An Extension of the Parallelism in X_n^{n-m} in X_n.

By

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§ 1. The parallelism in subspace has already been studied by many writers(1), but their methods have been based on the assumption that parallel vectors in X_n^{n-m} are obtained by a certain definite projection of the corresponding parallel vectors in X_n, accordingly the covariant derivatives in X_n^{n-m} are obtained by the projection of the corresponding covariant derivatives in X_n i.e. if we denote $\bar{v}'^\lambda$ is the projected vector of $\bar{v}^\lambda$ which is parallel (in X_n) to v^λ,

$$\bar{v}'^\lambda = B_\mu^\lambda \bar{v}^\mu \quad \text{and} \quad \nabla'_\omega v^\lambda = B_{\omega\nu}^{\mu\lambda} \nabla_\mu v^\nu \quad \ldots\ldots\ldots\ldots (1),$$

where
$$B_\mu^\lambda = \delta_\mu^\lambda - \sum_i \underset{i}{n}^\lambda \overset{i}{t}_\mu \text{ (2)} .$$

The parallelism in X_n^{n-m} so obtained does not become more extended than the original parallelism in X_n i.e. if the original parallelism is linear, affine, or Riemannian, the parallelism in X_n^{n-m} becomes at most the same.

Now the question arises: Is there any method by which the subspace X_n^{n-m} with a wider parallelism induced in X_n as we see in the case where any Riemannian space R_N can be induced in euclidean space $E_M\left(M \geqq \dfrac{N(N+1)}{2}\right)$? — for example, how can non-linear space, non-symmetric space or Weyl's space be induced in a Riemannian space?

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E. Bortollotti, Memorie Bologna (8) **7** (1929), 11–20.
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(2) In this paper we shall employ certain notations due to J.A. Schouten, *Der Ricci-Kalkül*, (1924).

In this paper, starting from three reasonable assumptions, we will investigate the inducing method and then find the relations between this method and the projection stated in (1).

We suppose that X_n^{n-m} is defined by $n-m$ covariant vectors $\overset{i}{t}_\lambda$ [1] $(i = n-m+1, \ldots, n)$, and let $E_{n-m}(x)$ and $E_{n-m}(x+dx)$ be tangential euclidean spaces to X_n^{n-m} at two neighbouring points $P(x)$ and $Q(x+dx)$ respectively ; and $\bar{v}^\lambda$, a vector at P parallel to v^λ lying in $E_{n-m}(x+dx)$ when the parallelism is regarded as taking place in X_n ; $\bar{\bar{v}}^\lambda$, a vector at P parallel to v^λ lying in $E_{n-m}(x+dx)$ when the parallelism is regarded as taking place in X_n^{n-m}.

Now we put the following three assumptions in three cases separately.

Case I. General assumptions :

(a) X_n is a space with general linear[2] parallel displacement of vector i.e. if $\bar{v}^\lambda$ at $P(x)$ is parallel to v^λ at $Q(x+dx)$, then $\bar{v}^\lambda = v^\lambda + dv^\lambda + \Gamma^\lambda_{\mu\nu}(x)v^\mu dx^\nu$.

In this case, we say that $\bar{v}^\lambda$ is parallel to v^λ with respect to X_n.

(b) the parallelism in X_n^{n-m} is a reversible one — to — one correspondence between the vector manifolds $E_{n-m}(x)$ and $E_{n-m}(x+dx) : \bar{\bar{v}}^\lambda \rightleftarrows v^\lambda$; in this case, we say that $\bar{\bar{v}}^\lambda$ is parallel to v^λ with respect to X_n^{n-m}.

(c) if $\bar{v}^\lambda$ is contained in $E_{n-m}(x)$ then $\bar{v}^\lambda$ is also parallel to v^λ with respect to X_n^{n-m}, e.i. if $\bar{v}^\lambda \overset{i}{t}_\lambda = 0$ then $\bar{\bar{v}}^\lambda = \bar{v}^\lambda$.

