ON THE t-EXTENDABILITY OF THE GENERALIZED PETERSEN GRAPHS

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Introduction. In an earlier paper (see [5]) we investigated the existence of full sets of states and full sets of dispersion free states on the orthomodular lattices and posets arising from the dualization of the generalized Petersen graphs. In particular, investigation of those orthomodular structures admitting full sets of dispersion free states led in a natural way to the question of 2-extendability (as defined by Plummer [4]) of the generalized Petersen graphs and was partially answered in [6] and [7].

In this paper we answer completely the question of t-extendability (also defined by Plummer [4]) on the generalized Petersen graphs where t is not equal to 2.

Terminology. All graphs in this paper are in the sense of Harary [2]. Recall that a $\underline{perfect\ matching}$ of a graph G is a set of independent (i.e., non-adjacent) edges which together cover all the points of G.

Following M. D. Plummer [4], we define a graph G to be $\underline{t - extendable}$ if it is connected, contains a set of t independent edges and every set of t independent edges extends to (i.e. is a subset of) a perfect matching.

Following Watkins [8] and Coxeter [1] we define the *generalized Petersen graphs* $G(n,k)$ as follows. The points are u_0, u_1, …, u_{n-1}, v_0, v_1, …, v_{n-1} and the edges are $[u_i, u_{i+1}]$ (called *outer edges*), $[v_i, v_{i+k}]$ (called *inner edges*), and $[u_i, v_i]$ (called *spokes*) where all subscripts are modulo n and $1 < 2k < n$. These edges will be denoted by a_i, b_i, and c_i respectively. Note that the Petersen graph is $G(5,2)$. Each connected component of the subgraph generated by $v_0, v_1, \ldots, v_{n-1}$ is called an *inner rim*.

Main Results.

Theorem 1. The generalized Petersen graphs $G(n,k)$ are 1-extendable and n-extendable.

Proof. For $i = 0, 1, 2, \ldots, n-1$ define perfect matchings M_i by the set $\{ a_i, a_{i+2}, \ldots, a_{i+2k-2}, b_i, b_{i+1}, \ldots, b_{i+k-1}, c_{i+2k}, c_{i+2k+1}, \ldots, c_{i+n-1} \}$. Since this set contains at least one outer edge, at least one inner edge, and at least one spoke, a given edge is covered by at least one of these perfect matchings. Thus every $G(n,k)$ is 1-extendable.

The first part of this theorem also follows from a result of Little, et al [3]. However, a proof using this approach depends on a characterization of d-covered graphs which is not needed here.

Since every generalized Petersen graph has a perfect matching,

$G(n,k)$ is n-extendable for all n and k.

It is worth noting here that because of Plummer's Theorem (Theorem 2.2 of [4]), all $G(n,k)$ which are 2-extendable are also 1-extendable. However, the question of 2-extendability has not been completely answered for all $G(n,k)$ and in fact it is known that some of $G(n,k)$ are not 2-extendable (see Schrag and Cammack [7]). For example, the edges a_2 and c_0 do not extend to a perfect matching in $G(5,2)$, the original Petersen graph. (see Figure 1.)

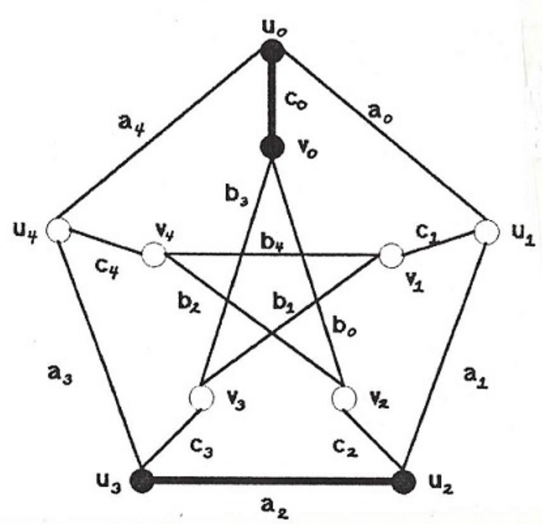

Figure 1.

Theorem 2. If $t > 2$, then $G(n,k)$ is t-extendable if and only if $t = n$.

Proof. No set of independent edges to which a_1, a_{n-2}, and b_0 extend can cover u_0. Furthermore these three edges are independent unless $u_{n-2} = u_2$ in which case $n = 4$, or $a_{n-2} = a_1$ in which case $n = 3$ (a_i and b_j are always independent for any i and j). Thus $G(n,k)$ is not 3-extendable if $n > 4$. By Plummer's Theorem $G(n,k)$ not (t-1)-extendable implies $G(n,k)$ not t-extendable for $t < n$. Thus no $G(n,k)$ is t-extendable for $2 < t < n$. Note that the fact that $G(n,k)$ is n-extendable but not (n-1)-extendable does not contradict Plummer's theorem since the hypothesis is not satisfied if $t = n$.

Now note that $n = 4$ implies that $G(n,k) = G(4,1)$ and $n = 3$ implies that $G(n,k) = G(3,1)$. That $G(3,1)$ is 3-extendable and that $G(4,1)$ is 4-extendable are restatements of Theorem 1 with $n = 3$ and $n = 4$. $G(4,1)$ is not 3-extendable since a_1, b_0, and c_3 do not extend to a perfect matching.

For the reader's convenience we conclude with a summary of the known results on t-extendability of the generalized Petersen graphs.

1. $G(n,k)$ is 1-extendable for all n and k.
2. $G(3k,k)$ is not 2-extendable for any k.
3. $G(n,1)$ is 2-extendable if and only if n is even.
4. $G(n,2)$ is 2-extendable if and only if $n \neq 5$, 6, or 8.

5. If $3 \leq k \leq 7$, $G(n,k)$ is 2-extendable if and only if $n \neq 3k$.
6. If $k \geq 4$, then any pair of independent edges of $G(n,k)$ at least one of which is a spoke can be extended to a perfect matching.
7. $G(n,k)$ is 2-extendable for all n and k such that $n \geq 3k+5$ and $k \geq 2$.
8. If $t > 2$, then $G(n,k)$ is t-extendable if and only if $t = n$.
9. $G(n,k)$ is n-extendable for all n and k.

It is worth noting that if the conditions of the definition of $G(n,k)$ are relaxed so as to allow $n = 2k$, then it is easy to show that $G(2k,k)$ is t-extendable if and only if $t = 1$ or $t = 2k$.

The only question remaining is that of 2-extendability of $G(n,k)$ for $k > 7$ and $n < 3k+5$ where $n \neq 3k$. It is conjectured that these are all 2-extendable.

References

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