>Proof</u>. By using the law of sines in $\triangle ABC$, $\triangle CDE$, $\triangle EFG$, $\triangle GHI$, and $\triangle IJA$, respectively, we have

$$\frac{AB}{BC} = \frac{\sin ACB}{\sin BAC}, \quad \frac{CD}{DE} = \frac{\sin CED}{\sin DCE}, \quad \frac{EF}{FG} = \frac{\sin EGF}{\sin FEG},$$

$$\frac{GH}{HI} = \frac{\sin GIH}{\sin HGI}, \quad \text{and} \quad \frac{IJ}{JA} = \frac{\sin IAJ}{\sin JIA}.$$

By multiplying these five equations, noting that vertical angles are congruent, and simplifying we obtain the result.

Finally, by combining the above theorems we have the following theorem.

Theorem 4. Let $ABCDEFGHIJ$ be a pentagram (Figure 5) in which the "diagonals" $\overline{BG}$, $\overline{DI}$, $\overline{FA}$, $\overline{HC}$, and $\overline{JE}$ are concurrent at some point P inside pentagon $ACEGI$, then

$$\frac{\sin\angle 1}{\sin\angle 2}\cdot\frac{\sin\angle 3}{\sin\angle 4}\cdot\frac{\sin\angle 5}{\sin\angle 6}\cdot\frac{\sin\angle 7}{\sin\angle 8}\cdot\frac{\sin\angle 9}{\sin\angle 10}=1,$$

and conversely, if four diagonals are concurrent and this formula holds, then the fifth diagonal is also concurrent at this point.

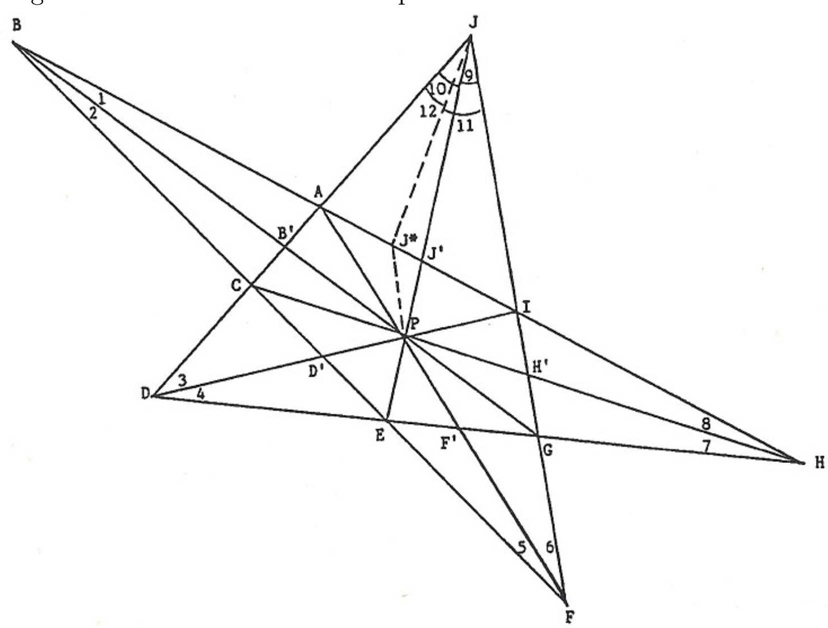

Figure 5.

Proof. Let B', D', F', H', and J' be the respective intersections of the diagonals of the pentagram with the corresponding sides of pentagon $ACEGI$ as shown in Figure 5. By Lemma 1 for $\triangle ABC$, $\triangle CDE$, $\triangle EFG$, $\triangle GHI$, and $\triangle IJA$, respectively, we have

$$\frac{AB'}{B'C} = \frac{AB\sin\angle 1}{BC\sin\angle 2}, \quad \frac{CD'}{D'E} = \frac{CD\sin\angle 3}{DE\sin\angle 4}, \quad \frac{EF'}{F'G} = \frac{EF\sin\angle 5}{FG\sin\angle 6},$$

$$\frac{GH'}{H'I} = \frac{GH\sin\angle 7}{HI\sin\angle 8}, \quad \text{and} \quad \frac{IJ'}{J'A} = \frac{IJ\sin\angle 9}{JA\sin\angle 10}.$$