Case II. Restricted assumptions :

(a) X_n is the same as before,

(b) $\bar{\bar{v}}^\lambda - v^\lambda$ is at most the same order with the difference dx^λ (between two coordinates of P and Q) and $k\bar{\bar{v}}^\lambda$ is parallel to kv^λ with respect to X_n^{n-m} where k is any multiplier,

(c) if $\bar{v}^\lambda \overset{i}{t}_\lambda = 0$, then $\bar{\bar{v}}^\lambda = \bar{v}^\lambda$.

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(1) We shall treat the general subspace which is holonomic or non -holonomic.

(2) When the connection $(\Gamma^\lambda(x, v, dx))$ of parallelism is linear with respect to v^λ and dx^λ($i, e, \Gamma^\lambda_{\mu\nu}(x)v^\mu dx^\nu$), we say the parallelism is linear.

We may conclude that in the case I and II subspaces with non-linear connection with respect to v^λ are dx^λ are induced in X_n .[1]

Case III.

(a) X_n is the same as before,

(b) the parallelism in X_n^{n-m} is linear with respect to v^λ and dx^λ.

(c) if $\bar{v}^\lambda \overset{i}{t}_\lambda = 0$ then $\bar{\bar{v}}^\lambda = \bar{v}^\lambda$.

§ 2. The treatment of cases I and II will be found to be nearly the same as that for case III, so we will first consider case III.

Let the coefficients of connection in X_n^{n-m} be $\bar{\Gamma}^\lambda_{\mu\nu}(x)$.
From (b) we can put

$$\bar{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}(x) + D^\lambda_{\mu\nu}(x)\,,$$

and if v^λ is contained in $E_{n-m}(x+dx)$; i.e.

$$v^\lambda(x+dx)\overset{i}{t}_\lambda(x+dx) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

then we have

$$\bar{\bar{v}}^\lambda = \bar{v}^\lambda + D^\lambda_{\mu\nu} v^\mu dx^\nu\,.$$

From the definition[2] of parallelism in X_n^{n-m}, we have $\overset{i}{t}_\lambda \bar{\bar{v}}^\lambda = 0$ or rewriting this,

$$\overset{i}{t}_\lambda(v^\lambda + dv^\lambda + \Gamma^\lambda_{\mu\nu} v^\mu dx^\nu + D^\lambda_{\mu\nu} v^\mu dx^\nu) = 0\,,$$

and from (2) this becomes

$$v^\mu \left\{ d\overset{i}{t}_\mu - (\Gamma^\lambda_{\mu\nu} + D^\lambda_{\mu\nu})\overset{i}{t}_\lambda dx^\nu \right\} = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (3).$$

Therefore, neglecting the term higher than the 1st order of dx^λ in (3), we have

$$D^\lambda_{\mu\nu} = \sum_i \underset{i}{n}^\lambda \overset{i}{a}_{\mu\nu} + \sum_a \underset{a}{T}^\lambda \overset{a}{D}_{\mu\nu} + \sum_i \left(\underset{1}{\overset{i}{V}}{}^\lambda_\mu \overset{i}{t}_\nu + \underset{2}{\overset{i}{V}}{}^\lambda_\nu \overset{i}{t}_\mu \right) \quad \ldots\ldots \quad (4),$$

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(1) In cases I and II, $\overset{a}{B}_{\mu\nu}$, $\underset{a}{T}^\lambda$ in (11) can depend on v^λ and dx^λ.

(2) C. f. (b) in case I on page 178.

where $$\overset{i}{a}_{\mu\nu} = \frac{\partial \overset{i}{t}_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} \overset{i}{t}_\lambda ;$$

and $\underset{i}{n}^\lambda$ and $\underset{a}{T}^\lambda$ are any independent vectors defined by the respective following equations:

$$\underset{i}{n}^\lambda \overset{j}{t}_\lambda = \delta^j_i ; \quad \underset{a}{T}^\lambda \overset{i}{t}_\lambda = 0 \qquad (a = 1, 2, \ldots\ldots, n-m),$$

and $\overset{a}{D}_{\mu\nu}$ is a certain tensor. (Here $\overset{a}{D}_{\mu\nu}$ may be arbitrary but later it will be restricted by condition (c)).

