On a new Definition of Vector and Parallel Displacement in Projective Differential Geometry.

By

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(Received Sept. 20, 1935.)

§ 1. Introduction.

In projective differential geometry as hitherto considered, as far as I know, a projective space was attached at every point in the base space, and the homogeneous coordinates were taken in each projective space. In this geometry the term "consecutive points" has no significance because of the homogeniality of the coordinates, nor, consequently, have the terms "differentiation," "infinitesimal quantity" etc. Besides, as to the relation between vectors and points in the projective space, we can find no necessary reason why this relation must be so considered.

In this paper, avoiding such considerations, we adopt the ordinary coordinates in each projective space instead of the homogeneous coordinates, and shall attempt to determine the relation between vectors and points in a rational manner.

For this purpose, we suppose that at every point in the base space, the projective space having a proper quadric is attached. And we give a new definition of summation of vectors such that the sum of the vectors is invariant for the projective transformations in the attached space. From this definition the relation between vectors and points will be determined as its necessary consequence. In this way we shall construct a system of projective differential geometry and find its various characteristics.

§ 2. The transformations of the coordinates in the base space and vector space, and those of the coefficients of quadric, in the projective space.

We consider an n-dimensional space X_n, in which we take the coordinates denoted by $x^1, \ldots, x^n$. At every point in this space X_n, we consider an N-dimensional projective space attached, which has a proper quadric. In this space we denote the coordinates by $X^1, \ldots, X^N$, (not homogeneous), and the quadric by

$$g_{\alpha\beta}X^\alpha X^\beta + 2g_\alpha X^\alpha + 1 = 0 \qquad (\alpha, \beta = 1, \ldots, N) \tag{1}$$

where g_α and $g_{\alpha\beta}$ are any given functions of $x^1, \ldots, x^n$.

Now we assume that any transformation of coordinates in x-space introduces correspondingly a projective transformation in X-space and that the point at which the projective space is attached to X_n, always has the coordinates $(0, \ldots, 0)$ in the projective space. Then the projective transformations in X-space are written as follows:

$$X'^\lambda = \frac{P^\lambda_\alpha X^\alpha}{Q_\omega X^\omega + 1} \qquad (\lambda, \alpha, \omega = 1, \ldots, N). \tag{2}$$

where P^λ_α and Q_ω are certain functions of x's which are related to the transformation in x-space.

From the transformation (2), we have the transformation of $g_{\mu\nu}$ and g_ν

$$\begin{cases} g_{\mu\nu} = g'_{\alpha\beta}P^\alpha_\mu P^\beta_\nu + 2g'_\alpha P^\alpha{}_{(\mu}Q_{\nu)} \\ g_\nu = g'_\alpha P^\alpha_\nu + Q_\nu \qquad (\mu, \nu = 1, \ldots, N) \end{cases}$$

or

$$\begin{cases} g'_\alpha = \overline{P}^\nu_\alpha (g_\nu - Q_\nu) \\ g'_{\alpha\beta} = \overline{P}^\mu_\alpha \cdot \overline{P}^\nu_\beta \{g_{\mu\nu} - 2g_{(\mu}Q_{\nu)}) + Q_\mu Q_\nu\} \end{cases} \tag{3}$$

where $\overline{P}^\mu_\alpha$ are defined by

$$\overline{P}^\mu_\alpha P^\beta_\mu = \delta^\beta_\alpha .$$

Then we see that the transformations (2) and (3) together form a group, and the infinitesimal transformations of this group can be obtained as follows:

$$\left.\begin{aligned} & X^\tau \frac{\partial}{\partial X^\sigma} - g_\sigma \frac{\partial}{\partial g_\tau} - 2g_{\alpha\sigma}\frac{\partial}{\partial g_{\alpha\tau}} \qquad (\tau, \sigma = 1, \ldots, N) \\ & X^\tau X^\alpha \frac{\partial}{\partial X^\alpha} + \frac{\partial}{\partial g_\tau} + 2g_\alpha \frac{\partial}{\partial g_{\alpha\tau}} \qquad (\tau = 1, \ldots, N) \end{aligned}\right\} \tag{4}$$

§ 3. Sum of vectors.

We suppose that a vector is represented by a point in X-space, the correspondence between vector and point cannot for the present be determined.

