# On the Space which admits a given Continuous Transformation Group.

By

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We shall consider the space in which the parallelism of vectors is unaltered by all the transformations of a given continuous transformation group.

First, we will express the parallel displacement of vectors by Lie's symbols of the infinitesimal transformations, and from this we will obtain the necessary and sufficient conditions for the existence of a space which admits a given continuous transformation group. Further, if such a space exists, we will find all such spaces.

Next, we will see how any vector-and tensor-fields are transformed by the given continuous group, and obtain the relation between the parallel displacement and the transformation of the vector-field.

Lastly, in the case when the operators of the given transformation group are all unconnected, we will obtain the most general space which admits it.

## I.

Let us consider a space $X_n^{(1)}$, and let the coordinates be $x^1, \ldots, x^n$, and the coefficients of connection be $\Gamma_{\mu\nu}^{\lambda}(x)$.

In this space let $v^\lambda(x)$ be a vector-field, and $v'^\lambda$ be the vector at a point $P(x)$ which is parallel to the vector $v^\lambda(x+dx)$ at a point $Q(x+dx)$ in the neighbourhood of the point $P(x)$. Then we have

$$v^\lambda(x+dx)-v'^\lambda = -\Gamma_{\mu\nu}^{\lambda} v'^\mu dx^\nu ,$$

neglecting the terms higher than the 2nd. The left hand of this equation expresses the change of vector $v'^\lambda$ when the vector $v'^\lambda$ at the point

(1) In this paper we shall employ certain notations due to J. A. Schouten, *Der Ricci-Kalkül*, (1924).

$P(x)$ is displaced in parallel to the point $Q(x+dx)$. Therefore, if we put $v'^{\lambda} = \dot{x}^{\lambda}$ in the above equation, the change of the vector $\dot{x}^{\lambda}$ is equal to $-\Gamma^{\lambda}_{\mu\nu}\dot{x}^{\mu}dx^{\nu}$. By using Lie's symbol of infinitesimal transformation, we can express the operation of finding this variation of a vector in the following form

$$dx^i\frac{\partial}{\partial x^i} - \Gamma^{\lambda}_{\mu i}\dot{x}^{\mu}dx^i\frac{\partial}{\partial \dot{x}^{\lambda}}. \tag{1}$$

Then if we put

$$T_i = \frac{\partial}{\partial x^i} - \Gamma^{\lambda}_{\mu i}\dot{x}^{\mu}\frac{\partial}{\partial \dot{x}^{\lambda}} \qquad (i = 1, \ldots\ldots, n), \tag{2}$$

$T_i$ expresses the operation in which, when a vector $\dot{x}^{\lambda}$ is displaced in parallel from a point $P(x^1, \ldots, x^i, \ldots, x^n)$ to the point $Q(x^1, \ldots, x^{i-1}, x^i+\delta t, x^{i+1}, \ldots, x^n)$, the change of the vector $\dot{x}^{\lambda}$ is equal to $-\Gamma^{\lambda}_{\mu i}\dot{x}^{\mu}\delta t$. And since (1) is given by $dx^iT_i$ we can define by (2) the parallelism of vectors in the space.

Now let an r-parameter continuous transformation group be given by the symbols of the infinitesimal transformations

$$S_k = \xi^i_k(x)\frac{\partial}{\partial x^i} \qquad (k = 1, \ldots\ldots, r). \tag{3}$$

When the parallelism of vectors, defined by (2), is unaltered by all the transformations of the group (3), we say that the space admits group (3).

Now we are going to find the conditions in which the space admits group (3). Let the extended infinitesimal transformations of (3) be denoted by

$$\dot{S}_k = \xi^i_k\frac{\partial}{\partial x^i} + \frac{\partial \xi^i_k}{\partial x^{\alpha}}\dot{x}^{\alpha}\frac{\partial}{\partial \dot{x}^i} \qquad (k = 1, \ldots\ldots, r), \tag{4}$$

then $T_i$ $(i = 1, \ldots, n)$ are transformed by (4) as follows

$$T'_i = T_i + t(\dot{S}_kT_i) + \frac{t^2}{2!}\left(\dot{S}_k(\dot{S}_kT_i)\right) + \ldots\ldots \qquad (i = 1, \ldots\ldots, n).$$

The necessary and sufficient conditions that the space, whose parallelism of vectors is defined by (2), admits the given group (3), are that $T'_i$ $(i = 1, \ldots, n)$ must be expressed linearly by $T_1, T_2, \ldots, T_n$, with coefficients independent of $x$ for all values of $t$. Therefore it must be that

$$(\dot{S}_k T_i) = \rho^l_{ki} T_l \qquad (k = 1, \ldots, r;\ i = 1, \ldots, n), \tag{5}$$

where $\rho^l_{ki}$ are functions of $x$. Then $(\dot{S}_k(\dot{S}_k T_i))$, etc., can be expressed linearly by $T_1, \ldots, T_n$, that is $T'_i$. But comparing the coefficients of $\frac{\partial}{\partial x^l}$ on both sides of the above equations, we have

$$\rho^l_{ki} = -\frac{\partial \xi^l_k}{\partial x^i}.$$

Therefore (5) become

$$(\dot{S}_k T_i) = -\frac{\partial \xi^l_k}{\partial x^i} T_l \qquad (k = 1, \ldots, r.\ i = 1, \ldots, n). \tag{6}$$

So we have the result: *the relations* (6) *are the necessary and sufficient conditions that the space admits the group* (3).

