A GENERALIZED EXPONENTIAL FUNCTION

Russell Euler

Northwest Missouri State University

Let p be a positive integer. Using the ratio test, it can be shown that

$$S(p) = \sum_{n=0}^{\infty} \frac{x^n}{(pn)!} \tag{1}$$

converges for $|x| < \infty$. This follows from the fact

$$\lim_{n\to\infty} \frac{(pn)!}{\big(p(n+1)\big)!} = 0.$$

The purpose of this paper is to express (1) in terms of a hypergeometric function. The special functions that will be used are reviewed first.

The *factorial function* is defined by

$$(a)_n = a(a+1)\cdots(a+n-1)$$

for any a and for $n \geq 1$ and, $(a)_0 = 1$, for $a \neq 0$. In particular, $n! = (1)_n$ and, from page 9 of [1],

$$(a)_{nk} = k^{nk} \prod_{i=1}^{k} \left(\frac{a+i-1}{k}\right)_n .$$

So,

$$
\begin{aligned}
(pn)! &= (1)_{pn} \\
&= p^{pn}\left(\frac{1}{p}\right)_n\left(\frac{2}{p}\right)_n\cdots\left(\frac{p-1}{p}\right)_n\left(\frac{p}{p}\right)_n \\
&= p^{pn}\left(\frac{1}{p}\right)_n\left(\frac{2}{p}\right)_n\cdots\left(\frac{p-1}{p}\right)_n(1)_n \\
&= p^{pn}\left(\frac{1}{p}\right)_n\left(\frac{2}{p}\right)_n\cdots\left(\frac{p-1}{p}\right)_n n!\,. \qquad (2)
\end{aligned}
$$

The *generalized hypergeometric function* is defined by

$$
{}_pF_q[a_1,\ldots,a_p;\ b_1,\ldots,b_q;\ x] = 1+\sum_{n=1}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{x^n}{n!}
$$

for all $b_i \neq 0,-1,-2,\ldots$. It is known that this series diverges for all $x \neq 0$ if $p > q+1$; converges for all x if $p \leq q$; and converges for $|x| < 1$ if $p = q+1$. These facts can be established by using the ratio test.

Substituting identity (2) into (1) gives

$$
\begin{aligned}
S(p) &= \sum_{n=0}^{\infty}\frac{x^n}{p^{pn}\left(\frac{1}{p}\right)_n\left(\frac{2}{p}\right)_n\cdots\left(\frac{p-1}{p}\right)_n n!} \\
&= \sum_{n=0}^{\infty}\frac{1}{\left(\frac{1}{p}\right)_n\left(\frac{2}{p}\right)_n\cdots\left(\frac{p-1}{p}\right)_n}\cdot\frac{(x/p^p)^n}{n!} \\
&= {}_0F_{p-1}\left(-;\ \frac{1}{p},\ \frac{2}{p},\ \cdots,\ \frac{p-1}{p};\ \frac{x}{p^p}\right). \qquad (3)
\end{aligned}
$$

In particular, if $p = 1$, then (1) is the series representation for e^x and (3) becomes

$$S(1) = {}_0F_0(-;\ -;\ x) = e^x.$$

This agrees with the identity $e^z = {}_0F_0(z)$ given on page 209 of [1].

<u>*Reference*</u>

1. Y. Luke, *The Special Functions and Their Approximations,* Vol. 1, Academic Press, 1969.