By substituting these five equations into the formula,

$$\frac{AB'}{B'C}\cdot\frac{CD'}{D'E}\cdot\frac{EF'}{F'G}\cdot\frac{GH'}{H'I}\cdot\frac{IJ'}{J'A} = 1$$

of Theorem 2, and rearranging factors we have

$$\left(\frac{AB}{BC}\cdot\frac{CD}{DE}\cdot\frac{EF}{FG}\cdot\frac{GH}{HI}\cdot\frac{IJ}{JA}\right)$$

$$\cdot\left(\frac{\sin\angle 1}{\sin\angle 2}\cdot\frac{\sin\angle 3}{\sin\angle 4}\cdot\frac{\sin\angle 5}{\sin\angle 6}\cdot\frac{\sin\angle 7}{\sin\angle 8}\cdot\frac{\sin\angle 9}{\sin\angle 10}\right) = 1.$$

Since the first set of factors is equal to one by Theorem 3, the result follows.

For the proof of the converse let diagonals $\overline{BG}$, $\overline{DI}$, $\overline{FA}$, $\overline{HC}$, and $\overline{JE}$ meet the sides of pentagon $ACEGI$ at B', D', F', H', and J', respectively. Suppose that the first four diagonals are concurrent at P and let J^* be the intersection of $\overline{EP}$ and $\overline{AI}$. Additionally, let $\angle 11 = \angle IJJ^*$ and $\angle 12 = \angle J^*JA$.

Since $\overline{AF'}$, $\overline{CH'}$, $\overline{EJ^*}$, $\overline{GB'}$, and $\overline{ID'}$ are concurrent at P, then by Theorem 2,

$$\frac{AB'}{B'C}\cdot\frac{CD'}{D'E}\cdot\frac{EF'}{F'G}\cdot\frac{GH'}{H'I}\cdot\frac{IJ^*}{J^*A} = 1.$$

But

$$\frac{AB'}{B'C}=\frac{AB\sin\angle 1}{BC\sin\angle 2},\quad \frac{CD'}{D'E}=\frac{CD\sin\angle 3}{DE\sin\angle 4},\quad \frac{EF'}{F'G}=\frac{EF\sin\angle 5}{FG\sin\angle 6},$$

$$\frac{GH'}{H'I}=\frac{GH\sin\angle 7}{HI\sin\angle 8},\quad \text{and}\quad \frac{IJ^*}{J^*A}=\frac{IJ\sin\angle 11}{JA\sin\angle 12}$$

by Lemma 1. By substituting these five formulas into the formula above and rearranging terms we have

$$\left(\frac{AB}{BC}\cdot\frac{CD}{DE}\cdot\frac{EF}{FG}\cdot\frac{GH}{HI}\cdot\frac{IJ}{JA}\right)$$

$$\cdot\left(\frac{\sin\angle 1}{\sin\angle 2}\cdot\frac{\sin\angle 3}{\sin\angle 4}\cdot\frac{\sin\angle 5}{\sin\angle 6}\cdot\frac{\sin\angle 7}{\sin\angle 8}\cdot\frac{\sin\angle 11}{\sin\angle 12}\right)=1.$$

But the first set of factors equals 1 by Theorem 3. By equating

$$\frac{\sin\angle 1}{\sin\angle 2}\cdot\frac{\sin\angle 3}{\sin\angle 4}\cdot\frac{\sin\angle 5}{\sin\angle 6}\cdot\frac{\sin\angle 7}{\sin\angle 8}\cdot\frac{\sin\angle 11}{\sin\angle 12}=1$$

with the formula in the hypothesis and simplifying we have

$$\frac{\sin\angle 9}{\sin\angle 10}=\frac{\sin\angle 11}{\sin\angle 12}.$$

Hence,

$$\frac{IJ\sin\angle 9}{JA\sin\angle 10}=\frac{IJ\sin\angle 11}{JA\sin\angle 12}.$$

Substituting via Lemma 1 yields

$$\frac{IJ'}{J'A}=\frac{IJ^*}{J^*A}.$$

Hence, $J'=J^*$. Therefore diagonal $\overline{EJ}$ passes through P so that all five diagonals are concurrent.

By the nature of the above proofs it should be clear that all the theorems can be generalized to polygrams with an *odd* number of sides and that some can be generalized to arbitrary polygrams. This is left to the interested reader.

References

1. H. Eves, *A Survey of Geometry* (rev. ed.), Allyn and Bacon, Inc., Boston, 1972.
2. L. Hoehn, "A Menelaus-type Theorem for the Pentagram," *Mathematics Magazine*, 66 (1993), 121–123.

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