Before we give a definition for the vectors, we will define the summation of points which represent the vectors in X-space. Let A^λ and B^λ $(\lambda = 1, \ldots, N)$ be any two points in X-space and let C^λ $(\lambda = 1, \ldots, N)$ be the *sum* of these two points. Then it is natural to suppose that C^λ $(\lambda = 1, \ldots, N)$ may in general be expressed as a function of A^α, B^α, g_α and $g_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots, N)$, i. e.

$$C^\lambda = \varphi^\lambda(A, B, g_\alpha, g_{\alpha\beta}) \qquad (\lambda = 1, \ldots, N), \tag{5}$$

which we express symbolically by

$$C = A+B$$

Now we make the following assumptions:

(a) The functions $\varphi^\lambda(A, B, g_\alpha, g_{\alpha\beta})$ are analytic in the arguments and equations (5) can be solved for A's and B's.

(b) The associative law holds for the *summation* of more than three points: If A^λ, B^λ and C^λ $(\lambda = 1, \ldots, N)$ are any three points in X-space then

$$(A+B)+C = A+(B+C)$$

(c) The *sum* of two points is invariant by the coordinate-transformations (2): When $C = A+B$ and the points A^λ, B^λ, C^λ $(\lambda = 1, \ldots, N)$ are transformed by the transformation into A', B', C' respectively, then it must be that

$$C' = A'+B' .$$

Under these assumptions we will determine the form of the functions $\varphi^\lambda(A, B, g_\alpha, g_{\alpha\beta})$.

The assumption (c) may be expressed as follows: In the equations

$$C^\lambda = \varphi^\lambda(A, B, g_\alpha, g_{\alpha\beta})$$

if we transform A, B, C, g_α and $g_{\alpha\beta}$ by the equations:

$$A'^{\lambda} = \frac{P^{\lambda}_{\alpha}A^{\alpha}}{O_{\omega}A^{\omega}+1}, \qquad B'^{\lambda} = \frac{P^{\lambda}_{\alpha}B^{\alpha}}{Q_{\omega}B^{\omega}+1}, \qquad C'^{\lambda} = \frac{P^{\lambda}_{\alpha}C^{\alpha}}{Q_{\omega}C^{\omega}+1} \tag{6}$$

$$\left.\begin{aligned} g'_{\alpha} &= \overline{P}^{\nu}_{\alpha}(g_{\nu}-Q_{\nu}) \\ g'_{\alpha\beta} &= \overline{P}^{\mu}_{\alpha}\overline{P}^{\nu}_{\beta}\{g_{\mu\nu}-2g_{(\mu}Q_{\nu)}+Q_{\mu}Q_{\nu}\} \end{aligned}\right\} \tag{3}$$

then (5) becomes

$$C'^{\lambda} = \varphi^{\lambda}(A', B', g'_{\alpha}, g'_{\alpha\beta})$$

This shows that the system of equations (5), in which $A^{\alpha}, B^{\alpha}, C^{\alpha}, g_{\alpha}, g_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots, N)$ are regarded as variables, admits the transformations (6) and (3). Since (6) and (3) together form a group whose infinitesimal transformations are obtained as follows:

$$T^{\tau}_{\sigma} \equiv A^{\tau}\frac{\partial}{\partial A^{\sigma}}+B^{\tau}\frac{\partial}{\partial B^{\sigma}}+C^{\tau}\frac{\partial}{\partial C^{\sigma}}-g_{\sigma}\frac{\partial}{\partial g_{\tau}}-2g_{\alpha\sigma}\frac{\partial}{\partial g_{\alpha\tau}}$$

$$T^{\tau} \equiv A^{\tau}A^{\alpha}\frac{\partial}{\partial A^{\alpha}}+B^{\tau}B^{\alpha}\frac{\partial}{\partial B^{\alpha}}+C^{\tau}C^{\alpha}\frac{\partial}{\partial C^{\alpha}}+\frac{\partial}{\partial g_{\tau}}+2g_{\alpha}\frac{\partial}{\partial g_{\alpha\tau}},$$

the condition that (5) admits the transformations (6) and (3), is expressed by the following equations:

$$T^{\tau}_{\sigma}(\varphi^{\lambda}-c^{\lambda}) = 0$$

$$T^{\tau}(\varphi^{\lambda}-c^{\lambda}) = 0$$

These are satisfied identically by (5). Namely $\varphi^{\lambda}(A, B, g_{\alpha}, g_{\alpha\beta})$ are the solutions of the following equations:

$$\left(A^{\tau}\frac{\partial}{\partial A^{\sigma}}+B^{\tau}\frac{\partial}{\partial B^{\sigma}}-g_{\sigma}\frac{\partial}{\partial g_{\tau}}-2g_{\alpha\sigma}\frac{\partial}{\partial g_{\alpha\tau}}\right)\varphi^{\lambda} = \varphi^{\tau}\delta^{\lambda}_{\sigma} \tag{7}$$

$$\left(A^{\tau}A^{\alpha}\frac{\partial}{\partial A^{\alpha}}+B^{\tau}B^{\alpha}\frac{\partial}{\partial B^{\alpha}}+\frac{\partial}{\partial g_{\tau}}+2g_{\alpha}\frac{\partial}{\partial g_{\alpha\tau}}\right)\varphi^{\lambda} = \varphi^{\tau}\varphi^{\lambda} \tag{8}$$