Further, comparing the coefficients of $\frac{\partial}{\partial \dot{x}^\lambda}$ on both sides of (6), we have[1]

$$\xi^\omega_k \frac{\partial \Gamma^\lambda_{\alpha\beta}}{\partial x^\omega} + \Gamma^\lambda_{\alpha\omega} \frac{\partial \xi^\omega_k}{\partial x^\beta} + \Gamma^\lambda_{\omega\beta} \frac{\partial \xi^\omega_k}{\partial x^\alpha} - \Gamma^\omega_{\alpha\beta} \frac{\partial \xi^\lambda_k}{\partial x^\omega} + \frac{\partial^2 \xi^\lambda_k}{\partial x^\alpha \partial x^\beta} = 0$$

$$(k = 1, \ldots, r). \tag{7}$$

Conversely, from (7) we can easily deduce (6). So we have the result: *the relations* (7) *are the necessary and sufficient conditions that the space admits the group* (3).

The problem of solving $\Gamma^\lambda_{\alpha\beta}$ from this system of differential equations (7), can be proved equivalent to the problem of finding a system of equations of the form

$$u^\lambda_{\alpha\beta} - \Gamma^\lambda_{\alpha\beta}(x) = 0 \qquad (\lambda, \alpha, \beta = 1, \ldots, n), \tag{8}$$

which admits the following operators

$$W_k = \xi^\omega_k \frac{\partial}{\partial x^\omega} - \left\{ \frac{\partial \xi^\omega_k}{\partial x^\nu} u^\varepsilon_{\mu\omega} + \frac{\partial \xi^\omega_k}{\partial x^\mu} u^\varepsilon_{\omega\nu} - \frac{\partial \xi^\varepsilon_k}{\partial x^\omega} u^\omega_{\mu\nu} + \frac{\partial^2 \xi^\varepsilon_k}{\partial x^\mu \partial x^\nu} \right\} \frac{\partial}{\partial u^\varepsilon_{\mu\nu}}$$

$$(k=1, \ldots, r),$$

(1) Eisenhart has obtained this equation from another point of view. (*Non-Riemannian Geometry* (1929), 125–126).

where $u^{\lambda}_{\alpha\beta}$ and $x^{\omega}$ are regarded as the independent variables. For, the conditions that (8) admits $W_k$, are written down as (7).

Since (3) form a group, namely

$$(S_iS_j) = c^k_{ij}S_k \qquad (i, j = 1, \ldots, r),$$

we can prove by actual calculation the following relations

$$(W_iW_j) = c^k_{ij}W_k \qquad (i, j = 1, \ldots, r).$$

Hence $W_1, \ldots, W_r$ form a group.

Now in the given group (3), take out all the unconnected operators, say $S_1, \ldots, S_m$, and express the other operators linearly by $S_1, \ldots, S_m$ as follows

$$S_{m+j} = \sum_{\nu=1}^{m} \varphi^{\nu}_j S_\nu \qquad (i = 1, \ldots, r-m),$$

then

$$\left.\begin{aligned} &\xi^i_{m+j} = \sum_{\nu=1}^{m} \varphi^{\nu}_j \xi^i_\nu \qquad (i = 1, \ldots, n), \\ &\frac{\partial \xi^i_{m+j}}{\partial x^\alpha} = \sum_{\nu=1}^{m} \left\{ \varphi^{\nu}_j \frac{\partial \xi^i_\nu}{\partial x^\alpha} + \frac{\partial \varphi^{\nu}_j}{\partial x^\alpha} \xi^i_\nu \right\} \qquad (\alpha = 1, \ldots, n). \end{aligned}\right\} \quad (9)$$

Hence by using (9), the equations of (7) are rewritten as follows

$$\left.\begin{aligned} &\xi^{\omega}_k \frac{\partial \Gamma^{\lambda}_{\alpha\beta}}{\partial x^{\omega}} + \Gamma^{\lambda}_{\alpha\omega} \frac{\partial \xi^{\omega}_k}{\partial x^\beta} + \Gamma^{\lambda}_{\omega\beta} \frac{\partial \xi^{\omega}_k}{\partial x^\alpha} - \Gamma^{\omega}_{\alpha\beta} \frac{\partial \xi^{\lambda}_k}{\partial x^{\omega}} + \frac{\partial^2 \xi^{\lambda}_k}{\partial x^\alpha \partial x^\beta} = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad (k = 1, \ldots, m) \\ &\sum_{\nu=1}^{m} \left[ \Gamma^{\lambda}_{\alpha\omega} \frac{\partial \varphi^{\nu}_j}{\partial x^\beta} \xi^{\omega}_\nu + \Gamma^{\lambda}_{\omega\beta} \frac{\partial \varphi^{\nu}_j}{\partial x^\alpha} \xi^{\omega}_\nu - \Gamma^{\omega}_{\alpha\beta} \frac{\partial \varphi^{\nu}_j}{\partial x^{\omega}} \xi^{\lambda}_\nu + \frac{\partial^2 (\varphi^{\nu}_j \xi^{\lambda}_\nu)}{\partial x^\alpha \partial x^\beta} \right. \\ &\qquad \left. - \varphi^{\nu}_j \frac{\partial^2 \xi^{\lambda}_\nu}{\partial x^\alpha \partial x^\beta} \right] = 0 \qquad (j = 1, \ldots, r-m). \end{aligned}\right\} \quad (10)$$

Therefore if the system of equations of the form of (8) admits $W_1, \ldots, W_r$, the system of equations

$$\sum_{\nu=1}^{m} \left[ u^{\lambda}_{\alpha\omega} \frac{\partial \varphi^{\nu}_j}{dx^\beta} \xi^{\omega}_\nu + u^{\lambda}_{\omega\beta} \frac{\partial \varphi^{\nu}_j}{dx^\alpha} \xi^{\omega}_\nu - u^{\omega}_{\alpha\beta} \frac{\partial \varphi^{\nu}_j}{\partial x^{\omega}} \xi^{\lambda}_\nu + \frac{\partial^2 (\varphi^{\nu}_j \xi^{\lambda}_\nu)}{\partial x^\alpha \partial x^\beta} - \varphi^{\nu}_j \frac{\partial^2 \xi^{\lambda}_\nu}{\partial x^\alpha \partial x^\beta} \right] = 0$$

$$(j = 1, \ldots, r-m), \qquad (11)$$

must be satisfied by the relations of (8). If the system (11) does not exist, then the system of equations of the form of (8), which admits the operators $W_1, \ldots, W_r$, cannot exist. If the system (11) exists, it can be proved that there exists the system of equations of the form of (8), which admits $W_1, \ldots, W_r$, by nearly the same method which *S.* Lie adopted in his treatise.(1) So we have