Since the last terms in (4) vanish while we are concerning v^λ and dx^λ lying in $E_{n-m}(x+dx)$, we can leave them out of consideration; and therefore we have

$$D^\lambda_{\mu\nu} = \sum_i \underset{i}{n}^\lambda \overset{i}{a}_{\mu\nu} + \sum_a \underset{a}{T}^\lambda \overset{a}{D}_{\mu\nu} \ldots\ldots\ldots\ldots\ldots \quad (5).$$

Thus we have

$$\bar{\bar{v}}^\lambda = \bar{v}^\lambda + \Big(\sum_i \underset{i}{n}^\lambda \overset{i}{a}_{\mu\nu} + \sum_a \underset{a}{T}^\lambda \overset{a}{D}_{\mu\nu}\Big) v^\mu dx^\nu \qquad (6),$$

and since

$$\begin{aligned} \overset{i}{t}_\lambda \bar{v}^\lambda &= (v^\lambda + dv^\lambda + \Gamma^\lambda_{\mu\nu} v^\mu dx^\nu) \overset{i}{t}_\lambda \\ &= (d\overset{i}{t}_\lambda v^\lambda + \Gamma^\lambda_{\mu\nu} v^\mu dx^\nu \overset{i}{t}_\lambda) . \qquad \text{(by (2))} \\ &= \overset{i}{a}_{\mu\nu} v^\mu dx^\nu , \end{aligned}$$

equation (6) becomes

$$\begin{aligned} \bar{\bar{v}}^\lambda &= \Big(\delta^\lambda_\mu - \sum_i \underset{i}{n}^\lambda \overset{i}{t}_\mu\Big) \bar{v}^\mu + \sum_a \underset{a}{T}^\lambda \overset{a}{B}_{\mu\nu} v^\mu dx^\nu \\ &= B^\lambda_\mu \bar{v}^\mu + \sum_a \underset{a}{T}^\lambda \overset{a}{B}_{\mu\nu} v^\mu dx^\nu \ldots\ldots\ldots\ldots\ldots\ldots \quad (7). \end{aligned}$$

Next, the assumption (c) is written down as follows:

$$\text{if} \quad \overset{i}{a}_{\mu\nu} v^\mu dx^\nu = 0 \quad \text{then} \quad D^\lambda_{\mu\nu} v^\mu dx^\nu = 0 \ldots\ldots\ldots\ldots\ldots \quad (8).$$

If u^λ and w^λ are any vectors in X_n, and we substitute (5) in (8), the above equation becomes as follows:

$$\text{if} \quad u^\mu w^\nu B^{\varepsilon\eta}_{\mu\nu} \overset{i}{a}_{\varepsilon\eta} = 0 \quad \text{then} \quad u^\mu w^\nu B^{\varepsilon\eta}_{\mu\nu} \overset{a}{D}_{\varepsilon\eta} = 0 \ldots\ldots\ldots \quad (9).$$

So, putting

$$B^{\varepsilon\eta}_{\mu\nu} \overset{i}{a}_{\varepsilon\eta} = \overset{i}{A}_{\mu\nu} \qquad \text{and} \qquad B^{\varepsilon\eta}_{\mu\nu} \overset{a}{D}_{\varepsilon\eta} = \overset{a}{B}_{\mu\nu} ,$$

and eliminating w^λ from (9), the condition (c) can be fasten in the following equation:

$$u^\mu \overset{a}{B}_{\mu\nu} = \sum_i u^\mu \overset{i}{A}_{\mu\nu} \underset{i}{\overset{a}{\rho}}(u^\varepsilon) \qquad\qquad (10),$$

where $\underset{i}{\overset{a}{\rho}}$ may be any function of $u^1, u^2, \ldots, u^n$.