Now we will solve these equations. The equations obtained by putting the left hand side of (8) equal to zero:

$$\left(A^{\tau}A^{\alpha}\frac{\partial}{\partial A^{\alpha}}+B^{\tau}B^{\alpha}\frac{\partial}{\partial B^{\alpha}}+\frac{\partial}{\partial g_{\tau}}+2g_{\alpha}\frac{\partial}{\partial g_{\alpha\tau}}\right)f = 0$$

have $2N+\frac{N(N+1)}{2}$ independent solutions:

$$\left.\begin{aligned} u^\lambda &\equiv \frac{A^\lambda}{1+g_a A^a}, \qquad v^\lambda \equiv \frac{B^\lambda}{1+g_a B^a}, \\ w_{\lambda\mu} &= g_\lambda g_\mu - g_{\lambda\mu}, \qquad (\lambda, \mu = 1, \ldots\ldots, N). \end{aligned}\right\} \tag{9}$$

Hence if we take u^λ, v^λ, and $w_{\lambda\mu}$ $(\lambda, \mu = 1, \ldots\ldots, N)$ instead of A^λ, B^λ, and $g_{\lambda\mu}$ as new variables, (7) and (8) become as follows:

$$\left(u^\tau \frac{\partial}{\partial u^\sigma} + v^\tau \frac{\partial}{\partial v^\sigma} - g_\sigma \frac{\partial}{\partial g_\tau} - w_{a\tau} \frac{\partial}{\partial w_{a\tau}}\right)\varphi^\lambda = \varphi^\tau \delta^\lambda_\sigma \tag{10}$$

$$\frac{\partial}{\partial g_\tau}\varphi^\lambda = \varphi^\tau \varphi^\lambda \tag{11}$$

From (11) we have

$$\frac{\partial}{\partial g_\tau} \log \varphi^\lambda = \varphi^\tau$$

i. e. φ^λ must have the form:

$$\varphi^\lambda = \rho \psi^\lambda \tag{12}$$

where ψ^λ does not contain g_a $(a = 1, \ldots\ldots, N)$. Substituting (12) into (11), we have

$$\frac{\partial \rho}{\partial g_\tau} = \rho^2 \psi^\tau$$

from which

$$-\rho^{-1} = \psi^\sigma g_\sigma + \Omega$$

where Ω is an arbitrary function not containing g_a $(a = 1, \ldots\ldots, N)$. Therefore (12) becomes

$$\varphi^\lambda = \frac{\psi^\lambda}{-(\psi^\sigma g_\sigma + \Omega)}$$

or, writing Φ^λ for $-\frac{\psi^\lambda}{\Omega}$,

$$\varphi^\lambda = \frac{\phi^\lambda}{1-g_a\phi^a} \tag{13}$$

(13) is the general solution of (8) where ϕ^λ are arbitrary functions not containing g_a $(a = 1, \ldots\ldots, N)$.

Substituting (13) into (7), we have

$$\left(u^{\tau}\frac{\partial}{\partial u^{\sigma}}+v^{\tau}\frac{\partial}{\partial v^{\sigma}}-w_{a\sigma}\frac{\partial}{\partial w_{a\tau}}\right)\phi^{\lambda}=\varphi^{\tau}\delta^{\lambda}_{\sigma}\,. \tag{14}$$

Since (14) has *2* independent particular solutions u^{λ} and v^{λ}, and since the equations obtained by putting the left hand side of (14) equal to zero:

$$\left(u^{\tau}\frac{\partial}{\partial u^{\sigma}}+v^{\tau}\frac{\partial}{\partial v^{\tau}}-w_{a\sigma}\frac{\partial}{\partial w_{a\tau}}\right)f=0 \tag{15}$$

have *3* independent solutions:

$$t_1\equiv w_{\alpha\beta}u^{\alpha}u^{\beta}\,,\qquad t_2=w_{\alpha\beta}u^{\alpha}v^{\beta}\,,\qquad t_3=w_{\alpha\beta}v^{\alpha}v^{\beta}\,, \tag{16}$$

therefore the general solution of (14) is given by

$$\phi^{\lambda}=f_1(t_1,t_2,t_3)u^{\lambda}+f_2(t_1,t_2,t_3)v^{\lambda} \tag{17}$$

where $f_1(t_1, t_2, t_3)$ and $f_2(t_1, t_2, t_3)$ are arbitrary functions of the arguments.[1] Substituting (17) into (13), we have the general solution of (7) and (8).