**Theorem 1.** *If*

$$S_k = \xi_k^i(x)\frac{\partial}{\partial x^i} \qquad (k = 1, \ldots, r) \tag{3}$$

*form an r-parameter group,* $S_1, \ldots, S_m$, *the unconnected operators, and the other operators* $S_{m+1}, \ldots, S_r$ *are expressed linearly by* $S_1, \ldots, S_m$ *as follows*

$$S_{m+j} = \sum_{\nu=1}^{m} \varphi_j^\nu S_\nu \qquad (j = 1, \ldots, r-m) .$$

*Then for the existence of the space which admits the given group* (3), *it is necessary and sufficient that the following system of equations should exist*

$$\sum_{\nu=1}^{m}\left[u_{\alpha\omega}^{\lambda}\frac{\partial\varphi_j^\nu}{\partial x^\beta}\xi_\nu^\omega + u_{\omega\beta}^{\lambda}\frac{\partial\varphi_j^\nu}{\partial x^\alpha}\xi_\nu^\omega - u_{\alpha\beta}^{\omega}\frac{\partial\varphi_j^\nu}{\partial x^\omega}\xi_\nu^\lambda + \frac{\partial^2(\varphi_j^\nu\xi_\nu^\lambda)}{\partial x^\alpha\partial x^\beta} - \varphi_j^\nu\frac{\partial^2\xi_\nu^\lambda}{\partial x^\alpha\partial x^\beta}\right] = 0$$
$$(j = 1, \ldots, r-m) . \tag{11}$$

N. B. If this system of equations exists, the coefficients of connection $\Gamma_{\mu\nu}^\lambda$ of the general space which admits the given group (3) can be obtained as follows:(2) Solve the system of equations (11) with respect to $u_{\mu\nu}^\lambda$. Suppose that the $l$ in such $u_{\mu\nu}^\lambda$ are expressed by the remaining $n^3-l$ $u_{\mu\nu}^\lambda$ and the $x$, and substituting the values of these $l$ $u_{\mu\nu}^\lambda$ into the operators

$$W_k = \xi_k^\omega\frac{\partial}{\partial x^\omega} - \left\{\frac{\partial\xi_k^\omega}{\partial x^\nu}u_{\mu\omega}^\varepsilon + \frac{\partial\xi_k^\omega}{\partial x^\mu}u_{\omega\nu}^\varepsilon - \frac{\partial\xi_k^\varepsilon}{\partial x^\omega}u_{\mu\nu}^\omega + \frac{\partial^2\xi_k^\varepsilon}{\partial x^\mu\partial x^\nu}\right\}\frac{\partial}{\partial u_{\mu\nu}^\varepsilon}$$
$$(k = 1, \ldots, r) ,$$

we obtain the operators $\overline{W}_1, \ldots, \overline{W}_r$ in $n^3+n-l$ variables $u_{\mu\nu}^\lambda, x$. As the $n^3+n-l-m$ independent solutions of the complete system $\overline{W}_1 f = 0, \ldots, W_m f=0$, we take the $n-m$ independent solutions $u_1(x), \ldots, u_{n-m}(x)$,

(1) S. Lie, *Theorie der Transformationsgruppe.* 1. (1930) 372–374.

(2) Cf. *ibid.*

of the complete system $S_1 f=0, \ldots, S_m f=0$ in $n$ variables $x^1, \ldots, x^n$, and the other $n^3-l$ independent solutions $V_\mu(u^\lambda_{\mu\nu}, x^i)(\mu=1, \ldots, n^3-l)$. Now consider the system of the following equations

$$V_\mu(u^\lambda_{\mu\nu}, x^i) = \Omega_\mu(u(x)) \qquad \mu = 1, \ldots, n^3-l) \,, \tag{12}$$

where $\Omega_\mu(u(x))$ are arbitrary functions of the $u_1, \ldots, u_{n-m}$. If we obtain the functions of $x$ by solving $n^3$ $u^\lambda_{\mu\nu}$ from (11) (12), the functions $u^\lambda_{\mu\nu}(x)$ give the most general form of $\Gamma^\lambda_{\mu\nu}$ of the space which admits the given group.

## II.

The extended infinitesimal transformatian of $S_k$

$$\dot{S}_k = \xi^i_k \frac{\partial}{\partial x^i} + \frac{\partial \xi^\lambda_k}{\partial x^\alpha} \dot{x}^\alpha \frac{\partial}{\partial \dot{x}^\lambda}$$

is interpreted as follows: when a point $P(x)$ is transformed by $S_k$ into a point $Q(x+\xi_k\delta t)$, a vector at the point $P(x)$ whose components are $\dot{x}^\lambda (\lambda=1, \ldots, n)$ is transformed into a vector at the point $Q(x+\xi_k\delta t)$, whose components are

$$\dot{x}^\lambda + \frac{\partial \xi^\lambda_k}{\partial x^\alpha} \dot{x}^\alpha \delta t \qquad (\lambda = 1, \ldots, n) \,. \tag{13}$$

Consider a vector-field $v^\lambda(x)$, and let the vector $v'^\lambda$ at the point $P(x)$ be the vector which may be transformed into the vector $v^\lambda(x+\xi_k\delta t)$ at the point $Q(x+\xi_k\delta t)$ by the infinitesimal transformation $S_k$. So we have, from (13)

$$v^\lambda(x+\xi_k\delta t) = v'^\lambda + \frac{\partial \xi^\lambda_k}{\partial x^\alpha} v'^\alpha \delta t \,;$$

or, expanding the left hand side in the power series of $\delta t$ and neglecting terms higher than the 2nd, we have

$$v'^\lambda - v^\lambda(x) = \left[ \xi^i_k \frac{\partial v^\lambda}{\partial x^i} - \frac{\partial \xi^\lambda_k}{\partial x^\alpha} v^\alpha \right] \delta t \,.$$

