Theory of Vector Valued Set Functions.

By

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Let $\mathfrak{H}$ be an abstract Hilbert space,[1] and $\{\mathfrak{f}_\nu\}$ be a sequence of elements of $\mathfrak{H}$. If there exists an element $\mathfrak{f}$ in $\mathfrak{H}$ such that

$$\lim_{\nu\to\infty} \|\mathfrak{f}_\nu - \mathfrak{f}\| = 0,$$

then I say that $\{\mathfrak{f}_\nu\}$ converges strongly to $\mathfrak{f}$, and I write thus

$$[\lim_{\nu\to\infty}]\,\mathfrak{f}_\nu = \mathfrak{f}.$$

If a series of elements

$$a_1\mathfrak{f}_1 + a_2\mathfrak{f}_2 + \ldots\ldots + a_\nu\mathfrak{f}_\nu + \ldots\ldots \tag{1}$$

be such that $$[\lim_{\nu\to\infty}]\,\mathfrak{s}_\nu = \mathfrak{f}$$

where $$\mathfrak{s}_\nu = a_1\mathfrak{f}_1 + a_2\mathfrak{f}_2 + \ldots\ldots + a_\nu\mathfrak{f}_\nu,$$

then I say that the series (1) converges strongly to $\mathfrak{f}$, and I write as follows

$$\mathfrak{f}\,[=]\sum_\nu a_\nu\mathfrak{f}_\nu.$$

In the strongly convergent series, the most important is the expansion of any element $\mathfrak{f}$ with respect to a complete normalized orthogonal system $\{\mathfrak{g}_\nu\}$ in $\mathfrak{H}$:

$$\mathfrak{f}\,[=]\sum_\nu (\mathfrak{f}, \mathfrak{g}_\nu)\,\mathfrak{g}_\nu. \tag{2}$$

(1) For the abstract Hilbert space, cf. J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik*, (1932); and M. H. Stone, *Linear Transformations in Hilbert Space*, (1932).

Now, the first problem that arises, is: can we not extend the expression (2) of $\mathfrak{f}$ into an integral expression?

Let H be a self-adjoint transformation. If, moreover, H is completely continuous, we can expand it as a series of projecting transformations:

$$H\mathfrak{f}\,[=]\sum_{\nu}\lambda_{\nu}\mathbf{E}_{\nu}\mathfrak{f}\,, \tag{3}$$

where $\frac{1}{\lambda_\nu}$ is a characteristic constant of H, and $\mathbf{E}_\nu$ is the projection on the linear manifold determined by the characteristic elements of H corresponding to $\frac{1}{\lambda_\nu}$.[1]

Now there is the second problem: can we not extend the expression (3) into an integral expression?

In my previous paper,[2] I extended the normalized orthogonal system $\{\psi_\nu(E)\}$ in the space of set functions, which has positive integer ν as parameter, to the normalized orthogonal system $\{\Psi_{(U)}(E)\}$ which has set U as parameter. And I showed that the expansions of set functions with repect to $\{\Psi_{(U)}(E)\}$ are integral expressions. Next, I defined the resolution of identity $\mathbf{E}(U)$ as a function of set U, and proved that a bounded self-adjoint transformation H is expressed as follows:

$$H\phi(E) = \int_I \lambda d_U \mathbf{E}(U)\,\phi(E)\,.$$

If we put
$$\mathbf{E}(U)\,\phi(E) = \Psi_{(U)}(E),$$

then $\{\Psi_{(U)}(E)\}$ is a normalized orthogonal system such as above cited.

Thus I have shown that in the space of set functions, the two problems are closely connected. In this paper, I consider these problems with respect to the abstract Hilbert space $\mathfrak{H}$. I first extend the normalized orthogonal system $\{\mathfrak{g}_\nu\}$ which has positive integer ν as parameter, to the normalized orthogonal system $\{\mathfrak{q}(U)\}$ which has set U as parameter. U may be a set in any metric space. Since $\mathfrak{q}(U)$ is a

(1) I have proved this theorem for the space of set functions, in my paper, "On the Space of Completely Continuous Transformations," this journal, **3** (1933), 156; but this property holds also in the abstract Hilbert space.

(2) "On Kernels and Spectra of Bounded Linear Transformations," *ibid.*, 243-273.

function of set U, whose functional value is an element of $\mathfrak{H}$, which we may describe as a vector, I call $\mathfrak{q}(U)$ a vector valued set function. I also define the integral of a point function with respect to this vector valued set function. Then I solve the first problem, so that when $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, any element $\mathfrak{f}$ can be expressed in the integral form:

$$\mathfrak{f} = \int_V D_{\sigma(U)}\,\xi(\lambda)\,d\mathfrak{q}(U)\,,$$

where $\xi(U) = (\mathfrak{f},\, \mathfrak{q}(U))$ and $\sigma(U) = \|\mathfrak{q}(U)\|^2$.

I also have additional theorems concerning the extended normalized orthogonal systems.

Next, I define the resolution of identity $E(U)$ as a function of set U in a metric space. Then $E(U)\mathfrak{f}$ is a vector valued set function. I prove a number of theorems respecting the properties of the resolution of identity.

Lastly, I consider the linear transformation expressed in the integral form

$$T\mathfrak{f} = \int_V f(\lambda)\,d\,E(U)\,\mathfrak{f}\,,$$

and I give brief discussions of the possiblity of integral expressions of self-adjoint tronsformations and others. Thus the second problem is solved, and any self-adjoint transformation H can be expressed as follows:

$$H\mathfrak{f} = \int_{R_1} \lambda d\,E(U)\,\mathfrak{f} \tag{4}$$

when $\int_{R_1} \lambda^2 d\,\|E(U)\mathfrak{f}\|^2$ is finite, where $E(U)$ is a resolution of identity defined in the Euclidean space R_1 of one dimension.

Ordinarily, it is proved that any self-adjoint transformation H can be expressed as follows:

$$(H\mathfrak{f},\, \mathfrak{g}) = \int_{-\infty}^{+\infty} \lambda d(E(\lambda)\,\mathfrak{f},\, \mathfrak{g}) \tag{5}$$

for any element $\mathfrak{g}$ in $\mathfrak{H}$, when $\int_{-\infty}^{+\infty} \lambda^2 d\,\|E(\lambda)\,\mathfrak{f}\|^2$ is finite.(1)

(1) Cf. Stone, *loc. cit.*, 180.

The expressions (4) and (5) are different in significance. Expression (5) has reference to weak convergence,[1] although the integral of (4) is defined by strong convergence.

Integrals with respect to a Vector Valued Set Function.

1. Let V be a Borel set in a metric space S which is half compact.[2] If for each Borel subset U of V, a vector $\mathfrak{q}(U)$ in $\mathfrak{H}$ be determined, then $\mathfrak{q}(U)$ may be called a *vector valued set function*. And $\mathfrak{q}(U)$ is said to be *completely additive*, if

$$(\mathfrak{q}(U),\ \mathfrak{q}(U')) = 0 \tag{1}$$

when $UU' = 0$, and

$$\mathfrak{q}(U)\ [=]\ \mathfrak{q}(U_1) + \mathfrak{q}(U_2) + \dots + \mathfrak{q}(U_n) + \dots \tag{2}$$

when $\qquad U = U_1 + U_2 + \dots + U_n + \dots\ .$

Since by (1), $\qquad (\mathfrak{q}(U_i),\ \mathfrak{q}(U_j)) = 0$

when $i \neq j$, we have, from (2),

$$\| \mathfrak{q}(U) \|^2 = \| \mathfrak{q}(U_1) \|^2 + \| \mathfrak{q}(U_2) \|^2 + \dots + \| \mathfrak{q}(U_n) \|^2 + \dots\ .$$

Hence, if we put $\qquad \| \mathfrak{q}(U) \|^2 = \sigma(U)\ ,$

then, $\sigma(U)$ is a completely additive set function defined for all Borel subsets of V.[3] And

$$\begin{aligned}(\mathfrak{q}(U),\ \mathfrak{q}(U')) &= (\mathfrak{q}(UU') + \mathfrak{q}(U - UU'),\ \mathfrak{q}(UU') + \mathfrak{q}(U' - UU')) \\ &= (\mathfrak{q}(UU'),\ \mathfrak{q}(UU'))\ ,\end{aligned}$$

(1) $\{\mathfrak{f}_\nu\}$ is said to converge weakly to $\mathfrak{f}$ when $\lim_{\nu\to\infty} (\mathfrak{f}_\nu,\ \mathfrak{g}) = (\mathfrak{f},\ \mathfrak{g})$ for any element $\mathfrak{g}$ in $\mathfrak{H}$.

