SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the editor.

10. *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Show

$$\sum_{\substack{k=1 \\ (k,45)=1}}^{45} \sin^8 \frac{k\pi}{45} = \frac{51}{8} \ .$$

Solution by the proposers.

Let $w = \exp \frac{\pi i}{n}$. Then

$$\begin{aligned}
\sum_{k=1}^{n} \sin^{4m} \frac{k\pi}{n} &= \sum_{k=1}^{n} \left(\frac{w^k - w^{-k}}{2i} \right)^{4m} \\
&= \frac{1}{2^{4m}} \sum_{k=1}^{n} \sum_{j=0}^{4m} (-1)^j \binom{4m}{j} \left(w^k\right)^{4m-j} \left(w^{-k}\right)^j \\
&= \frac{1}{2^{4m}} \sum_{k=1}^{n} \sum_{j=0}^{4m} (-1)^j \binom{4m}{j} \left(w^{4m-2j}\right)^k \\
&= \frac{1}{2^{4m}} \sum_{j=0}^{4m} (-1)^j \binom{4m}{j} \sum_{k=1}^{n} \left(w^{4m-2j}\right)^k \ .
\end{aligned}$$

Now

$$\sum_{k=1}^{n} \left(w^{4m-2j}\right)^k = \begin{cases} n, & \text{if } w^{4m-2j} = 1, \\ \frac{w^{4m-2j}(1-(w^n)^{4m-2j})}{1-w^{4m-2j}} = 0, & \text{if } w^{4m-2j} \neq 1. \end{cases}$$

But $w^{4m-2j} = 1$ iff $\frac{2m-j}{n}$ is an integer. Also, if $n \mid 2m - j$, then there exists an integer k such that $2m - j = kn$. Therefore,

$$0 \leq j = 2m - kn \leq 4m$$

so

$$-\frac{2m}{n} \le k \le \frac{2m}{n} \;.$$

Using the above,

$$\begin{aligned}
\sum_{k=1}^{n} \sin^{4m} \frac{k\pi}{n} &= \frac{1}{2^{4m}} \sum_{k=\lceil -\frac{2m}{n} \rceil}^{\lfloor \frac{2m}{n} \rfloor} (-1)^{2m-kn} \binom{4m}{2m-kn} n \\
&= \frac{n}{2^{4m}} \sum_{k=\lceil -\frac{2m}{n} \rceil}^{\lfloor \frac{2m}{n} \rfloor} (-1)^{-kn} \binom{4m}{2m-kn} \\
&= \frac{n}{2^{4m}} \sum_{k \ge 0} (-1)^{2m-n\lfloor \frac{2m}{n} \rfloor + kn} \binom{4m}{2m-n\lfloor \frac{2m}{n} \rfloor + kn} \;.
\end{aligned}$$

The last equality follows from examining limits, signs, and coefficients of the previous sum. Next, using Problem 14(a) from Chapter 2 [1] and the above results

$$\begin{aligned}
&\sum_{\substack{k=1 \\ (k,n)=1}}^{n} \sin^{4m} \frac{k\pi}{n} = \sum_{d|n} \mu\left(\frac{n}{d}\right) \left(\sum_{k=1}^{d} \sin^{4m} \frac{k\pi}{d} \right) \\
&= \sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{d}{2^{4m}} \sum_{k \ge 0} (-1)^{2m-d\lfloor \frac{2m}{d} \rfloor + kd} \binom{4m}{2m-d\lfloor \frac{2m}{d} \rfloor + kd} \\
&= \frac{1}{2^{4m}} \sum_{d|n} d \cdot \mu\left(\frac{n}{d}\right) \sum_{k \ge 0} (-1)^{2m-d\lfloor \frac{2m}{d} \rfloor + kd} \binom{4m}{2m-d\lfloor \frac{2m}{d} \rfloor + kd} \;.
\end{aligned}$$

Finally, applying this result when $n = 45$ and $m = 2$ we have

$$\sum_{\substack{k=1 \\ (k,45)=1}}^{45} \sin^{8} \frac{k\pi}{45} = \frac{51}{8} \;.$$

<u>*Reference*</u>

1. T. M. Apostol, *Introduction to Analytic Number Theory,* Springer-Verlag, 1984, p. 48.

11. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

A strip is the closed region bounded between two parallel lines in the plane. Prove that a finite number of strips cannot cover the entire plane.

Solution by the proposer.

Since there are only a finite number of strips, there must be some line, L, that is not parallel to the bounding lines of any of the strips. Each strip intersects L in a finite line segment. These line segments cannot completely cover L, so there must be some point of L that is not in any of these line segments. This point is not covered by any of the strips.

The proposer also submitted a second solution.

13. *Proposed by James H. Taylor, Central Missouri State University, Warrensburg, Missouri.*

(a) Show

$$\sum_{n=0}^{l} \sum_{m=0}^{n} \frac{(-1)^m}{m!(l-n)!} = 1$$

for any non-negative integer l.

(b) For any positive integer n, define

$$(2n-1)!! = 1 \cdot 3 \cdot \ \cdots \ \cdot (2n-1)$$

and

$$(2n)!! = 2 \cdot 4 \cdot \ \cdots \ \cdot (2n)\ .$$

Show

$$\sum_{k=1}^{n} \frac{(2k-1)!!}{(2k)!!} = \frac{(2n+1)!!}{(2n)!!} - 1\ .$$

Solution I to part (a) by Joseph E. Chance, Pan American University, Edinburg, Texas.