The left hand side of the above is interpreted as the variation of $v^\lambda(x)$ by the infinitesimal transformation $S_k$; and we call $\xi^i_k \dfrac{\partial v^\lambda}{\partial x^i} - \dfrac{\partial \xi^\lambda_k}{\partial x^\alpha} v^\alpha$ on the right hand side, *the transformation-derivative of a contravariant vector* $v^\lambda(x)$ *with respect to the infinitesimal transformation* $S_k$, and denote it by the symbol $\varDelta_k v^\lambda$. Namely

$$\varDelta_k v^\lambda = \xi_k^i \frac{\partial v^\lambda}{\partial x^i} - \frac{\partial \xi_k^\lambda}{\partial x^\alpha} v^\alpha .$$

When $v^\lambda(x)$ satisfies the relations

$$\varDelta_k v^\lambda = 0 \qquad (k = 1, \ldots, r) ,$$

we call it the *invariant vector (contravariant) by the group* $S_1, \ldots, S_r$.

Similarly we can define *the transformation-derivative of a covariant vector* $w_\lambda(x)$ *with respect to* $S_k$, by considering another extended infinitesimal transformation

$$\overline{S}_k = \xi_k^i \frac{\partial}{\partial x^i} - \frac{\partial \xi_k^\alpha}{\partial x^\lambda} \bar{x}_\alpha \frac{\partial}{\partial \bar{x}_\lambda} ,$$

where $\bar{x}_\alpha (= 1, \ldots, n)$ are the componente of a covariant vector, and we denote it by the symbol $\varDelta_k w_\lambda$, where

$$\varDelta_k w_\lambda = \xi_k^i \frac{\partial v^\lambda}{\partial x^i} + \frac{\partial \xi_k^\alpha}{\partial x^\lambda} w_\alpha .$$

When $w_\lambda(x)$ satisfies the relations

$$\varDelta_k w^\lambda = 0 \qquad (k = 1, \ldots, r) ,$$

we call it *the invariant vector (covariant) by the group* $S_1, \ldots, S_r$.

**N. B.** $\varDelta_k v^\lambda$ and $\varDelta_k w_\lambda$ are respectively contravariant and covariant vectors with the suffixes $\lambda$, for they have been introduced by vector-differences.

Now a question arises: Under what conditions may an invariant vector (contravariant) exist? To answer this question, put

$$V = v^\lambda \frac{\partial}{\partial x^\lambda}$$

and take account of the identities

$$(S_k V) = (\varDelta_k v^\lambda) \frac{\partial}{\partial x^\lambda} \qquad (k = 1, \ldots, r) ;$$

then we see that if $v^\lambda$ is an invariant vector that is $\Delta_k v^\lambda = 0$ ($k = 1, \ldots, r$), there exists an operator $V$ such that $(S_k V) = 0$ ($k = 1, \ldots, r$); and vice versa. Hence the problem of finding an invariant vector $v^\lambda$ is equivalent to that of finding an operator $V$ which satisfies the relations $(S_k V) = 0$ ($k = 1, \ldots, r$). But from the Lie's theorem(1) the condition for the existence of such an operator $V$, is that $S_1, \ldots, S_r$ form a stationary group. So we have

**Theorem 2.** *The necessary and sufficient condition for the existence of an invariant vector (contravariant) by a given group, is that the given group is stationary.*

Next, the problem as to the existence of invariant vector (covariant) can be solved by a method almost identical to that employed by Lie in obtaining his theorem;(2) so we will omit the proof, stating only the final result, i. e.

**Theorem 3.** *In an r-parameter group*

$$S_k = \xi_k^i \frac{\partial}{\partial x^i} \qquad (k = 1, \ldots, r), \tag{3}$$

*$S_1, \ldots, S_m$ being the unconnected operators and the other operators $S_{m+1}, \ldots, S_r$ being expressed linearly by $S_1, \ldots, S_m$,*

$$S_{m+j} = \sum_{\nu=1}^{m} \varphi_j^\nu S_\nu \qquad (j = 1, \ldots, r-m).$$

*For the existence of invariant vector (covariant) by the group, it is necessary and sufficient that the rank of the matrix*

$$\left\| \begin{matrix} \sum_{\nu=1}^{m} \dfrac{\partial \varphi_j^\nu}{\partial x^k} \xi_\nu^1, & \ldots\ldots\ldots\ldots, & \sum_{\nu=1}^{m} \dfrac{\partial \varphi_j^\nu}{\partial x^k} \xi_\nu^m \\ (j = 1, \ldots, r-m; & & k = 1, \ldots, n) \end{matrix} \right\|$$

*is less than $n$. If the matrix has the rank $l\,(l < n)$, there exist $n-l$ linearly independent invariant vectors (covariant).*

The idea of the transformation-derivative of vectors, can be extended to tensors. If we denote by $\Delta_k A^{\lambda_1, \ldots, \lambda_p}_{\mu_1, \ldots, \mu_q}$ the transformation-derivative of a tensor $A^{\lambda_1, \ldots, \lambda_p}_{\mu_1, \ldots, \mu_q}$ by the infinitesimal transformation $S_k$, we can accomplisch our purpose by the following definition:

(1) S. Lie, *loc. cit.*, 510.