(2) A metric space is said to be half compact, when it is the sum of an enumerably infinite number of point sets which are compact in themselves. The Euclidean space of n dimensions is half compact. (Cf. H. Hahn, *Reelle Funktionen* I, *Punktfunktionen*, (1932), 95.)

(3) In what follows, the derivatives with respect to $\sigma(U)$ being treated when S is not an Euclidean space, I shall consider the case where $\sigma(U)$ is uniformly monotone almost everywhere (σ) in V. But when S is an Euclidean space, such a restriction is superfluous. (Cf. F. Maeda, this journal, **1** (1931), 3; and **2** (1932), 33.)

that is, $$(\mathfrak{q}(U),\ \mathfrak{q}(U')) = \sigma(UU')\,.$$

I will call $\sigma(U)$ the *base* of $\mathfrak{q}(U)$.

Let $\boldsymbol{U}$ be an unitary transformation. Then, since

$$(\boldsymbol{U}\mathfrak{q}(U),\ \boldsymbol{U}\mathfrak{q}(U')) = (\mathfrak{q}(U),\ \mathfrak{q}(U'))\,,$$

we have, from (1),

$$(\boldsymbol{U}\mathfrak{q}(U),\ \boldsymbol{U}\mathfrak{q}(U')) = 0 \qquad \text{when} \quad UU' = 0,$$

and $$\|\,\boldsymbol{U}\mathfrak{q}(U)\,\|^2 = \|\,\mathfrak{q}(U)\,\|^2\,.$$

But, from (2), it is evident that

$$\boldsymbol{U}\mathfrak{q}(U)\,[=]\,\boldsymbol{U}\mathfrak{q}(U_1) + \boldsymbol{U}\mathfrak{q}(U_2) + \dots + \boldsymbol{U}\mathfrak{q}(U_n) + \dots$$

Hence $\boldsymbol{U}\mathfrak{q}(U)$ *is also a completely additive vector valued set function with base* $\sigma(U)$.

It will be convenient to agree upon the notation for a vector valued set function and its base once and for all. I will denote a vector valued set function by $\mathfrak{q}(U)$ and its base by $\sigma(U)$, if several vector valued set functions denoted by the same letter and distinguished by subscripts or other suitable marks are to be considered simultaneously, the letters denoting the corresponding bases will be distinguished by affixing corresponding subscripts or marks. For example,

$$\|\,\mathfrak{q}_i(U)\,\|^2 = \sigma_i(U), \qquad \|\,\mathfrak{q}^{(\nu)}(U)\,\|^2 = \sigma^{(\nu)}(U)\,.$$

2. Let $f(\lambda)$ be a complex valued point function defined in a Borel set V in S, and $g(\lambda)$ and $h(\lambda)$ be its real and imaginary parts respectively that is

$$f(\lambda) = g(\lambda) + ih(\lambda)\,.$$

When $g(\lambda)$ and $h(\lambda)$ are Baire's functions, I will call $f(\lambda)$ a Baire's function.

First assume that Baire's function $f(\lambda)$ is bounded in V, that is, there exists a positive number M such that

$$|\,f(\lambda)\,| < M$$

for all points λ in V. Divide the interval $(-M, M)$ by the insertion of the intermediate points, such that

$$-M = l_1 < l_2 < \ldots . < l_{n+1} = M$$

and $$l_{p+1} - l_p \leqq \varepsilon \qquad (p = 1, 2, \ldots , n),$$

where ε is a given positive number.

Let V_{pq} be the subset of V, for all points of which

$$l_p \leqq g(\lambda) < l_{p+1}, \qquad l_q \leqq h(\lambda) < l_{q+1},$$

where $p, q = 1, 2, \ldots , n$. Then V_{pq} are Borel sets. If λ_{pq} is a point in V_{pq}, then

$$\sum_{p,q} f(\lambda_{pq})\, \mathfrak{q}(V_{pq}) \tag{1}$$

is a vector in $\mathfrak{H}$.

Let $\{\varepsilon_\nu\}$ be a sequence of positive numbers, which converges to zero. Denote the vector (1), when $\varepsilon = \varepsilon_\nu$, by $\mathfrak{f}^{(\nu)}$. Then

$$\| \mathfrak{f}^{(\mu)} - \mathfrak{f}^{(\nu)} \| \leqq \sqrt{2}\,(\varepsilon_\mu + \varepsilon_\nu) \{\sigma(V)\}^{\frac{1}{2}}.$$ [1]

Hence, $\mathfrak{H}$ being complete, $\{\mathfrak{f}^{(\nu)}\}$ converges strongly to a vector in $\mathfrak{H}$, say $\mathfrak{f}$. Since the vector so defined is independent of the particular mode in which the interval $(-M, M)$ has been successively sub-divided, I say that $\mathfrak{f}$ is *the integral of* $f(\lambda)$ *with respect to* $\mathfrak{q}(U)$ *over* V, and I write

$$\mathfrak{f} = \int_V f(\lambda)\, d\mathfrak{q}(U).$$

If we divide V into the sum of Borel sets

$$V = V_1 + V_2 + \ldots . + V_i + \ldots . ,$$

such that $$|f(\lambda) - f_i| < \varepsilon \qquad \text{for all } \lambda \text{ in } V_i,$$

f_i being complex numbers $(i = 1, 2, \ldots .)$; then, it is evident that

$$\int_V f(\lambda)\, d\mathfrak{q}(U) = \underset{\varepsilon \to 0}{[\lim]} \sum_i f_i\, \mathfrak{q}(V_i).$$ [2]

(1) Since $|f(\lambda) - f(\lambda')| < \sqrt{2}\varepsilon$ for any two points λ, λ' in V_{pq}.

(2) Since $f(\lambda)$ is bounded in V, $\sum_i f_i\, \mathfrak{q}(V_i)$ is a strongly convergent series.

3. Let $f_1(\lambda)$ and $f_2(\lambda)$ be two bounded Baire's functions defined in V, and write

$$\mathfrak{f}_1 = \int_V f_1(\lambda)\, d\mathfrak{q}(U)\,, \qquad \mathfrak{f}_2 = \int_V f_2(\lambda)\, d\mathfrak{q}(U)\,.$$

Define $V_{pq}^{(1)}$ and $V_{rs}^{(2)}$ for $f_1(\lambda)$ and $f_2(\lambda)$ respectively as the preceding section. Then, since

$$\mathfrak{f}_1 = [\lim_{\varepsilon\to 0}] \sum_{p,q}\sum_{r,s} f_1(\lambda_{pqrs})\, \mathfrak{q}(V_{pq}^{(1)} V_{rs}^{(2)})\,,$$

$$\mathfrak{f}_2 = [\lim_{\varepsilon\to 0}] \sum_{p,q}\sum_{r,s} f_2(\lambda_{pqrs})\, \mathfrak{q}(V_{pq}^{(1)} V_{rs}^{(2)})\,,$$

where λ_{pqrs} is a point in $V_{pq}^{(1)} V_{rs}^{(2)}$, we have

$$(\mathfrak{f}_1, \mathfrak{f}_2) = \lim_{\varepsilon\to 0} \sum_{p,q}\sum_{r,s} f_1(\lambda_{pqrs})\, \overline{f_2(\lambda_{pqrs})}\, (\mathfrak{q}(V_{pq}^{(1)} V_{rs}^{(2)}),\ \mathfrak{q}(V_{pq}^{(1)} V_{rs}^{(2)}))\,.$$

But $$(\mathfrak{q}(V_{pq}^{(1)} V_{rs}^{(2)}),\ \mathfrak{q}(V_{pq}^{(1)} V_{rs}^{(2)})) = \sigma(V_{pq}^{(1)} V_{rs}^{(2)})\,;$$

hence $$(\mathfrak{f}_1, \mathfrak{f}_2) = \int_V f_1(\lambda)\overline{f_2(\lambda)}\, d\sigma(U)\,.$$

Especially, $$\|\mathfrak{f}\|^2 = \int_V |f(\lambda)|^2\, d\sigma(U)\,.$$

Similarly, we can easily prove that for any vector $\mathfrak{g}$

$$(\mathfrak{f}, \mathfrak{g}) = \int_V f(\lambda)\, d(\mathfrak{q}(U), \mathfrak{g})\,,$$

where $(\mathfrak{q}(U), \mathfrak{g})$ is a completely additive, complex valued set function defined for all Borel sets in V.(1) And, since, by sec. 1, $U\mathfrak{q}(U)$ is also a completely additive vector valued set function, we can easily prove that

$$U\mathfrak{f} = \int_V f(\lambda)\, dU\mathfrak{q}(U)\,.$$

(1) We will define the integral of a complex valued point function with respect to a complex valued set function as follows:

$$\int_V \{g(\lambda)+ih(\lambda)\}\, d\,\{\phi(U)+i\psi(U)\} = \int_V g(\lambda)d\phi(U) - \int_V h(\lambda)d\psi(U) + i\Big\{\int_V g(\lambda)\,d\psi(U) + \int_V h(\lambda)\,d\phi(U)\Big\}$$

where g, h, ϕ, ψ are real valued functions.