The result follows from successive Cauchy Products of the series

$$(1) \qquad e^{-x} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} x^l \ ,$$

$$(2) \qquad \frac{1}{1-x} = \sum_{l=0}^{\infty} 1 \cdot x^l \ , \ \text{for } |x| < 1 \ , \quad \text{and}$$

$$(3) \qquad e^{x} = \sum_{l=0}^{\infty} \frac{1}{l!} x^l \ .$$

The Cauchy Product of series (1) and (2) is

$$(4) \qquad \frac{e^{-x}}{1-x} = \sum_{l=0}^{\infty} \left(\sum_{m=0}^{l} \frac{(-1)^m}{m!} \right) x^l \quad \text{for } |x| < 1 \ .$$

The Cauchy Product of series (3) and (4) is

$$\begin{aligned} \frac{1}{1-x} &= e^x \cdot \frac{e^{-x}}{1-x} \\ &= \sum_{l=0}^{\infty} \left(\sum_{n=0}^{l} \frac{1}{(l-n)!} \left(\sum_{m=0}^{n} \frac{(-1)^m}{m!} \right) \right) x^l \ , \ \text{for } |x| < 1 \ . \end{aligned}$$

The last series and series (2) represent the same function, thus must have identical coefficients. Therefore

$$\sum_{n=0}^{l} \sum_{m=0}^{n} \frac{(-1)^m}{m!(l-n)!} = 1 \ .$$

Solution II to part (a) by the proposer.

We prove this result by induction on l. The result is true for $l = 0$. Suppose the result is true for some $l = k \geq 0$. Then by rearranging terms and using the induction hypothesis

$$\begin{aligned}
\sum_{n=0}^{k+1}\sum_{m=0}^{n}\frac{(-1)^m}{m!(k+1-n)!} &= \sum_{n=1}^{k+1}\sum_{m=0}^{n-1}\frac{(-1)^m}{m!(k+1-n)!} \\
&\quad + \sum_{n=0}^{k+1}\frac{(-1)^n}{n!(k+1-n)!} \\
&= \sum_{n=0}^{k}\sum_{m=0}^{n}\frac{(-1)^m}{m!(k-n)!} \\
&\quad + \sum_{n=0}^{k+1}\frac{(-1)^n}{n!(k+1-n)!} \\
&= 1 + \sum_{n=0}^{k+1}\frac{(-1)^n}{n!(k+1-n)!}\ .
\end{aligned}$$

But

$$\begin{aligned}
\sum_{n=0}^{k+1}\frac{(-1)^n}{n!(k+1-n)!} &= \frac{1}{(k+1)!}\sum_{n=0}^{k+1}\frac{(-1)^n(k+1)!}{n!(k+1-n)!} \\
&= \frac{1}{(k+1)!}\sum_{n=0}^{k+1}(-1)^n\binom{k+1}{n} \\
&= \frac{1}{(k+1)!}(1-1)^{k+1} \\
&= 0\ .
\end{aligned}$$

Thus

$$\sum_{n=0}^{k+1}\sum_{m=0}^{n}\frac{(-1)^m}{m!(k+1-n)!} = 1$$

so the result is true for $l = k+1$. Hence by induction,

$$\sum_{n=0}^{l}\sum_{m=0}^{n}\frac{(-1)^m}{m!(l-n)!} = 1$$

for any non-negative integer l.

Solution I to part (b) by Joseph E. Chance, Pan American University, Edinburg, Texas.

From the binomial theorem

$$(5) \qquad (1-x)^{-\frac{1}{2}} = 1 + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot \ \cdots \ \cdot (2k-1)}{2 \cdot 4 \cdot \ \cdots \ \cdot (2k)} x^k = 1 + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} x^k \ , \text{ for } |x| < 1$$

The Cauchy Product of series (2) and (5) is

$$\frac{1}{1-x}(1-x)^{-\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} \left(1 + \sum_{k=1}^{n} \frac{(2k-1)!!}{(2k)!!} \right) x^n \ .$$

But

$$\frac{1}{1-x}(1-x)^{-\frac{1}{2}} = (1-x)^{-\frac{3}{2}} \ ,$$

and from the binomial theorem

$$(1-x)^{-\frac{3}{2}} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \ \cdots \ \cdot (2n+1)}{2 \cdot 4 \cdot \ \cdots \ \cdot (2n)} x^n = 1 + \sum_{n=1}^{\infty} \frac{(2n+1)!!}{(2n)!!} x^n$$

Equating coefficients of the respective series representations for $(1-x)^{-\frac{3}{2}}$ gives

$$\frac{(2n+1)!!}{(2n)!!} = 1 + \sum_{k=1}^{n} \frac{(2k-1)!!}{(2k)!!} \ .$$

Composite Solution II to part (b) by Russell Euler, Northwest Missouri State University, Maryville, Missouri and the proposer.

The result will be established by using mathematical induction on n. For $n = 1$, both sides of the alleged identity equal $\frac{1}{2}$.

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{(2k-1)!!}{(2k)!!} &= \sum_{k=1}^{n} \frac{(2k-1)!!}{(2k)!!} + \frac{(2n+1)!!}{(2n+2)!!} \\
&= \frac{(2n+1)!!}{(2n)!!} - 1 + \frac{(2n+1)!!}{(2n+2)!!} \\
&= \frac{(2n+1)!!(2n+2) + (2n+1)!!}{(2n+2)!!} - 1 \\
&= \frac{(2n+1)!!(2n+3)}{(2n+2)!!} - 1 \\
&= \frac{(2n+3)!!}{(2n+2)!!} - 1 \ ,
\end{aligned}$$

as desired.