(2) *Ibid.*, 376.

$$\Delta_k A^{\lambda_1, \dots, \lambda_p}_{\mu_1, \dots, \mu_q} = \xi^i_k \frac{\partial A^{\lambda_1, \dots, \lambda_p}_{\mu_1, \dots, \mu_q}}{\partial x^i}$$

$$- \frac{\partial \xi^\lambda_k}{\partial x^\alpha} A^{\alpha, \lambda_2, \dots, \lambda_p}_{\mu_1, \dots, \mu_q} - \dots - \frac{\partial \xi^\lambda_k}{\partial x^\alpha} A^{\lambda_1, \dots, \lambda_{p-1}, \alpha}_{\mu_1, \dots, \mu_q}$$

$$+ \frac{\partial \xi^\alpha_k}{\partial x^{\mu_1}} A^{\lambda_1, \dots, \lambda_p}_{\alpha, \mu_2, \dots, \mu_q} + \dots + \frac{\partial \xi^\alpha_k}{\partial x^{\mu_q}} A^{\lambda_1, \dots, \lambda_p}_{\mu_1, \dots, \mu_{q-1}, \alpha} .$$

And as in the case of vectors, when

$$\Delta_k A^{\lambda_1, \dots, \lambda_p}_{\mu_1, \dots, \mu_q} = 0 \qquad (k = 1, \dots, r),$$

we call such a tensor $A^{\lambda_1, \dots, \lambda_p}_{\mu_1, \dots, \mu_q}$ *an invariant tensor by the group* $S_1, \dots, S_r$.

If in a space admitting the given group $S_1, \dots, S_r$, the antisymmetric part of the coefficient of connection $\Gamma^\lambda_{\alpha\beta}$ is denoted by $\Omega^\lambda_{\alpha\beta}$, and the curvature tensor by $R^\lambda_{\alpha\beta\gamma}$, we can easily see that

$$\Delta_k \Omega^\lambda_{\alpha\beta} = 0 , \quad \Delta_k R^\lambda_{\alpha\beta\gamma} = 0 \qquad (k = 1, \dots, r) .$$

So we have

**Theorem 4.** *The antisymmetric part of the coffecients of connection and the curvature tensor of the space which admits a given group, are the invariant tensors by the group.*

From theorem 4 we have

**Theorem 5.** *The space which admits an* $n+1$*-parameter group transitive and non-stationary, is not other than an euclidean space.*

**Proof.** If the group is denoted by

$$S_k = \xi^i_k \frac{\partial}{\partial x^i} \qquad (k = 1, \dots, n+1),$$

then from the transitivity of the group, we can suppose $S_1, \dots, S_n$ to be the unconnected operators, while $S_{n+1}$ is expressed by

$$S_{n+1} = \sum_{\nu=1}^{n} \varphi^\nu S_\nu .$$

From this we have

$$\left.\begin{aligned} &\xi^i_{n+1} = \sum_{\nu=1}^{n} \varphi^\nu \xi^i_\nu , \\ &\frac{\partial \xi^i_{n+1}}{\partial x^j} = \sum_{\nu=1}^{n} \left\{ \frac{\partial \varphi^\nu}{\partial x^j} \xi^i_\nu + \varphi^\nu \frac{\partial \xi^i_\nu}{\partial x^j} \right\} \qquad (i, j = 1, \dots, n) . \end{aligned}\right\} \quad (14)$$

On the other hand we have, from theorem 4

$$\Delta_k \Omega^\lambda_{\alpha\beta} = 0 \qquad (k = 1, \ldots, n+1) \tag{15}$$

$$\Delta_k R^\lambda_{\alpha\beta\gamma} = 0 \qquad (k = 1, \ldots, n+1)\,. \tag{16}$$

And therefore by using (14), the equations (15) and (16) can be rewritten in the following forms respectively,

$$\left.\begin{aligned} &\Delta_h \Omega^\lambda_{\alpha\beta} = 0 \qquad (h = 1, \ldots, n)\,, \\ &\Omega^\lambda_{\alpha\omega}\frac{\partial\varphi^\nu}{\partial x^\beta}\xi^\omega_\nu + \Omega^\lambda_{\omega\beta}\frac{\partial\varphi^\nu}{\partial x^\alpha}\xi^\omega_\nu - \Omega^\omega_{\alpha\beta}\frac{\partial\varphi^\nu}{\partial x^\omega}\xi^\lambda_\nu = 0\,, \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} &\Delta_h R^\lambda_{\alpha\beta\gamma} = 0 \qquad (h = 1, \ldots, n)\,, \\ &R^\lambda_{\alpha\beta\omega}\frac{\partial\varphi^\nu}{\partial x^\gamma}\xi^\omega_\nu + R^\lambda_{\alpha\omega\gamma}\frac{\partial\varphi^\nu}{\partial x^\beta}\xi^\omega_\nu + R^\lambda_{\omega\beta\gamma}\frac{\partial\varphi^\nu}{\partial x^\alpha}\xi^\omega_\nu - R^\omega_{\alpha\beta\gamma}\frac{\partial\varphi^\nu}{\partial x^\omega}\xi^\lambda_\nu = 0\,. \end{aligned}\right\} \tag{18}$$

By assuming that the group is stationary, we see that there is no functional relation between $\varphi^1, \ldots, \varphi^n$, and therefore that the determinant $\left\|\frac{\partial\varphi^\nu}{\partial x^\mu}\right\|$ $(\nu, \mu = 1, \ldots, n)$ does not vanish identically. But since $\|\xi^\lambda_\nu\| \neq 0$ $(\lambda, \nu = 1, \ldots, n)$, we have

$$\left\|\frac{\partial\varphi^\nu}{\partial x^\mu}\xi^\lambda_\nu\right\| \neq 0 \qquad (\lambda, \mu = 1, \ldots, n)\,.$$

Hence at *any* point $x = x_0$, by suitably choosing the coordinate system, we can so normalize that

$$\left(\frac{\partial\varphi^\nu}{\partial x^\mu}\xi^\lambda_\nu\right)_{x=x_0} = \delta^\lambda_\mu \begin{cases} = 1 & \lambda = \mu\,, \\ = 0 & \lambda \neq \mu\,, \end{cases}$$

and substituting the above in (17) and (18) we have, at the point $x = x_0$,

$$\Omega^\lambda_{\alpha\beta} = 0\,, \qquad R^\lambda_{\alpha\beta\gamma} = 0\,.$$

But since this point may be any point whatever, the above relations hold good at every point. So the space is an euclidean space. Q.E.D.