4. *If $f_1(\lambda)$ and $f_2(\lambda)$ are two bounded Baire's functions, defined in V, then*

$$\int_V \left\{f_1(\lambda)+f_2(\lambda)\right\} d\mathfrak{q}(U) = \int_V f_1(\lambda)\, d\mathfrak{q}(U) + \int_V f_2(\lambda)\, d\mathfrak{q}(U)\,.$$

Define $V_{pq}^{(1)}$ and $V_{rs}^{(2)}$ for $f_1(\lambda)$ and $f_2(\lambda)$ respectively as sec. 2. Then

$$\int_V f_1(\lambda)\, d\mathfrak{q}(U) + \int_V f_2(\lambda) d\mathfrak{q}(U) = [\lim_{\varepsilon\to 0}] \sum_{p,q}\sum_{r,s} \left\{ f_1(\lambda_{pq}^{(1)}) + f_2(\lambda_{rs}^{(2)})\right\} \mathfrak{q}(V_{pq}^{(1)}\, V_{rs}^{(2)}),$$

where $\lambda_{pq}^{(1)}$ and $\lambda_{rs}^{(2)}$ are points in $V_{pq}^{(1)}$ and $V_{rs}^{(2)}$ respectively. But, since

$$\left| f_1(\lambda) + f_2(\lambda) - \left\{ f_1(\lambda_{pq}^{(1)}) + f_2(\lambda_{rs}^{(2)}) \right\} \right| < 2\sqrt{2}\,\varepsilon$$

for all λ in $V_{pq}^{(1)}\, V_{rs}^{(2)}$, we have by sec. 2

$$\int_V \left\{f_1(\lambda)+f_2(\lambda)\right\} d\mathfrak{q}(U) = [\lim_{\varepsilon\to 0}] \sum_{p,q}\sum_{r,s} \left\{ f_1(\lambda_{pq}^{(1)}) + f_2(\lambda_{rs}^{(2)})\right\} \mathfrak{q}(V_{pq}^{(1)}\, V_{rs}^{(2)})\,.$$

Hence, we have (1).

5. When $$f(\lambda) = g(\lambda) + ih(\lambda)$$

is a non-bounded Baire's function defined in V, I will define a bounded Baire's function $f^N(\lambda)$ as follows:

$$f^N(\lambda) = g^N(\lambda) + ih^N(\lambda)\,,$$

where
$$\begin{aligned} g^N(\lambda) &= g(\lambda) && \text{when} \quad |g(\lambda)| \leqq N, \\ &= N && \text{when} \quad g(\lambda) > N, \\ &= -N && \text{when} \quad g(\lambda) < -N, \end{aligned}$$
and
$$\begin{aligned} h^N(\lambda) &= h(\lambda) && \text{when} \quad |h(\lambda)| \leqq N, \\ &= N && \text{when} \quad h(\lambda) > N, \\ &= -N && \text{when} \quad h(\lambda) < -N. \end{aligned}$$

If $\int_V f^N(\lambda) d\mathfrak{q}(U)$ converges strongly to a vector, say $\mathfrak{f}$, as N becomes infinite, then I say that *the integral of $f(\lambda)$ with respect to $\mathfrak{q}(U)$ over V exists*, and I write

$$\mathfrak{f} = \int_V f(\lambda)\, d\mathfrak{q}(U)\,.$$

In this case, of course, $\int_V |f(\lambda)|^2 d\sigma(U)$ exists, and is equal to $\|\mathfrak{f}\|^2$. Hence, $f(\lambda)$ belongs to the class $\mathfrak{L}_2(\sigma)$.(1)

Conversely, if $f(\lambda)$ belongs to $\mathfrak{L}_2(\sigma)$, then $\int_V f(\lambda)\, d\mathfrak{q}(U)$ exists. For, since

$$\lim_{\substack{M\to\infty \\ N\to\infty}} \int_V |f^M(\lambda)-f^N(\lambda)|^2 d\sigma(U) = 0 ,$$

and $$\left\| \int_V f^M(\lambda)\, d\mathfrak{q}(U) - \int_V f^N(\lambda)\, d\mathfrak{q}(U) \right\|^2 = \int_V |f^M(\lambda)-f^N(\lambda)|^2\, d\sigma(U),$$

$\int_V f^N(\lambda) d\mathfrak{q}(U)$ converges strongly in $\mathfrak{H}$ as N becomes infinite. Hence $\int_V f(\lambda)\, d\mathfrak{q}(U)$ exists.

Consequently, $\int_V f(\lambda) d\mathfrak{q}(U)$ *exists when and only when* $f(\lambda)$ *belongs to* $\mathfrak{L}_2(\sigma)$.(2)

Let $f_1(\lambda)$ *and* $f_2(\lambda)$ *be two Baire's functions belonging to* $\mathfrak{L}_2(\sigma)$, *and*

$$\mathfrak{f}_1 = \int_V f_1(\lambda)\, d\mathfrak{q}(U) , \qquad \mathfrak{f}_2 = \int_V f_2(\lambda)\, d\mathfrak{q}(U) ;$$

then $$(\mathfrak{f}_1 , \mathfrak{f}_2) = (f_1 , f_2) .$$

For, put $\mathfrak{f}_1^N = \int_V f_1^N(\lambda)\, d\mathfrak{q}(U)$, $\mathfrak{f}_2^N = \int_V f_2^N(\lambda)\, d\mathfrak{q}(U)$.

Then $$(\mathfrak{f}_1 , \mathfrak{f}_2) = \lim_{N\to\infty} (\mathfrak{f}_1^N , \mathfrak{f}_2^N)$$

by sec. 3, $$= \lim_{N\to\infty} \int_V f_1^N(\lambda)\overline{f_2^N(\lambda)}\, d\sigma(U)$$

$$= \int_V f_1(\lambda)\overline{f_2(\lambda)}\, d\sigma(U) = (f_1 , f_2) .$$

(1) When $\int_V |f(\lambda)|^2 d\sigma(U)$ exists, it is said that $f(\lambda)$ belongs to $\mathfrak{L}_2(\sigma)$. $\mathfrak{L}_2(\sigma)$ is a Hilbert space, where the inner product is defined as follows:

$$(f_1 , f_2) = \int_V f_1(\lambda)\overline{f_2(\lambda)} d\sigma(U).$$

(Cf. Stone, *loc. cit.*, 23.)

(2) When $f(\lambda)$ is a bounded Baire's function, $\int_V f(\lambda) d\mathfrak{q}(U)$ always exists, and $f(\lambda)$ belongs to $\mathfrak{L}_2(\sigma)$.

Similarly, we can easily extend the following theorems for bounded Baire's functions[1] to the case of any Baire's functions belonging to $\mathfrak{L}_2(\sigma)$.

When $$\mathfrak{f} = \int_V f(\lambda)\, d\mathfrak{q}(U),$$

then $$(\mathfrak{f}, \mathfrak{g}) = \int_V f(\lambda)\, d(\mathfrak{q}(U), \mathfrak{g})$$

for any vector $\mathfrak{g}$.

And *if* $\boldsymbol{U}$ *is a unitary transformation, then*

$$\boldsymbol{U}\mathfrak{f} = \int_V f(\lambda)\, d\boldsymbol{U}\mathfrak{q}(U).$$

6. If $f(\lambda)$ belongs to $\mathfrak{L}_2(\sigma)$, then $\int_V |f(\lambda)|^2 d\sigma(U)$ exists, and therefore $\int_U |f(\lambda)|^2 d\sigma(U)$ exists for any Borel subset U of V. Hence $\int_U f(\lambda) d\mathfrak{q}(U)$ exists, and

$$\int_U f(\lambda)\, d\mathfrak{q}(U) = \int_V f(\lambda)\, h_{(U)}(\lambda)\, d\mathfrak{q}(U),$$

where $h_{(U)}(\lambda)$ is de la Vallée Poussin's characteristic function of U, that is

$$h_{(U)}(\lambda) = 1 \qquad \text{when } \lambda \text{ is a point of } U,$$
$$= 0 \qquad \text{when } \lambda \text{ is not a point of } U.$$

Put $$\mathfrak{p}(U) = \int_U f(\lambda)\, d\mathfrak{q}(U),$$

then, from the preceding section,

$$(\mathfrak{p}(U), \mathfrak{p}(U')) = \int_V f(\lambda)\, h_{(U)}(\lambda) \overline{f(\lambda)\, h_{(U')}(\lambda)}\, d\sigma(U) = \int_{UU'} |f(\lambda)|^2\, d\sigma(U). \quad (1)$$

Especially, $$\|\mathfrak{p}(U)\|^2 = \int_U |f(\lambda)|^2 d\sigma(U). \quad (2)$$

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(1) Cf. sec. 3.