Since u^λ is any vector in X_n and $\overset{a}{B}_{\mu\nu}$ and $\overset{i}{A}_{\mu\nu}$ are independent of u^λ, $\underset{i}{\overset{a}{\rho}}$ is homogeneous zero th order with respect to u^ε .

Consequently, in the case of III the most general parallelism induced in X_n^{n-m} is given by the following equation:

$$\boxed{\bar{\bar{v}}^\lambda = B^\lambda_\mu \bar{v}^\mu + \sum_a \underset{a}{T^\lambda} \overset{a}{B}_{\mu\nu} v^\mu \, dx^\nu} \qquad\qquad (11),$$

where $\overset{a}{B}_{\mu\nu}$ is given by equation (9).

The process of inducing $\bar{\bar{v}}^\lambda$ from v^λ may be called *a "pseudo-projection".*

In cases I and II, $\underset{a}{T^\lambda}$ and $\overset{a}{B}_{\mu\nu}$ can depend on v^λ and dx^λ, but by a little modification, the cases are treated in the same way as the case III.

§ 3. In the general case (10), I have not succeeded in finding the general form of $\overset{a}{B}_{\mu\nu}$. So we shall treat a somewhat restricted case.

We assume that $\underset{i}{\overset{a}{\rho}}$ has the form:

$$\underset{i}{\overset{a}{\rho}}(u) = \frac{\underset{i\ \varepsilon_1 \ldots \varepsilon_l}{\overset{a}{p}(x)} \quad u^{\varepsilon_1} \ldots u^{\varepsilon_l}}{\underset{i\ \varepsilon'_1 \ldots \varepsilon'_l}{\overset{a}{q}(x)} \quad u^{\varepsilon'_1} \ldots u^{\varepsilon'_l}} \qquad (l \text{ is any positive integer}).$$

Substituting this into (10) and comparing the power of u^λ, we have

$$\underset{n-m+1}{\overset{a}{q}}(x)_{(\varepsilon_{1,1}\ldots\varepsilon_{1,l}}\ldots\underset{n}{\overset{a}{q}}(x)_{\varepsilon_{m,1}\ldots\varepsilon_{m,l}}\overset{a}{B}_{\mu)\nu}$$
$$=\sum_{r=1}^{m}\underset{n-m+1}{\overset{a}{q}}(x)_{(\varepsilon_{1,1}\ldots\varepsilon_{1,l}}\ldots\underset{n-m+r}{\overset{a}{p}}(x)_{\varepsilon_{r,1}\ldots\varepsilon_{r,l}}\ldots\underset{n}{\overset{a}{q}}(x)_{\varepsilon_{m,1}\ldots\varepsilon_{m,l}}\overset{n-m+r}{A}_{\mu)\nu}\ \ldots \quad (12).$$

So, if we can find $\underset{i}{\overset{a}{p}}_{\varepsilon_1\ldots\varepsilon_l}$, $\underset{i}{\overset{a}{q}}_{\varepsilon_1\ldots\varepsilon_l}$ and $\overset{a}{B}_{\mu\nu}$ from the above, $\overset{a}{B}_{\mu\nu}$ is a solution of (10).

$\overset{a}{B}_{\mu\nu}$ so obtained is not in general linear in $\overset{i}{A}_{\mu\nu}\left(\text{i.e. } \sum_i \underset{a}{\overset{i}{h}}\overset{i}{A}_{\mu\nu}\right)$. We will illustrate this by an example. Namely, in the case when

$$\overset{n}{A}_{12}(x)=\overset{n}{A}_{21}=\overset{n-1}{A}_{11}=\overset{n-1}{A}_{22} \quad \text{and other} \quad A_{\mu\nu}=0\,,$$

if we take $\overset{a}{B}_{\mu\nu}$ so that

$$\overset{a}{B}_{21}-\overset{a}{B}_{12}=-(\overset{a}{B}_{11}-\overset{a}{B}_{22}) \quad \text{and} \quad \overset{a}{B}_{\mu\nu}=0 \qquad (\text{for } \mu,\nu\neq 1,2)\,,$$