Now we will proceed to investigate the relations between the covariant derivative $\Delta_i v^\lambda$ and the transformation-derivative $\Delta_k v^\lambda$ of a vector $\nu^\lambda(x)$. We have seen that the space defined by $T_1, \ldots, T_n$, (see (2)) admits the group $S_1, \ldots, S_r$ when, and only when,

$$(\dot{S}_k T_i) = -\frac{\partial \xi^l_k}{\partial x^i} T_l \quad (k = 1, \ldots, r;\ i = 1, \ldots, n). \tag{6}$$

If we apply the operators on both sides of this relation to an arbitrary system of equations of the form

$$f^\lambda \equiv \dot{x}^\lambda - v^\lambda(x) = 0 \qquad (\lambda = 1, \ldots, n)$$

and substituting $\dot{x}^\lambda = v^\lambda(x)$ $(\lambda = 1, \ldots, n)$ in the results, we have

$$\nabla_i \Delta_k v^\lambda - \Delta_k \nabla_i v^\lambda = 0 \quad (k = 1, \ldots, r;\ i = 1, \ldots, n). \tag{19}$$

Conversely from (19) we can easily deduce the relations (6). So we have

**Theorem 6.** *A space admits a given transformation group when and only when for an arbitary vector-field the covariant derivative and the transformation-derivative are interchangeable with each other.*

In the special case when $\Delta_i v_\lambda = 0$, we have from (19)

$$\nabla_i \Delta_k v^\lambda = 0 .$$

So we have

**Corollary 1.** *In the space which admits a given group, the transformation-derivative of a vector defining a parallel vector-field, gives also a parallel vector-field.*

When $\Delta_k v^\lambda = 0$ we have from (19)

$$\Delta_k \nabla_i v^\lambda = 0 .$$

So we have

**Corollary 2.** *In the space which admits a given group, the covariant derivative of an invariant vector (contravariant) by the group is an invariant tensor by the group.*

When $\xi^\lambda(x)$ is an arbitary invariant vector (contravariant) by the group $S_1, \ldots, S_r$, we have from (6)

$$(\dot{S}_k \zeta^i T_i) = 0 \qquad (k = 1, \ldots, r).$$

If we apply the operators on both sides of the above to an arbitrary system of equations of the form

$$f^\lambda \equiv \dot{x}^\lambda - v^\lambda(x) = 0$$

and substituting $\dot{x}^\lambda = v^\lambda(x)$ $(\lambda = 1, \ldots, n)$ in the results, we have

$$\zeta^i \nabla_i (\Delta_k v^\lambda) - \Delta_k (\zeta^i \Delta_i v^\lambda) = 0 \qquad (k = 1, \ldots, r) \tag{20}$$

So we have

**Theorem 7.** *In a space which admits a given group, for an arbitrary vector (contravariant) the parallel displacament in the direction of an invariant vector and the transformation-derivative are interchangeable with each other.*

In the special case when $\Delta_k v^\lambda = 0$ $(k = 1, \ldots, r)$, we have from (20)

$$\Delta_k \zeta^i \nabla_i v^\lambda = 0 \qquad (k = 1, \ldots, r).$$

So we have

**Corollary 1.** *In a space which admits a given group, if $v^\lambda(x)$ and $\zeta^\lambda(x)$ give any invariant vector-fields $\zeta^i \Delta_i v^\lambda$ also gives an invariant vector-field.*

Lastly, we consider the case in which a vector is displaced in parallel along a sub-space determined by a complete system which admits the given group $S_1, \ldots, S_r$. Let the complete system be

$$Y_h f \equiv \eta_h^i(x) \frac{\partial f}{\partial x^i} = 0 \qquad (h = 1, \ldots, q), \tag{21}$$

then we have

$$(S_k Y_h) = \sum_{\nu=1}^{q} \rho_{kh}^\nu Y_\nu \quad (k = 1, \ldots, r;\ h = 1, \ldots, q). \tag{22}$$

where $\rho_{kh}^\nu$ are certain functions of $x$.

Hence from (6) and (22), we have

$$(\dot{S}_k \eta_h^i T_i) = \sum_{\nu=1}^{q} \rho_{kh}^\nu \eta_\nu^i T_i \quad (k = 1, \ldots, r;\ h = 1, \ldots, q).$$

If we apply the operators on both sides of the above, to an arbitrary system of equations of the form

$$f^\lambda \equiv \dot{x}^\lambda - v^\lambda(x) = 0 \qquad (\lambda = 1, \ldots, n)$$

and substituting $\dot{x}^\lambda = v^\lambda(x)$ $(\lambda = 1, \ldots, n)$ in the results, we have

$$\Delta_k(\eta_h^i \nabla_i v^\lambda) - \eta_h^i \nabla_i (\Delta_k v^\lambda) = \sum_{\nu=1}^{q} \rho_k^\nu (\eta_\nu^i \nabla_i v^\lambda)$$
$$(k = 1, \ldots, r;\ h = 1, \ldots, q).$$

more especially, when a vector $v^\lambda(x)$ gives a parallel vector-field along the sub-space determined by the complete system (21) namely $\xi^i_h \Delta_i v^\lambda = 0$, we have from the above relations

$$\eta^i_h \nabla_i (\Delta_k v^\lambda) = 0 \qquad (k = 1, \ldots, r;\ h = 1, \ldots, q).$$

So we have

**Theorem** 8. *In the space which admits a given group, the transformation-derivative of a vector defining a parallel vector-field along a sub-space determined by a complete system which admits the group, also gives a parallel vector-field along the same sub-space.*

## III.

In this section we will consider the case where the operators of our group

$$S_k = \xi^i_k \frac{\partial}{\partial x^i} \qquad (k = 1, \ldots, r), \tag{23}$$

are all unconnected.