Let $U_1 + U_2 = U_3$, and put

$$f(\lambda)\, h_{(U_i)}(\lambda) = f_i(\lambda) \qquad (i = 1, 2, 3)\,.$$

And define $f^N(\lambda)$, $f_i^N(\lambda)$ $(i = 1, 2, 3)$ as sec. 5. Then

$$f_i^N(\lambda) = f^N(\lambda)\, h_{(U_i)}(\lambda) \qquad (i = 1, 2, 3)\,,$$

and

$$f_1^N(\lambda) + f_2^N(\lambda) = f_3^N(\lambda)\,.$$

Then by sec. 4 $\displaystyle\int_V f_1^N(\lambda)\, d\mathfrak{q}(U) + \int_V f_2^N(\lambda)\, d\mathfrak{q}(U) = \int_V f_3^N(\lambda)\, d\mathfrak{q}(U)$.

But, since $\displaystyle\mathop{[\lim]}_{N\to\infty} \int_V f_i^N(\lambda)\, d\mathfrak{q}(U) = \int_V f_i(\lambda)\, d\mathfrak{q}(U) = \int_{U_i} f(\lambda)\, d\mathfrak{q}(U)$

$$(i = 1, 2, 3)\,,$$

we have $\displaystyle\int_{U_1} f(\lambda)\, d\mathfrak{q}(U) + \int_{U_2} f(\lambda)\, d\mathfrak{q}(U) = \int_{U_3} f(\lambda)\, d\mathfrak{q}(U)$.

That is,

$$\mathfrak{p}(U_1) + \mathfrak{p}(U_2) = \mathfrak{p}(U_1 + U_2)\,. \tag{3}$$

Next, let $U = U_1 + U_2 + \dots + U_i + \dots$,

and put $M_i = U_1 + U_2 + \dots + U_i$.

Then, by (2) and (3)

$$\| \mathfrak{p}(U) - \mathfrak{p}(M_i) \|^2 = \| \mathfrak{p}(U - M_i) \|^2 = \int_{U-M_i} | f(\lambda) |^2\, d\sigma(U)\,.$$

Hence

$$\mathop{[\lim]}_{i\to\infty} \mathfrak{p}(M_i) = \mathfrak{p}(U)\,.$$

That is $\mathfrak{p}(U) \,[=]\, \mathfrak{p}(U_1) + \mathfrak{p}(U_2) + \dots + \mathfrak{p}(U_i) + \dots$.

But, from (1), $(\mathfrak{p}(U),\ \mathfrak{p}(U')) = 0$

when $UU' = 0$. Consequently, *when $f(\lambda)$ is a Baire's function in $\mathfrak{L}_2(\sigma)$,*

$$\mathfrak{p}(U) = \int_U f(\lambda)\, d\mathfrak{q}(U)$$

is a completely additive vector valued set function defined for all Borel subsets U of V.

7. *Let $f_1(\lambda)$ and $f_2(\lambda)$ be two Baire's functions in $\mathfrak{L}_2(\sigma)$. Then*

$$\int_U \left\{ f_1(\lambda) + f_2(\lambda) \right\} d\mathfrak{q}(U) = \int_U f_1(\lambda)\, d\mathfrak{q}(U) + \int_U f_2(\lambda) d\mathfrak{q}(U)\,.$$

Since $f_3(\lambda) = f_1(\lambda) + f_2(\lambda)$ belongs to $\mathfrak{L}_2(\sigma)$, $\int_U f_3(\lambda) d\mathfrak{q}(U)$ exists. Let U_N be the set of all points λ of U, where

$$|f_1(\lambda)| \leqq N \qquad \text{and} \qquad |f_2(\lambda)| \leqq N\,.$$

Since $f_i(\lambda) h_{(U_N)}(\lambda)$ $(i = 1, 2, 3)$ are bounded, we have by sec. 4

$$\int_V f_3(\lambda)\, h_{(U_N)}(\lambda)\, d\mathfrak{q}(U) = \int_V f_1(\lambda)\, h_{(U_N)}(\lambda)\, d\mathfrak{q}(U) + \int_V f_2(\lambda)\, h_{(U_N)}(\lambda)\, d\mathfrak{q}(U)\,;$$

that is
$$\int_{U_N} f_3(\lambda)\, d\mathfrak{q}(U) = \int_{U_N} f_1(\lambda)\, d\mathfrak{q}(U) + \int_{U_N} f_2(\lambda)\, d\mathfrak{q}(U)\,.$$

But, by the complete additivity of $\int_U f_i(\lambda)\, d\mathfrak{q}(U)$, we have

$$\int_U f_i(\lambda)\, d\mathfrak{q}(U) = \underset{N\to\infty}{[\lim]} \int_{U_N} f_i(\lambda)\, d\mathfrak{q}(U) \qquad (i = 1, 2, 3)\,.$$

Consequently,

$$\int_U f_3(\lambda)\, d\mathfrak{q}(U) = \int_U f_1(\lambda)\, d\mathfrak{q}(U) + \int_U f_2(\lambda)\, d\mathfrak{q}(U)\,.$$

8. *Let $f(\lambda)$ and $f_\nu(\lambda)$ $(\nu = 1, 2, \ldots.)$ be Baire's function in $\mathfrak{L}_2(\sigma)$. If*

$$\underset{\nu\to\infty}{[\lim]}\, f_\nu(\lambda) = f(\lambda) \qquad [\textit{in } \mathfrak{L}_2(\sigma)]\,, \qquad (1)$$

then
$$\underset{\nu\to\infty}{[\lim]} \int_U f_\nu(\lambda)\, d\mathfrak{q}(U) = \int_U f(\lambda)\, d\mathfrak{q}(U) \qquad [\textit{in } \mathfrak{H}]\,. \qquad (2)$$

From (1), we have $\displaystyle\lim_{\nu\to\infty} \int_V |f_\nu(\lambda) - f(\lambda)|^2\, d\sigma(U) = 0\,.$

But
$$\left\| \int_U f_\nu(\lambda)\, d\mathfrak{q}(U) - \int_U f(\lambda)\, d\mathfrak{q}(U) \right\|^2 = \int_U |f_\nu(\lambda) - f(\lambda)|^2\, d\sigma(U)$$
$$\leqq \int_V |f_\nu(\lambda) - f(\lambda)|^2\, d\sigma(U)\,.$$

Hence, we have (2).

9. Let $f_1(\lambda)$ and $f_2(\lambda)$ be two Baire's functions in $\mathfrak{L}_2(\sigma)$, and $f_1(\lambda)$ be equal to $f_2(\lambda)$ at all points of V except those which belong to a Borel set H where $\sigma(H)=0$. Then, since

$$\int_H f_1(\lambda)\,d\mathfrak{q}(U)=0 \qquad \text{and} \qquad \int_H f_2(\lambda)\,d\mathfrak{q}(U)=0,$$

we have
$$\int_V f_1(\lambda)\,d\mathfrak{q}(U)=\int_V f_2(\lambda)\,d\mathfrak{q}(U).$$

Thus the integral $\int_V f(\lambda)d\mathfrak{q}(U)$ is independent of the value of $f(\lambda)$ at H.

Now, I will extend the definition of the integral of Baire's functions to the integral of any point functions. Let $f(\lambda)$ be a point function. If $f(\lambda)$ is equal almost everywhere (σ) to a Baire's function $f'(\lambda)$ which belongs to $\mathfrak{L}_2(\sigma)$, then I say that

$$\int_V f(\lambda)\,d\mathfrak{q}(U)$$

exists and I shall define its value by $\int_V f'(\lambda)d\mathfrak{q}(U)$. In this case, the value of $f(\lambda)$ may or may not be defined at the points in H, where $\sigma(H)=0$.

Extensions of Normalized Orthogonal Systems.