and take p_λ, q_λ such that

$$q_2+q_1=1\,,\ p_2-p_1=B_{22}-B_{21} \quad \text{and} \quad p_\lambda\,,\ q_\lambda\,,\ =0 \quad \text{for} \quad \lambda\neq 1,2,$$

then $\overset{a}{B}_{\mu\nu}$, p_λ, q_λ satisfy (12). Accordingly $\overset{a}{B}_{\mu\nu}$ is a solution of (10). Hence $\overset{a}{B}_{\mu\nu}$ is not in general expressed linearly by $\overset{i}{A}_{\mu\nu}$.

But when and only when $\overset{a}{B}_{\mu\nu}$ are linear forms of $\overset{i}{A}_{\mu\nu}$, we can identify $\bar{v}'^\lambda$ with $\bar{\bar{v}}^\lambda$ by suitably choosing the normal vectors of X_n^{n-m}(1).

So we have the result: *Since $\overset{a}{B}_{\mu\nu}$ is not, in general, linear in $\overset{i}{A}_{\mu\nu}$ we know that, in fact, $\sum_i \underset{i}{n}^\lambda \overset{i}{a}_{\mu\nu}+\sum_a \underset{a}{T}^\lambda \overset{a}{B}_{\mu\nu}$* in (6) *cannot be expressed in the form $\sum_i {}'\underset{i}{n}^\lambda a_{\mu\nu}$ but gives more complicated connection of sub-space, i.e. our "pseudo-projection" is a generalization of projection.*

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(1) If $\overset{a}{B}_{\mu\nu}=\sum_i \underset{a}{\overset{i}{h}}\overset{i}{A}_{\mu\nu}$, then from (6) and (7) we have

$$\bar{\bar{v}}^\lambda=\bar{v}^\lambda+\sum_i(\underset{i}{n}^\lambda+\sum_a \underset{a}{T}^\lambda \underset{a}{\overset{i}{h}})\overset{i}{a}_{\mu\nu}v^\mu dx^\nu\,. \quad (\text{within the } X_n^{n-m})$$
$$=(\delta^\lambda_\mu-\sum_i {}'\underset{i}{n}^\lambda \overset{i}{t}_\mu)\bar{v}^\mu$$
$$\therefore\ \bar{\bar{v}}^\lambda={}'B^\lambda_\mu\bar{v}^\mu={}'(\bar{v}'^\lambda)\,,\quad \text{where } {}'\underset{i}{n}^\lambda=\underset{i}{n}^\lambda+\sum_a \underset{a}{T}^\lambda \underset{a}{\overset{i}{h}}\,.$$

Therefore by means of the "pseudo-projection", non-symmetric subspace can be induced in symmetric space, or Weyl's subspace can be induced in euclidean space.

§ 4. In the case of X_n^{n-1} the most general tensor $\overset{a}{B}_{\mu\nu}$ satisfying (10) is obtained as $\rho\overset{n}{A}_{\mu\nu}$, as can be easily seen by normalizing the bilinear form $\overset{a}{A}_{\mu\nu} u^\mu w^\nu$ by a coorninate tranformation at a point.

Hence in X_n^{n-1} the pseudo-projection becomes a projection.

Now, a question arises:

(d). In X_n when X_n^l is contained in X_n^m, in what case does it occur that the parallelism in X_n^l induced from X_n^m in which the parallelism first induced from X_n, and the parallelism in X_n^l induced directly from X_n, are always equivalent for each value of m less than n.

In this case $\overset{a}{B}_{u\nu}$ becomes a linear form of $\overset{i}{A}_{\mu\nu}$ by means of the iteration of hypermanifolds, X_p^{p-1} therefore our pseudo-projection becomes projection. Conversely, in the case of projection (a), (b), (c) and (d) are clearly satisfied. So we have the result:

The conditions (a), (b), (c) and (d) characterize the projection.

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