$$\varphi^{\lambda}=\frac{f_1u^{\lambda}+f_2v^{\lambda}}{1-g_a\{f_1u^{a}+f_2v^{a}\}} \tag{18}$$

Therefore the equation which defines the sum of two points A^{λ} and B^{λ} satisfying the assumption (c), is given by

$$C^{\lambda}=\frac{f_1u^{\lambda}+f_2v^{\lambda}}{1-g_a\{f_1u^{a}+f_2u^{a}\}} \tag{19}$$

where $u^{\lambda}, v^{\lambda}, f_1, f_2$ are given by (9), (16) and (17).

Lastly we will determine f_1 and f_2 so that (19) also satisfies the assumption (b). For this purpose we introduce $u^{\lambda}, v^{\lambda}, w^{\lambda}$ instead of $A^{\lambda}, B^{\lambda}, C^{\lambda}$ ($\lambda=1,\ldots\ldots,N$), by the relations:

$$u^{\lambda}=\frac{A^{\lambda}}{1-g_aA^{a}}\,,\qquad v^{\lambda}=\frac{B^{\lambda}}{1-g_aB^{a}}\,,\qquad w^{\lambda}=\frac{C^{\lambda}}{1-g_aC^{a}} \tag{20}$$

Then (19) becomes

$$w^{\lambda}=f_1u^{\lambda}+f_2v^{\lambda} \tag{21}$$

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(1) See **Note.**

which we express symbolically by

$$w = u+v \tag{22}$$

Since the equations (19) must satisfy the assumption (b), the same assumption must also be satisfied by (22) and conversely. Therefore it must be that

$$(u+v)+w = u+(v+w)\,.$$

From this condition we can prove that

$$f_1 = f_2 = 1\,. \tag{23}$$

So that (19) and (21) become as follows:

$$C^\lambda = \frac{u^\lambda+v^\lambda}{1-g_a\{u^a+v^a\}} \tag{24}$$

where u^a and v^a are given by (20), and

$$w^\lambda = u^\lambda+v^\lambda\,, \tag{25}$$

Thus, (24) is the expression for the *sum* of two points A and B, satisfying the assumptions (a), (b) and (c).

Next we will define vector in our space, since a vector corresponds to a given point as we have assumed at the beginning. If a vector $\overline{A}^\lambda$ $(\lambda = 1, \ldots\ldots, N)$ corresponds to a point A^λ, the equation of correspondence is written in the form:

$$\overline{A}^\lambda = V^\lambda(A, g_a, g_{a\beta}) \qquad (\lambda = 1, \ldots\ldots, N)\,, \tag{26}$$

or, inversely

$$A^\lambda = \overset{\bullet}{V}{}^{-1\lambda}(\overline{A}, g_a, g_{\beta a})\,. \tag{26'}$$

For the transformation of vector we make the assumption that, when point A^λ is transformed to the point A'^λ by the projective transformation, the corresponding vector $\overline{A}^\lambda$ is transformed to $\overline{A}'^\lambda$; i. e. analytically, when $A^\lambda \to A'^\lambda$, $\overline{A}'^\lambda = V^\lambda(A', g'_a, g'_{a\beta})$.

For the *summation* of vectors we make the following assumptions: When we have two given vectors $\overline{A}^\lambda$ and $\overline{B}^\lambda$ corresponding to two points A and B respectively, we say that vector $\overline{C}^\lambda$ is the *sum* of vectors $\overline{A}^\lambda$ and $\overline{B}^\lambda$, the assumptions being

(a) The relation which defines the *summation* is analytic and can be solved for $\overline{A}$'s and $\overline{B}$'s.
(b) The associative law holds for the *summation* of more than three vectors.
(c) The *sum* of two vectors is invariant by the vector-transformations.

Analytically express, the equation of *summation* of two vectors $\overline{A}^\lambda$ and $\overline{B}^\lambda$ can be written thus

$$\overline{C}^\lambda = S^\lambda(\overline{A}, \overline{B}, g_\alpha, g_{\alpha\beta}) \tag{27}$$

where $S^\lambda(\overline{A}, \overline{B}, g_\alpha, g_{\alpha\beta})$ are analytic functions of the arguments.