10. Let $\{\mathfrak{g}_\nu\}$ be a normalized orthogonal system in $\mathfrak{H}$. That is,

$$(\mathfrak{g}_\mu, \mathfrak{g}_\nu)=\delta_{\mu\nu}, \tag{1}$$

where
$$\delta_{\mu\nu}=1 \quad \text{when} \quad \mu=\nu, \qquad =0 \quad \text{when} \quad \mu\neq\nu.$$

And let $\mathfrak{M}$ be the closed linear manifold determined by $\{\mathfrak{g}_\nu\}$. Let $\mathfrak{f}$ be any vector in $\mathfrak{M}$. Then

$$\mathfrak{f}\,[=]\sum_\nu c_\nu \mathfrak{g}_\nu \tag{2}$$

where
$$c_\nu=(\mathfrak{f},\ \mathfrak{g}_\nu) \qquad (\nu=1,2,\ldots\ldots). \tag{3}$$

$\sum_\nu |c_\nu|^2$ is convergent, and

$$\sum_\nu |c_\nu|^2=\|\mathfrak{f}\|^2.$$

When $\mathfrak{f}$ is any vector in $\mathfrak{H}$, we have Bessel's inequality

$$\sum_{\nu} | c_\nu |^2 \leqq || \mathfrak{f} ||^2 . \tag{4}$$

I will denote the class or space of all sequences $\{ c_\nu \}$ of complex numbers such that $\sum_{\nu} | c_\nu |^2$ is convergent, by $\mathfrak{L}_2(Z)$. If we define the inner product of two elements $\mathfrak{c} = \{ c_\nu \}$ and $\mathfrak{c}' = \{ c'_\nu \}$ of $\mathfrak{L}_2(Z)$, by

$$(\mathfrak{c} , \mathfrak{c}') = \sum_{\nu} c_\nu \overline{c'_\nu} ,$$

then $\mathfrak{L}_2(Z)$ is a Hilbert space. By the relation (3), for any vector $\mathfrak{f}$ in $\mathfrak{M}$ there is a corresponding unique element $\{ c_\nu \}$ of $\mathfrak{L}_2(Z)$. Conversely, by the generalization of Riesz-Fischer's theorem, for any element $\{ c_\nu \}$ of $\mathfrak{L}_2(Z)$ by the relation (2), there is a correponding unique vector $\mathfrak{f}$ in $\mathfrak{M}$.

Thus, by relations (2) and (3), there exists a one-to-one correspondence between $\mathfrak{M}$ and $\mathfrak{L}_2(Z)$. Let $\mathfrak{f}$ and $\mathfrak{f}'$ be two vectors in $\mathfrak{M}$ which correspond to $\mathfrak{c} = \{ c_\nu \}$ and $\mathfrak{c}' = \{ c'_\nu \}$ in $\mathfrak{L}_2(Z)$. Then, $a\mathfrak{f}$ and $\mathfrak{f}+\mathfrak{f}'$ correspond to $a\mathfrak{c} = \{ ac_\nu \}$ and $\mathfrak{c}+\mathfrak{c}' = \{ c_\nu + c'_\nu \}$. Hence, the transformations (2) and (3) are linear. Moreover, by the extension of Parseval's theorem,

$$(\mathfrak{f} , \mathfrak{f}') = (\mathfrak{c} , \mathfrak{c}') .$$

Hence the transformations (2) and (3) are isometric.(1)

11. $\{ \mathfrak{g}_\nu \}$ is a normalized orthogonal system which has positive integer ν as paramater. I will now extend this system which has Borel set U as paramater. Let $\mathfrak{q}(U)$ be a completely additive vector valued set function. Then, from sec. 1

$$(\mathfrak{q}(U), \mathfrak{q}(U)) = \sigma(UU') .$$

Since this relation is similar to (1) of the preceding section, considering U as parameter, I say that $\{\mathfrak{q}(U)\}$ is a *normalized orthogonal system in* $\mathfrak{H}$ *with base* $\sigma(U)$.

Let $\mathfrak{M}(\mathfrak{q})$ be the closed linear manifold determined by the system $\mathfrak{q}\{(U)\}$.(2) Then any vector $\mathfrak{g}$ is orthogonal to $\mathfrak{M}(\mathfrak{q})$ when and only when

$$(\mathfrak{q}(U) , \mathfrak{g}) = 0 \tag{1}$$

(1) Cf. Stone, *loc. cit.*, 76.

(2) That is, $\mathfrak{M}(\mathfrak{q})$ is the smallest closed linear manifold which contains all the vectors of the system $\{ \mathfrak{q}(U) \}$. Cf. Stone, *loc. cit.*, 7.

for all Borel sets U. Hence, when $\mathfrak{q}_1(U)$ and $\mathfrak{q}_2(U)$ are two vector valued set functions such that

$$(\mathfrak{q}_1(U), \mathfrak{q}_2(U')) = 0$$

for any Borel sets, U and U', then $\mathfrak{M}(\mathfrak{q}_1)$ and $\mathfrak{M}(\mathfrak{q}_2)$ are orthogonal.

Any vector expressed in the integral form

$$\mathfrak{f} = \int_V f(\lambda)\, d\mathfrak{q}(U)$$

where $f(\lambda)$ is a Baire's function in $\mathfrak{L}_2(\sigma)$, belongs to $\mathfrak{M}(\mathfrak{q})$. For, let $\mathfrak{g}$ be any vector which is orthogonal to $\mathfrak{M}(\mathfrak{q})$, that is,

$$(\mathfrak{q}(U), \mathfrak{g}) = 0$$

for all Borel set U. Then, by sec. 5

$$(\mathfrak{f}, \mathfrak{g}) = \int_V f(\lambda)\, d(\mathfrak{q}(U), \mathfrak{g}) = 0 .$$

Therefore, $\mathfrak{f}$ belongs to $\mathfrak{M}(\mathfrak{q})$.

Let $\mathfrak{f}$ be any vector in $\mathfrak{M}(\mathfrak{q})$. And put

$$\xi(U) = (\mathfrak{f}, \mathfrak{q}(U)) . \tag{2}$$

When $\quad U = U_1 + U_2 + \dots + U_i + \dots ,$

since $\quad \mathfrak{q}(U) [=] \mathfrak{q}(U_1) + \mathfrak{q}(U_2) + \dots + \mathfrak{q}(U_i) + \dots ,$

we have $\quad (\mathfrak{f}, \mathfrak{q}(U)) = (\mathfrak{f}, \mathfrak{q}(U_1)) + (\mathfrak{f}, \mathfrak{q}(U_2)) + \dots + (\mathfrak{f}, \mathfrak{q}(U_i)) + \dots .$

That is, $\xi(U)$ is a completely additive set function. And, it is obvious that $\xi(U)$ is absolutely continuous with respect to $\sigma(U)$.

Divide V into the sum of Borel sets, i.e.

$$V = U_1 + U_2 + \dots + U_i + \dots .$$

Then, since $\quad (\mathfrak{q}(U_i), \mathfrak{q}(U_j)) = 0 \quad$ when $\quad i \neq j,$

$\left\{ \dfrac{\mathfrak{q}(U_i)}{\sqrt{\sigma(U_i)}} \right\} \quad (i = 1, 2, \dots) \quad$ is a normalized orthogonal system in $\mathfrak{H}$.

Moreover the coefficients of the expansion of $\mathfrak{f}$ with respect to this system are

$$\left(\mathfrak{f},\ \frac{\mathfrak{q}(U_i)}{\sqrt{\sigma(U_i)}}\right) = \frac{\xi(U_i)}{\sqrt{\sigma(U_i)}} \qquad (i = 1, 2, \ldots.).$$

Hence, by Bessel's inequality

$$\sum_i \frac{|\xi(U_i)|^2}{\sigma(U_i)} \leqq ||\mathfrak{f}||^2.$$

But, since this inequality holds for any division of V, $\xi(U)$ belongs to $\mathfrak{L}_2(\sigma)$,[1] and

$$||\xi|| \leqq ||\mathfrak{f}||.$$ [2]

Now $D_{\sigma(U)}\xi(\lambda)$ is equal to a Baire's function in $\mathfrak{L}_2(\sigma)$ almost everywhere (σ); hence by sec. 5

$$\int_V D_{\sigma(U)}\,\xi(\lambda)\,d\mathfrak{q}(U)$$

exists. Denote this integral by $\mathfrak{f}'$, then $\mathfrak{f}'$ belongs to $\mathfrak{M}(\mathfrak{q})$, and

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(1) For the sake of simplicity I prove this theorem in the case where $\xi(E)\geqq 0$. Denote the set of points, where $p\varepsilon \leqq [D_{\sigma(U)}\xi(\lambda)]^2 \leqq (p+1)\varepsilon$, by U_p. Then

$$\xi(U_p) = \int_{U_p} D_{\sigma(U)}\xi(\lambda)d\sigma(U) \geqq \sqrt{p\varepsilon}\,\sigma(U_p).$$

Hence

$$p\varepsilon\,\sigma(U_p) \leqq \frac{[\xi(U_p)]^2}{\sigma(U_p)}.$$

Since $\sum_p \frac{[\xi(U_p)]^2}{\sigma(U_p)} \leqq ||\mathfrak{f}||^2$, $\sum_p p\varepsilon\sigma(U_p)$ is convergent. Therefore, $\int_V [D_{\sigma(U)}\xi(\lambda)]^2\,d\sigma(U)$ exists and is not greater than $||\mathfrak{f}||^2$. In the general case where $\xi(U)$ is a complex valued set function, a slight modification is needed to prove the theorem.