By Lie's theorem,[1] we know that there exist $n$ linearly independent invariant vector-fields (contravariant) by the group, which we denote by

$$\zeta^\lambda_{(\varepsilon)} \qquad (\epsilon = 1, \ldots, n).$$

When a space admits the group $S_1, \ldots, S_r$, we know by corollary 2 of theorem 6, that $\nabla_i \zeta^\lambda_{(\varepsilon)}$ $(\epsilon = 1, \ldots, n)$ are $n$ invariant tensors by the group. Conversely in a certain space, if $\nabla_i \zeta^\lambda_{(\varepsilon)}$ $(\epsilon = 1, \ldots, n)$ are $n$ invariant tensors by the group when $\zeta^\lambda_{(\varepsilon)}$ $(\epsilon = 1, \ldots, n)$ are $n$ invariant vectors (contravariant), the space admits the group. For, putting

$$\nabla_i \zeta^\lambda_{(\varepsilon)} = \frac{\partial \zeta^\lambda_{(\varepsilon)}}{\partial x_i} + \Gamma^\lambda_{\alpha i} \zeta^\alpha_{(\varepsilon)} = A^\lambda_{(\varepsilon) i} \qquad (\epsilon = 1, \ldots, n)$$

and solving for $\Gamma^\lambda_{\alpha\beta}$, we have

$$\Gamma^\lambda_{\alpha\beta} = -\bar{\zeta}^{(\varepsilon)}_\alpha \frac{\partial \zeta^\lambda_{(\varepsilon)}}{\partial x^\beta} + \bar{\zeta}^{(\varepsilon)}_\alpha A^\lambda_{(\varepsilon)\beta}, \tag{24}$$

where $\bar{\zeta}^{(\varepsilon)}_\alpha$ are related by the equations

$$\bar{\zeta}^{(\varepsilon)}_\alpha \zeta^\lambda_{(\varepsilon)} = \delta^\lambda_\alpha.$$

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(1) S. Lie, *loc. cit.*, 376.

And if we subtitute the $\Gamma^{\lambda}_{\alpha\beta}$ in (7) which are the equations of condition that the space admits the group, and take account of the invariancy of tensor $A^{\lambda}_{(\varepsilon)i}$ and vector $\zeta^{\lambda}_{(\varepsilon)}$, we easily see that (7) is satisfied identically. So we have

**Theorem 9.** *When the operators of a given group are all unconnected with each other, the coefficients of connection $\Gamma^{\lambda}_{\alpha\beta}$ of the space which admits the group, are given by the equations*

$$\Gamma^{\lambda}_{\alpha\beta} = -\bar{\zeta}^{(\varepsilon)}_{\alpha}\frac{\partial\zeta^{\lambda}_{(\varepsilon)}}{\partial x^{\beta}} + \bar{\zeta}^{(\varepsilon)}_{\alpha}A^{\lambda}_{(\varepsilon)\beta}, \tag{24}$$

*where $A^{\lambda}_{(\varepsilon)\beta}$ and $\zeta^{\lambda}_{(\varepsilon)}$ are $n$ arbitrary invariant tensors and linearly independent invariant vectors by the group respectively.*

In particular, when $A^{\lambda}_{(\varepsilon)\mu} = 0$ $(\epsilon = 1, \ldots, p)$, the $p$ invariant vectors $\zeta^{\lambda}_{(\varepsilon)}$ $(\epsilon = 1, \ldots, p)$ give $p$ parallel vector-fields in the space defined by (24).

In general when $A^{\lambda}_{(\varepsilon)\mu}$ are $n$ arbitrary invariant tensors by the group, we shall find all the parallel vector-fields in the space defined by (24).

Lemma. When an invariant tensor $A^{\lambda}_{\mu}$ by the group $S_1, \ldots, S_r$ is given, it can always be expressed in the form

$$A^{\lambda}_{\mu} = w^{(\varepsilon)}_{\mu}\zeta^{\lambda}_{(\varepsilon)}, \tag{25}$$

where $w^{(\varepsilon)}_{\mu}$ $(\epsilon = 1, \ldots, n)$ are certain invariant vectors (covariant).

Proof. Since $\zeta^{\lambda}_{(\varepsilon)}$ $(\epsilon = 1, \ldots, n)$ are $n$ linearly independent vectors, $A^{\lambda}_{\mu}$ can be expressed linearly by $\zeta^{\lambda}_{(1)}, \ldots, \zeta^{\lambda}_{(n)}$, namely

$$A^{\lambda}_{\mu} = w^{(\varepsilon)}_{\mu}\zeta^{\lambda}_{(\varepsilon)}$$

Then $w^{(\varepsilon)}_{\mu}$ $(\epsilon = 1, \ldots, n)$ may be proved to be an invariant vector (covariant); for, in the equation of invariancy of $A^{\lambda}_{\mu}$:

$$\Delta_k A^{\lambda}_{\mu} = 0 \qquad (k = 1, \ldots, r).$$

Substituting (25), we have

$$\zeta^{\lambda}_{(\varepsilon)}\Delta_k w^{(\varepsilon)}_{\mu} + w^{(\varepsilon)}_{\mu}\Delta_k\zeta^{\lambda}_{(\mu)} = 0,$$

but since $\Delta_k\zeta^{\lambda}_{(\varepsilon)} = 0 (\varepsilon = 1, \ldots, n)$, we have

$$\zeta^{\lambda}_{(\varepsilon)}\Delta_k w^{(\varepsilon)}_{\mu} = 0 \qquad (k = 1, \ldots, r)$$

And from the independency of the $n$ vectors $\zeta^{\lambda}_{(\varepsilon)}(=1, \ldots, n)$, we have

$$\Delta_k w^{(\varepsilon)}_{\mu} = 0 \qquad (k = 1, \ldots, r;\ \epsilon = 1, \ldots, n).$$

This means that $w^{(\varepsilon)}_{\mu}(\varepsilon = 1, \ldots, n)$ are invariant vectors (covariant).