Then we can easily prove the following theorem: If we are given the *sum* $\overline{C}^\lambda$ of two vectors $\overline{A}^\lambda$ and $\overline{B}^\lambda$, the *sum* of the two corresponding points A and B is a point C corresponding to $\overline{C}^\lambda$. Therefore we have

$$\overline{A}^\lambda = V^\lambda(A, g_\alpha, g_{\alpha\beta}), \qquad \overline{B}^\lambda = V^\lambda(B, g_\alpha, g_{\alpha\beta}), \qquad \overline{C}^\lambda = V^\lambda(C, g_\alpha, g_{\alpha\beta}) \tag{28}$$

where A, B and C are related by (24).

Substituting (28) into (27), we have

$$V^\lambda(C, g_\alpha, g_{\alpha\beta}) = S^\lambda\Big(V(A, g_\alpha, g_{\alpha\beta}), V(B, g_\alpha, g_{\alpha\beta}), g_\alpha, g_{\alpha\beta}\Big). \tag{29}$$

If we solve C^λ from the above, the resulting equation must coincide with (24). So we see that (29) takes the same form as (25) by the change of variables (20): In other words, by suitable transformation of vectors, the equation for the *sum* of vectors can be reduced to the same form as (25).

So we have the theorem: *If we make a vector correspond to each point and make the summation of vectors satisfy the assumptions* (*a*), (*b*), (*c*), *the* <u>*sum*</u> *of vectors can be reduced to the ordinary sum, by suitable transformation of vectors.*

By this theorem we are enabled to treat the case where the *sum* of two vectors is the ordinary sum instead of the general case.

Next we shall obtain the relation between vectors and points when the *summation* of vectors is ordinary.

From the result stated in my previous papers,[1] (24) can be inter-

(1) Cf. T. Sibata, this journal, **5** (1935), 84.

preted as the relation of the parameter group for the three sets of values $A^\lambda, B^\lambda, C^\lambda$ $(\lambda = 1, \ldots, N)$ of parameters of an original group, and (25) is the expression reduced into the canonical form from (24). In other words, (20) is the transformation by which (24) is reduced to the canonical form.

Since, on the other hand, (27) must be obtained from (24) by the transformation (28), the latter must also be the transformation which transforms (24) into the canonical form. Thereform from the property of the canonical parameter we know that the functions V^λ in (26) must be of the form:

$$V^\lambda = \Psi^\lambda_\mu(g_\alpha, g_{\alpha\beta})\frac{A^\mu}{1+g_\nu A^\nu} \tag{30}$$

where $\Psi^\lambda_\mu(g_\alpha, g_{\alpha\beta})$ are arbitrary functions of the arguments with non-vanishing determinant.

So combining the result above obtained with the preceding theorem, we have the following important theorem: *If we make a vector correspond to each point and make the summation of vectors satisfy the assumptions* (*a*), (*b*), (*c*), *the relation between point* Λ^λ *and the corresponding vector* $\overline{A}^\lambda$ *is reduced to the form:*

$$\overline{A}^\lambda = \Psi^\lambda_\mu(g_\alpha, g_{\alpha\beta})\frac{A^\mu}{1+g_\nu \Lambda^\nu},$$

the case supplying the ordinary summation for vectors.

Since the above theorem is stated for the case in which the expression of vector is reduced by suitable transformation of vector from the general case, we can state it as follows by transforming back to the general case (here Ψ^λ_α can be chosen as δ^λ_α without loss of generality): If we make a vector correspond to each point and make the summation of vectors satisfy the assumptions (a), (b), (c), the relation between point A^λ and the corresponding vector $\overline{A}^\lambda$ is expressed in the form:

$$\overline{A}^\lambda = \Omega^\lambda\left(\frac{A^\alpha}{1+g_\omega A^\omega}, g_\alpha, g_{\alpha\beta}\right)$$

where Ω^λ are arbitrary functions of the arguments, and the sum of vectors $\overline{A}^\lambda$ and $\overline{B}^\lambda$ is given by the equations

$$\Omega^{-1\lambda}(\overline{C}, g_a, g_{a\beta}) = \Omega^{-1\lambda}(A, g_a, g_{a\beta}) + \Omega^{-1\lambda}(B, g_a, g_{a\beta})$$

where $\Omega^{-1\lambda}$ are inverse functions of Ω^{λ}.