When $\xi(U)$ is absolutely continuous with respect to $\sigma(U)$, and $\int_V |D_{\sigma(U)}\xi(\lambda)|^2 d\sigma(U)$ exists, it is said that $\xi(U)$ belongs to the class $\mathfrak{L}_2(\sigma)$. $\mathfrak{L}_2(\sigma)$ is a Hilbert space, where the inner product is defined as follows:

$$(\xi_1, \xi_2) = \int_V D_{\sigma(U)}\xi_1(\lambda)\,\overline{D_{\sigma(U)}\xi_2(\lambda)}\,d\sigma(U).$$

(Cf. F. Maeda, this journal, **3** (1933), 3-13, 243.)

(2) This inequality holds for any vector in $\mathfrak{H}$, and we may call it *Bessel's inequality* with respect to the normalized orthogonal system $\{\mathfrak{q}(U)\}$ with base $\sigma(U)$, for it corresponds to (4) of the preceding section.

$$\| \mathfrak{f}' \| = \| \xi \| .$$

Since $$\mathfrak{q}(U) = \int_V h_{(U)}(\lambda)\, d\mathfrak{q}(U) ,$$

by sec. 5, we have

$$(\mathfrak{f}', \mathfrak{q}(U)) = \int_V D_{\sigma(U)}\, \xi(\lambda)\, h_{(U)}(\lambda)\, d\sigma(U) = \xi(U) .$$

Combining with (2), we have

$$(\mathfrak{f}-\mathfrak{f}', \mathfrak{q}(U)) = \xi(U)-\xi(U) = 0 .$$

Hence, by (1), $\mathfrak{f}-\mathfrak{f}'$ is orthogonal to $\mathfrak{M}(\mathfrak{q})$. Therefore,

$$\mathfrak{f} = \mathfrak{f}' .$$

Consequently, *any vector* $\mathfrak{f}$ *in* $\mathfrak{M}(\mathfrak{q})$ *is expressed as follows:*

$$\mathfrak{f} = \int_V D_{\sigma(U)}\, \xi(\lambda)\, d\mathfrak{q}(U) \tag{3}$$

where $$\xi(U) = (\mathfrak{f}, \mathfrak{q}(U)) . \tag{4}$$

(3) and (4) correspond to (2) and (3) of the preceding section.

When $\mathfrak{f}'$ is another vector in $\mathfrak{M}(\mathfrak{q})$, then

$$\mathfrak{f}' = \int_V D_{\sigma(U)}\, \xi'(\lambda)\, d\mathfrak{q}(U) ,$$

where $\xi'(U) = (\mathfrak{f}', \mathfrak{q}(U))$. And by sec. 5,

$$(\mathfrak{f}, \mathfrak{f}') = \int_V D_{\sigma(U)}\, \xi(\lambda)\, \overline{D_{\sigma(U)}\, \xi'(\lambda)}\, d\sigma(U) = (\xi, \xi') .$$

This expression corresponds to the generalized form of Parseval's theorem.

Corresponding to any set function $\xi(U)$ in $\mathfrak{L}_2(\sigma)$, there exists a vector

$$\mathfrak{f} = \int_V D_{\sigma(U)}\, \xi(\lambda)\, d\mathfrak{q}(U)$$

in $\mathfrak{M}(\mathfrak{q})$, and $$\| \mathfrak{f} \| = \| \xi \| .$$

This statement corresponds to the generalized form of Riesz-Fischer's theorem.

Thus, as in the case of $\{\mathfrak{g}_\nu\}$, (3) and (4) represent isometric transformations between $\mathfrak{M}(\mathfrak{q})$ and $\mathfrak{L}_2(\sigma)$.

12. Let $\qquad \mathfrak{q}_1(U),\ \mathfrak{q}_2(U),\ \ldots\ldots,\ \mathfrak{q}_i(U),\ \ldots\ldots$

be a sequence of completely additive vector valued set functions, such that

$$\mathfrak{M}(\mathfrak{q}_1),\ \mathfrak{M}(\mathfrak{q}_2),\ \ldots\ldots,\ \mathfrak{M}(\mathfrak{q}_i),\ \ldots\ldots \tag{1}$$

are mutually orthogonal. In this case I say that $\{\mathfrak{q}_1(U), \mathfrak{q}_2(U), \ldots\ldots, \mathfrak{q}_i(U), \ldots\ldots\}$ is an *orthogonal system.* If there exists no vector orthogonal to all the closed linear manifolds of (1), then I say that the orthogonal system $\{\mathfrak{q}_1(U), \mathfrak{q}_2(U), \ldots\ldots, \mathfrak{q}_i(U)\ldots\ldots\}$ is *complete in* $\mathfrak{H}$.

The three following assertions concerning the orthogonal system $\{\mathfrak{q}_1(U), \mathfrak{q}_2(U), \ldots\ldots, \mathfrak{q}_i(U), \ldots\ldots\}$ *are equivalent.*

(α) $\{\mathfrak{q}_1(U), \mathfrak{q}_2(U), \ldots\ldots, \mathfrak{q}_i(U), \ldots\ldots\}$ *is complete in* $\mathfrak{H}$;

(β) *for every vector* $\mathfrak{f}$ *in* $\mathfrak{H}$,

$$\mathfrak{f}\,[=]\sum_i \int_V D_{\sigma_i(U)}\,\xi_i(\lambda)\,d\mathfrak{q}_i(U)\,,$$

where $\qquad \xi_i(U) = (\mathfrak{f}, \mathfrak{q}_i(U)) \qquad (i = 1, 2, \ldots\ldots)$;

(γ) *for every pair* $\mathfrak{f}$, $\mathfrak{g}$ *in* $\mathfrak{H}$, *the identity*

$$(\mathfrak{f}, \mathfrak{g}) = \sum_i (\xi_i, \eta_i)$$

is true, where

$$\left.\begin{aligned}\xi_i(U) &= (\mathfrak{f}, \mathfrak{q}_i(U))\\ \eta_i(U) &= (\mathfrak{g}, \mathfrak{q}_i(U))\end{aligned}\right. \qquad (i = 1, 2, \ldots\ldots).$$

Now, I shall show that the following inferences are possible:

$$(\alpha) \longrightarrow (\beta) \longrightarrow (\gamma) \longrightarrow (\alpha)\,,$$

each arrow being directed from hypothesis to conclusion. The equivalence of the three assertions is then obvious.

Assume that (α) holds. Let $\mathfrak{f}$ be any vector in $\mathfrak{H}$. Denote the component of $\mathfrak{f}$ which is contained in $\mathfrak{M}(\mathfrak{q}_i)$ by $\mathfrak{f}_i$. Then

$$\mathfrak{f}\,[=]\sum_i \mathfrak{f}_i\,.$$

But, by the preceding section

$$\mathfrak{f}_i = \int_V D_{\sigma_i(U)}\,\xi_i(\lambda)\,d\mathfrak{q}_i(U)$$

where $$\xi_i(U) = (\mathfrak{f}_i\,,\,\mathfrak{q}_i(U))\,.$$

But, since $$(\mathfrak{f},\,\mathfrak{q}_i(U)) = (\mathfrak{f}_i\,,\,\mathfrak{q}_i(U))\,,$$

we have (β).