Q. E. D.

Conversely, if $w^{(\varepsilon)}_{\mu}(\varepsilon = 1, \ldots, n)$ and $\zeta^{\lambda}_{(\varepsilon)}(\varepsilon = 1, \ldots, n)$ are any invariant vectors (covariant and contravariant), then $w^{(\varepsilon)}_{\mu}\zeta^{\lambda}_{(\varepsilon)}$ is also an invariant tensor.

Therefore, if we substitute in (24) the relation

$$A^{\lambda}_{(\varepsilon)\beta} = w^{(\omega)}_{(\varepsilon)\beta}\zeta^{\lambda}_{(\omega)},$$

we have

$$\Gamma'^{\lambda}_{\alpha\beta} = -\bar{\zeta}^{(\varepsilon)}_{\alpha}\frac{\partial\zeta^{\lambda}_{(\varepsilon)}}{\partial x^{\beta}} + \bar{\zeta}^{(\varepsilon)}_{\alpha} w^{(\omega)}_{(\varepsilon)\beta}\zeta^{\lambda}_{(\omega)}. \tag{26}$$

This may be regarded as the general form of coefficients of connection of the space which admits the group.

Thus the problem of finding all the parallel vector-fields, is reduced to that of finding such vector-fields in the space defined by (26) instead of (24).

Let $v^{\lambda}(x)$ give a parallel vector-field in the space defined by (26), and express it linearly by $\zeta^{\lambda}_{(1)}, \ldots, \zeta^{\lambda}_{(n)}$, namely

$$v^{\lambda} = a^{\varepsilon}\zeta^{\lambda}_{(\varepsilon)}.$$

Then substituting $\Gamma'^{\lambda}_{\alpha\beta}$ given by (26), into $\nabla_i v^{\lambda} = 0$, we have

$$\zeta^{\lambda}_{(\varepsilon)}\left\{\frac{\partial a^{\varepsilon}}{\partial x^i} + a^{\omega} w^{(\varepsilon)i}_{(\omega)}\right\} = 0 \qquad (i = 1, \ldots, n).$$

Since $n$ vectors $\zeta^{\lambda}_{(1)}, \ldots, \zeta^{\lambda}_{(n)}$ are linearly independent, it must be that

$$\frac{\partial a^{\varepsilon}}{\partial x^i} + w^{(\varepsilon)}_{(\omega)i} a^{\omega} = 0 \qquad (i = 1, \ldots, n).$$

So we know that $a^{\varepsilon}$ gives a parallel vector-field in the space whose coefficients of connection are $w^{(\varepsilon)}_{(\omega)i}$. Conversely if $a^{\varepsilon}$ gives such a parallel vector-field, the relations $\nabla_i(a^{\varepsilon}\zeta^{\lambda}_{(\varepsilon)})=0$ $(i = 1, \ldots, n)$ hold for $\Gamma'^{\lambda}_{\alpha\beta}$, defined by (26). So we have

**Theorem 10.** *The coefficients of connection $\Gamma^{\lambda}_{\alpha\beta}$ in a space which admits the graup with all unconnected operators, are given by* (26). *Moreover there exists a parallel vector-field in this space if, and only if,* *a parallel vector-field exists in the space whose coefficients of connection are* $w^{(\varepsilon)}_{(\omega)i}$.

*If there exist p linearly independent parallel vector-fields in the latter space, say* $a^{\lambda}_{(1)}, \ldots, a^{\lambda}_{(p)}$, *then in the former space p linearly independent parallel vector-fields are obtained in the forms*

$$a^{\varepsilon}_{1}\zeta^{\lambda}_{(\varepsilon)}, \ldots\ldots, a^{\varepsilon}_{(p)}\zeta^{\lambda}_{(\varepsilon)} .$$

Lastly, we shall find all the parallel vector-fields along a sub-space determined by a complete system

$$\eta^{i}_{h}(x)\frac{\partial f}{\partial x^{i}} = 0 \qquad (h = 1, \ldots\ldots, q) .$$

Let $v^{\lambda}(x)$ give a parallel vector-field along the sub-space, and let it be expressed linearly by $\zeta^{\lambda}_{(1)}, \ldots, \zeta^{\lambda}_{(n)}$, namely

$$v^{\lambda} = a^{\varepsilon}\zeta^{\lambda}_{(\varepsilon)} .$$

Then substituting $\Gamma^{\lambda}_{\alpha\beta}$ given by (26) into

$$\zeta^{i}_{h}\nabla_{i}(a^{\varepsilon}\zeta^{\lambda}_{(\varepsilon)}) = 0$$

and taking account of the fact that the $n$ vectors $\zeta^{\lambda}_{(1)}, \ldots, \zeta^{\lambda}_{(n)}$ are linearly independent, we have

$$\eta^{i}_{h}\left\{\frac{\partial a^{\varepsilon}}{\partial x^{i}} + w^{(\varepsilon)}_{(\omega)i}a^{\omega}\right\} = 0 .$$

Hence by the same procedure as was used in obtaining theorem 10, we have

**Theorem 11.** *In the space defined by (26) which admits a given group with all unconnected operators, there exists a parallel vector-field along a sub-space determined by a complete system, if and only if in the space whose coefficients of connection are* $w^{(\varepsilon)}_{(\omega)i}$, *there exists a parallel vector-field along the sub-space determined by the same complete system.*

*If in the latter space there exist p linearly independent parallel vector-fields, say* $a^{\lambda}_{(1)}, \ldots, a^{\lambda}_{(p)}$, *then in the former space p linearly independent parallel vector-fields are obtained in the forms*

$$a^{\varepsilon}_{(1)}\zeta^{\lambda}_{(\varepsilon)}, \ldots\ldots, a^{\varepsilon}_{(p)}\zeta^{\lambda}_{(\varepsilon)} .$$

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