§ 4. Transformations of vectors.

We will consider from now on the transformations of vectors in the reduced form (31). When X^{λ} are transformed into X'^{λ} by the transformation:

$$X'^{\lambda} = \frac{P^{\lambda}_{a} X^{a}}{1+Q_{\omega}X^{\omega}}, \tag{2}$$

and the vector v^{λ} corresponding to the point X^{λ} by the equation

$$v^{\lambda} = \Psi^{\lambda}_{\mu}(g_a, g_{a\beta}) \frac{X^{\mu}}{1+g_{\omega}X^{\omega}},$$

is transformed into v'^{λ}, it must be that

$$v'^{\lambda} = \Psi^{\lambda}_{\mu}(g'_a, g'_{a\beta}) \frac{X'^{\mu}}{1+g'_{\omega}X'^{\omega}}.$$

Then, substituting (2) and (3) into (32), we can easily see that

$$v'^{\lambda} = \overline{\Psi}^{\lambda}_{\mu}(g'_a, g'_{a\beta}) \cdot P^{\mu}_{\nu} \overline{\Psi}^{\nu}_{\omega}(g_a, g_{a\beta}) v^{\omega}$$

where $\overline{\Psi}^{\lambda}_{\mu}$ are defined by $\overline{\Psi}^{\lambda}_{\mu}\Psi^{\mu}_{\nu} = \delta^{\lambda}_{\nu}$.

This gives the transformation of vector by the transformation (2) in X-space.

§ 5. Parallel displacement.

We define parallel displacement of vectors, making the following assumptions.

(A). Parallelism between two vectors at a point $P(x)$ in x-space and any neighbouring point $Q(x+dx)$, is a reversively one-to-one correspondence.

(B). The *sum* of any two vectors at $P(x)$ is parallel to the *sum* of the vectors at $Q(x+dx)$ which are parallel to the former respectively.

We will consider the vectors in the reduced form, and let a vector

$\bar{v}^\lambda$ at $P(x)$ be parallel to a vector v^λ at $Q(x+dx)$; then from the above assumptions, we see that[1] the relation is expressed by the equation:

$$\bar{v}^\lambda = v^\lambda + \Gamma^\lambda_{ai}(x) v^a dx^i$$

where $\Gamma^\lambda_{ai}(x)$ may be any functions of x's.

Next we will find the transformation of Γ^λ_{ai}. For this purpose we suppose that by the coordinate-transformation $x'^i = f^i(x)$ $(i = 1, \ldots, n)$, $\bar{v}^\lambda, v^\lambda, \Gamma^\lambda_{ai}$ and dx^i are transformed into $\bar{v}'^\lambda, v'^\lambda, \Gamma'^\lambda_{ai}$ and dx'^i respectively. Then it must be that

$$\bar{v}'^\lambda = v'^\lambda + \Gamma'^\lambda_{ai} v'^a dx^i$$

If we substitute the following relations: putting

$$R^\lambda_a(x) \equiv \overline{\Psi}^\lambda_\mu(g'_a, g'_{a\beta}) P^\mu_\nu \cdot \overline{\Psi}(g_a, g_{a\beta}),$$

$$\bar{v}'^\lambda = R^\lambda_a(x) \bar{v}^a, \qquad v'^\lambda = R^\lambda_a(x+dx) v^a,$$

into (28), then after a little calculation we have the following relation between Γ^λ_{ai} and Γ'^λ_{ai}:

$$R^\lambda_a \Gamma^a_{\mu i} = \frac{\partial R^\lambda_\mu}{\partial x^i} + \Gamma'^\lambda_{aj} R^a_\mu \frac{\partial x'^j}{\partial x^i},$$

which shows the transformation of Γ'^λ_{ai} by the coordinate-transformation in x-space.

This is the general statement of the parallel displacement. But we can specialize and classify it from the standpoint of the absolute quadric in each vector space.

Note.

We will prove that the general solution of the equation (14):

$$\left(u^\tau \frac{\partial}{\partial u^\sigma} + v^\tau \frac{\partial}{\partial v^\sigma} - w_{a\sigma} \frac{\partial}{\partial w_{a\tau}}\right) \phi^\lambda = \phi^\tau \delta^\lambda_\sigma \tag{N.1}$$

is given by (17).