Next, assume that (β) holds. Then

$$\mathfrak{f}\,[=]\sum_i \mathfrak{f}_i\,,$$

$$\mathfrak{g}\,[=]\sum_i \mathfrak{g}_i\,,$$

where $$\mathfrak{f}_i = \int_V D_{\sigma_i(U)}\,\xi_i(\lambda)\,d\mathfrak{q}_i(U)\,,\qquad \xi_i(U) = (\mathfrak{f},\,\mathfrak{q}_i(U))\,,$$

$$\mathfrak{g}_i = \int_V D_{\sigma_i(U)}\,\eta_i(\lambda)\,d\mathfrak{q}_i(U)\,,\qquad \eta_i(U) = (\mathfrak{g},\,\mathfrak{q}_i(U))\,.$$

But, since $$(\mathfrak{f}_i\,,\,\mathfrak{g}_j) = 0 \qquad \text{when} \qquad i \neq j\,,$$

and $$(\mathfrak{f}_i\,,\,\mathfrak{g}_i) = \int_V D_{\sigma_i(U)}\,\xi_i(\lambda)\,\overline{D_{\sigma_i(U)}\,\eta_i(\lambda)}\,d\sigma_i(U) = (\xi_i\,,\,\eta_i)\,,$$

we have $$(\mathfrak{f},\,\mathfrak{g}) \doteq \sum_i (\mathfrak{f}_i\,,\,\mathfrak{g}_i) = \sum_i (\xi_i\,,\,\eta_i)\,.$$

That is, (γ) holds.

Next, assume that (γ) holds. Let $\mathfrak{f}$ be a vector orthogonal to all of (1). Then

$$\xi_i(U) = (\mathfrak{f},\,\mathfrak{q}_i(U)) = 0 \qquad (i = 1, 2, \ldots.)$$

for all sets U. But from (γ)

$$(\mathfrak{f},\,\mathfrak{f}) = \sum_i (\xi_i\,,\,\xi_i)\,.$$

Consequently, $$\mathfrak{f} = 0\,.$$

That is, (α) holds.

13. *Let* $$\{\mathfrak{q}_1(U),\ \mathfrak{q}_2(U),\ \ldots.,\ \mathfrak{q}_i(U),\ \ldots.\} \tag{1}$$

be an orthogonal system. If

$$\sigma_1(U)+\sigma_2(U)+\ldots.+\sigma_i(U)+\ldots. \tag{2}$$

converges to a finite value, say $\sigma(U)$, *then*

$$\mathfrak{q}_1(U)+\mathfrak{q}_2(U)+\ldots.+\mathfrak{q}_i(U)+\ldots. \tag{3}$$

converges strongly to a completely additive vector valued set function, say $\mathfrak{q}(U)$, *with base* $\sigma(U)$.[1]

$\sigma_i(U)$ being non-negative, it is evident that $\sigma(U)$ is a completely additive set function.

Since (2) converges, it is evident that (3) converges strongly to a vector, say $\mathfrak{q}(U)$, for any set U. And

$$(\mathfrak{q}(U),\ \mathfrak{q}(U')) = \sum_i (\mathfrak{q}_i(U),\ \mathfrak{q}_i(U')) = \sum_i \sigma_i(UU') = \sigma(UU'). \tag{4}$$

It is evident that

$$\mathfrak{q}(U_1+U_2) = \mathfrak{q}(U_1)+\mathfrak{q}(U_2).$$

Now, let $$U = U_1+U_2+\ldots.+U_i+\ldots.,$$

and put $$M_i = U_1+U_2+\ldots.+U_i.$$

Then $$\|\mathfrak{q}(U)-\mathfrak{q}(M_i)\|^2 = \|\mathfrak{q}(U-M_i)\|^2 = \sigma(U-M_i).$$

Hence $$\mathfrak{q}(U) = [\lim_{i\to\infty}]\,\mathfrak{q}(M_n),$$

that is, $$\mathfrak{q}(U)\,[=]\,\mathfrak{q}(U_1)+\mathfrak{q}(U_2)+\ldots.+\mathfrak{q}(U_i)+\ldots.$$

Combining with (4), we conclude that $\mathfrak{q}(U)$ is a completely additive vector valued set function with base $\sigma(U)$.

14. *Let* $f(\lambda)$ *and* $g(\lambda)$ *be Baire's functions. If*

$$\mathfrak{p}(U) = \int_U f(\lambda)\,d\mathfrak{q}(U), \tag{1}$$

(1) Cf. F. Maeda, this journal, **3** (1933), 266.

then
$$\int_U g(\lambda)\, d\mathfrak{p}(U) = \int_U g(\lambda) f(\lambda)\, d\mathfrak{q}(U)\,,$$

when the left hand integral exists.

Put
$$\mathfrak{f} = \int_V g(\lambda)\, d\mathfrak{p}(U)\,, \tag{2}$$

then $\mathfrak{f}$ belongs to $\mathfrak{M}(\mathfrak{p})$. Hence $\mathfrak{f}$ may be expressed as follows:

$$\mathfrak{f} = \int_V D_{\rho(U)}\, \zeta(\lambda)\, d\mathfrak{p}(U)\,,$$

where
$$\zeta(U) = (\mathfrak{f}, \mathfrak{p}(U))\,, \tag{3}$$

and
$$\rho(U) = \| \mathfrak{p}(U)\|^2 = \int_U |f(\lambda)|^2\, d\sigma(U)\,. \tag{4}$$

Therefore, it must be that

$$g(\lambda) = D_{\rho(U)}\, \zeta(\lambda) \tag{5}$$

almost everywhere (ρ).(1)

But, since $\mathfrak{M}(\mathfrak{p}) \subseteqq \mathfrak{M}(\mathfrak{q})$, $\mathfrak{f}$ belongs to $\mathfrak{M}(\mathfrak{q})$. Hence $\mathfrak{f}$ may also be expressed as follows:

$$\mathfrak{f} = \int_V D_{\sigma(U)}\, \eta(\lambda)\, d\mathfrak{q}(U) \tag{6}$$

where
$$\eta(U) = (\mathfrak{f}, \mathfrak{q}(U))\,.$$

Hence, from (1), (3) and (6),

$$\zeta(U) = (\mathfrak{f}, \mathfrak{p}(U)) = \int_U D_{\sigma(U)}\, \eta(\lambda) \overline{f(\lambda)}\, d\sigma(U)\,. \tag{7}$$

But, from (5) and (4), we have

$$\zeta(U) = \int_U g(\lambda)\, d\rho(U) = \int_U g(\lambda)\, |f(\lambda)|^2\, d\sigma(U)\,.^{(2)} \tag{8}$$

(1) Since $\rho(U)$ is absolutely continuous with respect to $\sigma(U)$, $\rho(U)$ is uniformly monotone almost everywhere (ρ) in V. (Cf. F. Maeda, this journal, **3** (1933), 257 footnote.)

(2) By the theorem proved in my paper, *ibid.*, 258 footnote.

From (7) and (8), we have

$$D_{\sigma(U)}\,\eta(\lambda) = g(\lambda)f(\lambda)$$

almost everywhere (σ). Hence, from (2) and (6)

$$\int_V g(\lambda)\,d\mathfrak{p}(U) = \int_V g(\lambda)f(\lambda)\,d\mathfrak{q}(U)\,. \tag{9}$$

If we put $g(\lambda)\,h_{(U)}(\lambda)$ instead of $g(\lambda)$ in (9), we have

$$\int_U g(\lambda)\,d\mathfrak{q}(U) = \int_U g(\lambda)f(\lambda)\,d\mathfrak{q}(U)\,.$$

Resolution of Identity.

15. Let $\mathrm{E}(U)$ be a self-adjoint transformation which depends on Borel subset U of a Borel set V in the metric space S. If $\mathrm{E}(U)$ satisfies the following conditions, then I say that $\mathrm{E}(U)$ is a *resolution of identity*.(1)

(α) $$\mathrm{E}(U)\,\mathrm{E}(U')\,\mathfrak{f} = \mathrm{E}(UU')\,\mathfrak{f}\,;$$

(β) $$\mathrm{E}(U)\,\mathfrak{f}\,[=]\,\mathrm{E}(U_1)\,\mathfrak{f}+\mathrm{E}(U_2)\,\mathfrak{f}+\ldots\ldots+\mathrm{E}(U_i)\,\mathfrak{f}+\ldots\ldots,$$

where $$U = U_1+U_2+\ldots\ldots+U_i+\ldots\ldots;$$

(γ) $$\mathrm{E}(V)\,\mathfrak{f} = \mathfrak{f}\,;$$

all for any vector $\mathfrak{f}$.

From (α), we have

$$\mathrm{E}(U)\,\mathrm{E}(U)\,\mathfrak{f} = \mathrm{E}(U)\,\mathfrak{f}\,.$$

Hence $\mathrm{E}(U)$ is a projection on some closed linear manifold which depends on U.(2) From (β), if L is an empty set, then

$$\mathrm{E}(L)\,\mathfrak{f} = 0\,;$$

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(1) Ordinarily, resolutions of identity are defined as self-adjoint transformations which depend on real or complex numbers. (Cf. Stone, *loc. cit.*, 174 and 314.) Here, I extend this conception.