Proof. First we take $w^{\omega\sigma}$ which are defined by the equation $w^{\omega\sigma} w_{\nu\sigma} = \delta^\omega_\nu$ and multiply both sides of (N.1) by $w^{\omega\sigma}$. Then contracting for σ we have

(1) Cf. T. Sibata, this journal, **5** (1935), 93.

$$\left(u^{\tau}w^{\omega\sigma}\frac{\partial}{\partial u^{\sigma}}+v^{\tau}w^{\omega\sigma}\frac{\partial}{\partial v^{\sigma}}-\frac{\partial}{\partial w_{\omega\tau}}\right)\phi^{\lambda}=\phi^{\tau}w^{\omega\lambda} \tag{N.2}$$

Since $w_{\omega\tau}=w_{\tau\omega}$, it follows that

$$\left(u^{[\tau}w^{\omega]\sigma}\frac{\partial}{\partial u^{\sigma}}+v^{[\tau}w^{\omega]\sigma}\frac{\partial}{\partial v^{\sigma}}\right)\phi^{\lambda}=\phi^{[\tau}w^{\omega]\lambda} \tag{N.3}$$

Moreover, multiplying the above equation by $w_{\tau\mu}w_{\omega\nu}$, and contracting for τ and ω, and writing u_{μ} for $u^{\tau}w_{\tau\mu}$, v_{μ} for $v^{\tau}w_{\tau\mu}$, ϕ_{μ} for $\phi^{\tau}w_{\tau\mu}$, we have

$$u_{[\mu}\frac{\partial\phi^{\lambda}}{\partial u^{\nu]}}+v_{[\mu}\frac{\partial\phi^{\lambda}}{\partial v^{\nu]}}=\phi_{[\mu}\delta^{\lambda}_{\nu]} \tag{N.4}$$

When $\lambda\neq\mu$, $\lambda\neq\nu$, (N.4) becomes

$$u_{[\mu}\frac{\partial\phi^{\lambda}}{\partial u^{\nu]}}+v_{[\mu}\frac{\partial\phi^{\lambda}}{\partial v^{\nu]}}=0 \tag{N.5}$$

Eliminating $\dfrac{\partial\phi^{\lambda}}{\partial v^{\nu}}$ $(\nu=1,\ldots\ldots,N)$, we have

$$\begin{vmatrix}\dfrac{\partial\phi^{\lambda}}{\partial u^{\mu}} & \dfrac{\partial\phi^{\lambda}}{\partial u^{\nu}} & \dfrac{\partial\phi^{\lambda}}{\partial u^{\omega}}\\ u_{\mu} & u_{\nu} & u_{\omega}\\ v_{\mu} & v_{\nu} & v_{\omega}\end{vmatrix}=0 \qquad (\lambda\neq\mu,\nu,\omega)\,. \tag{N.6}$$

therefore, $\dfrac{\partial\phi^{\lambda}}{\partial u^{\mu}}$ must have the form

$$\frac{\partial\phi^{\lambda}}{\partial u^{\mu}}=\rho_1u_{\mu}+\rho_2v_{\mu} \qquad (\lambda\neq\mu) \tag{N.7}$$

Similarly, we see that $\dfrac{\partial\phi^{\lambda}}{\partial v^{\mu}}$ has the form

$$\frac{\partial\phi^{\lambda}}{\partial v^{\mu}}=\rho_3v_{\mu}+\rho_4u_{\mu} \qquad (\lambda\neq\mu)\,. \tag{N.8}$$

Substituting (N.7) and (N.8) into (N. 5), we have

$$\rho_2v_{[\nu}u_{\mu]}+\rho_4u_{[\nu}v_{\mu]}=0\,;$$

hence,

$$\rho_2 = \rho_4 ,$$

therefore, (N.8) becomes

$$\frac{\partial \phi^\lambda}{\partial v^\mu} = \rho_3 v_\mu + \rho_2 u_\mu \qquad (\lambda \neq \mu) . \tag{N.9}$$

When $\lambda \neq \mu$, $\lambda = \nu$, using (N.7) and (N.9), (N.4) becomes

$$\phi_\mu = \left(\frac{\partial \phi^\lambda}{\partial u^\lambda} - \rho_1 u_\lambda - \rho_2 v_\lambda\right) u_\mu + \left(\frac{\partial \phi^\lambda}{\partial v^\lambda} - \rho_2 u_\lambda - \rho_3 v_\lambda\right) v_\mu \qquad (\lambda : \text{not summed})$$

or putting

$$P^\lambda \equiv \frac{\partial \phi^\lambda}{\partial u^\lambda} - \rho_1 u_\lambda - \rho_1 v_\lambda , \qquad Q^\lambda \equiv \frac{\partial \phi^\lambda}{\partial v^\lambda} - \rho_2 u_\lambda - \rho_3 v_\lambda ,$$

we have

$$\phi_\mu = P^\lambda u_\mu + Q^\lambda v_\mu \tag{N.10}$$

Since the left hand sides of these equations are independent of λ, we have

$$(P^\lambda - P^\nu) u_\mu + (Q^\lambda - Q^\nu) v_\mu = 0$$

$$(P^\lambda - P^\nu) u_\omega + (Q^\lambda - Q^\nu) v_\omega = 0$$

From these equations it must be that

$$P^\lambda - P^\nu = 0 , \qquad Q^\lambda - Q^\nu = 0 .$$

Hence P^λ and Q^λ are independent of λ. So writing P, Q for P^λ, Q^λ, (N.10) becomes

$$\phi_\mu = P u_\mu + Q v_\mu$$

or

$$\phi^\mu = P u^\mu + Q v^\mu \tag{N.11}$$

The solution of (N.11) must be of the form (N.11), namely the equotian (N.1) has two independent solutions. On the other hand, (N.1) has two particular solutions u^λ and v^λ; therefore the general solution of (N.1) is given by

$$f_1 u^\lambda + f_2 v^\lambda$$

where f_1 and f_2 are the general solutions of the equation obtained by putting the left hand side of (N.1) equal to zero:

$$\left(u^{\tau}\frac{\partial}{\partial u^{\sigma}}+v^{\tau}\frac{\partial}{\partial v^{\sigma}}-w_{\alpha\sigma}\frac{\partial}{\partial w_{\alpha\tau}}\right)f=0\,. \tag{N.12}$$

Now we will find the general solution of (N.12). We can easily see that the system of equations (N.12) constitutes a complete system and the number of independent variables is $2N+\frac{N(N+1)}{2}$. In order to obtain the number of independent solutions of (N.12), we have to find the number of independent solutions of (N.12). Using the method by which (N.2), (N.3) and (N.6) were obtained from (N.1), we have the following equations from (N.1)

$$\left(u^{\tau}w^{\omega\sigma}\frac{\partial}{\partial u^{\sigma}}+v^{\tau}w^{\omega\sigma}\frac{\partial}{\partial v_{\sigma}}-\frac{\partial}{\partial w_{\omega\tau}}\right)f=0 \tag{N.13}$$

$$\left(u^{[\tau}w^{\omega]\sigma}\frac{\partial}{\partial u^{\sigma}}+v^{[\tau}w^{\omega]\sigma}\frac{\partial}{\partial v^{\sigma}}\right)f=0 \tag{N.14}$$

$$\begin{vmatrix} \dfrac{\partial\phi^{\lambda}}{\partial u^{\mu}} & \dfrac{\partial\phi^{\lambda}}{\partial u^{\nu}} & \dfrac{\partial\phi^{\lambda}}{\partial u^{\omega}} \\ u_{\mu} & u_{\nu} & u_{\omega} \\ v_{\mu} & v_{\nu} & v_{\omega} \end{vmatrix}=0 \tag{N.15}$$

Among the operators: $\frac{\partial}{\partial w_{\omega\tau}}$ $(\omega, \tau=1,\ldots\ldots, N)$ in (N.13) the number of unconnected operators is $\frac{N(N+1)}{2}$; among the operators: $v^{[\tau}w^{\omega]}\frac{\partial}{\partial v^{\sigma}}$ in (N.14), $(N-1)$ operators $v^{[1}w^{\alpha]\sigma}\frac{\partial}{\partial v^{\sigma}}$ $(\alpha=2,\ldots\ldots, N)$ are unconnected, and among the equations of (N.15), $(N-2)$ equations (for the values of $\mu=1,\ \nu=2,\ \omega=3,\ldots\ldots, N$) are independent. Therefore (N.1) has $\frac{N(N+1)}{2}+(N-1)+(N-2)$ independent equations. Namely the number of independent solutions of (N.1) is

$$\frac{N(N+1)}{2}+2N-\left[\frac{N(N+1)}{2}+(N-1)+(N-2)\right]=3\,.$$

On the other hand, (N.1) has *3* independent solutions:

$$t_1 \equiv g_{\alpha\beta}u^\alpha u^\beta\,, \qquad t_2 = g_{\alpha\beta}u^\alpha v^\beta\,, \qquad t_3 = g_{\alpha\beta}v^\alpha v^\beta\,.$$

Hence the general solution of (N.1) is obtained as an arbitrary function of t_1, t_2, and t_3. Therefore the proposition is proved.