(2) Stone, *ibid.*, 71.

and if $\operatorname*{Lim}_{n\to\infty} V_n = U$, then

$$\operatorname*{[lim]}_{n\to\infty} E(V_n)\,\mathfrak{f} = E(U)\,\mathfrak{f}\,.$$

Let $\mathfrak{b}$ be a vector, and put

$$\mathfrak{q}(U) = E(U)\,\mathfrak{b}\,,$$

Then, from (α)

$$(\mathfrak{q}(U),\, \mathfrak{q}(U')) = (E(U)\,\mathfrak{b},\, E(U')\,\mathfrak{b}) = (\mathfrak{b},\, E(U)\,E(U')\,\mathfrak{b}) = (\mathfrak{b},\, E(UU')\,\mathfrak{b})\,.$$

Hence $\qquad (\mathfrak{q}(U),\, \mathfrak{q}(U')) = 0 \qquad$ when $\qquad UU' = 0\,.$

And, from (β) we have

$$\mathfrak{q}(U)\,[=]\,\mathfrak{q}(U_1) + \mathfrak{q}(U_2) + \ldots\ldots + \mathfrak{q}(U_i) + \ldots\ldots$$

Therefore, $\mathfrak{q}(U) = E(U)\,\mathfrak{b}$ *is a completely additive vector valued set function.* In this case, I say that $\mathfrak{q}(U)$ is *generated by* $E(U)$.

16. *When* $\mathfrak{q}(U)$ *is generated by* $E(U)$, *if*

$$\mathfrak{f} = \int_V f(\lambda)\, d\mathfrak{q}(U)\,,$$

then

$$E(U)\,\mathfrak{f} = \int_U f(\lambda)\, d\mathfrak{q}(U)\,.$$

When $f(\lambda)$ is bounded, by the definition of the integral(1)

$$\mathfrak{f} = \operatorname*{[lim]}_{\nu\to\infty} \mathfrak{f}^{(\nu)}\,,$$

where $\mathfrak{f}^{(\nu)}$ is expressed in the form

$$\sum_{p,q} f(\lambda_{pq})\, \mathfrak{q}(V_{pq})\,.$$

Then

$$E(U)\,\mathfrak{f} = \operatorname*{[lim]}_{\nu\to\infty} E(U)\,\mathfrak{f}^{(\nu)}\,,$$

where $E(U)\,\mathfrak{f}^{(\nu)}$ is expressed in the form

$$\sum_{p,q} f(\lambda_{pq})\, E(U)\, \mathfrak{q}(V_{pq})\,.$$

---

(1) Sec. 2.

But $\quad \mathrm{E}(U)\,\mathfrak{q}(V_{pq}) = \mathrm{E}(U)\,\mathrm{E}(V_{pq})\,\mathfrak{b} = \mathrm{E}(UV_{pq})\,\mathfrak{b} = \mathfrak{q}(UV_{pq})\,.$

Hence $\mathrm{E}(U)\,\mathfrak{f}$ is expressed in the integral form

$$\int_U f(\lambda)\, d\mathfrak{q}(U)\,.$$

Next, when $f(\lambda)$ is not bounded, by the definition of the integral,[1]

$$\mathfrak{f} = \underset{N\to\infty}{[\lim]} \int_V f^N(\lambda)\, d\mathfrak{q}(U)\,.$$

Hence $$\mathrm{E}(U)\,\mathfrak{f} = \underset{N\to\infty}{[\lim]}\, \mathrm{E}(U) \int_V f^N(\lambda)\, d\mathfrak{q}(U) = \underset{N\to\infty}{[\lim]} \int_U f^N(\lambda)\, d\mathfrak{q}(U)$$
$$= \int_U f(\lambda)\, d\mathfrak{q}(U)\,.$$

17. *If* $\quad \mathfrak{f} = \int_V f(\lambda)\, d\mathrm{E}(U)\,\mathfrak{a} \quad$ *and* $\quad \mathfrak{g} = \int_V g(\lambda)\, d\mathrm{E}(U)\,\mathfrak{b}\,,$

then
$$(\mathfrak{f}, \mathfrak{g}) = \int_V f(\lambda)\, \overline{g(\lambda)}\, d(\mathrm{E}(U)\,\mathfrak{a}, \mathfrak{b})\,. \tag{1}$$

By sec. 5,
$$(\mathfrak{f}, \mathfrak{g}) = \int_V f(\lambda)\, d\zeta(U)\,, \tag{2}$$

where
$$\zeta(U) = (\mathrm{E}(U)\,\mathfrak{a}, \mathfrak{g}) = (\mathfrak{a}, \mathrm{E}(U)\,\mathfrak{g})\,.$$

But, since
$$\mathrm{E}(U)\,\mathfrak{g} = \int_U g(\lambda)\, d\mathrm{E}(U)\,\mathfrak{b}\,,$$

by sec. 5, we have
$$\zeta(U) = \int_U \overline{g(\lambda)}\, d(\mathfrak{a}, \mathrm{E}(U)\,\mathfrak{b}) \tag{3}$$

Therefore, from (2) and (3), we have (1).

18. *Let* $\mathrm{E}(U)$ *be a resolution of identity. Then there exists an orthogonal system* $\{\mathfrak{q}_1(U), \mathfrak{q}_2(U), \ldots\ldots, \mathfrak{q}_i(U), \ldots\ldots\}$ *which is complete in* $\mathfrak{H}$, *such that* $\mathfrak{q}_i(U)$ $(i = 1, 2, \ldots\ldots)$ *are generated by* $\mathrm{E}(U)$.

Since $\mathfrak{H}$ is separable, there exists a sequence $\{\mathfrak{b}_n\}$ which is dense in $\mathfrak{H}$. Put

$$\mathfrak{q}_1(U) = \mathrm{E}(U)\,\mathfrak{b}_1\,.$$

(1) Cf. sec. 5.

Let $\mathfrak{b}_{n_1}$ be the first element of the sequence $\{\mathfrak{b}_n\}$ which is not contained in $\mathfrak{M}(\mathfrak{q}_1)$. And let $\mathfrak{c}_{n_1}$ be the component of $\mathfrak{b}_{n_1}$ which is orthogonal to $\mathfrak{M}(\mathfrak{q}_1)$. And put

$$\mathfrak{q}_2(U) = \mathbf{E}(U)\,\mathfrak{c}_{n_1}.$$

Then, since $$(\mathfrak{q}_1(U),\ \mathfrak{q}_2(U')) = (\mathbf{E}(U)\,\mathfrak{b}_1,\ \mathbf{E}(U')\,\mathfrak{c}_{n_1}) = (\mathbf{E}(UU')\,\mathfrak{b}_1,\ \mathfrak{c}_{n_1})$$
$$= (\mathfrak{q}_1(UU'),\ \mathfrak{c}_{n_1}) = 0$$

for any Borel sets, U and U', $\mathfrak{M}(\mathfrak{q}_1)$ and $\mathfrak{M}(\mathfrak{q}_2)$ are orthogonal.

Let $\mathfrak{b}_{n_2}$ be the first element of the sequence $\{\mathfrak{b}_n\}$ which is not contained in $\mathfrak{M}(\mathfrak{q}_1) \oplus \mathfrak{M}(\mathfrak{q}_2)$.(1) Of course $n_2 > n_1$. And let $\mathfrak{c}_{n_2}$ be the component of $\mathfrak{b}_{n_2}$ which is orthogonal to $\mathfrak{M}(\mathfrak{q}_1) \oplus \mathfrak{M}(\mathfrak{q}_2)$. And put

$$\mathfrak{q}_3(U) = \mathbf{E}(U)\,\mathfrak{c}_{n_2}.$$

Then, as above, $\mathfrak{M}(\mathfrak{q}_3)$ is orthogonal to $\mathfrak{M}(\mathfrak{q}_1)$ and $\mathfrak{M}(\mathfrak{q}_2)$.

Continuing this process indefinitely, we have an at most denumerably infinite orthogonal system $\{\mathfrak{q}_1(U),\ \mathfrak{q}_2(U),\ \ldots,\ \mathfrak{q}_i(U),\ \ldots\}$. Then this orthogonal system must be complete in $\mathfrak{H}$. For, if there exists a vector $\mathfrak{f}$ which is orthogonal to $\mathfrak{M}(\mathfrak{q}_1) \oplus \mathfrak{M}(\mathfrak{q}_2) \oplus \ldots \oplus \mathfrak{M}(\mathfrak{q}_i) \oplus \ldots$, then $\mathfrak{f}$ must be orthogonal to all the vectors of $\{\mathfrak{b}_n\}$. Hence $\mathfrak{f}$ must be the null vector.

19. Conversely, *let* $\{\mathfrak{q}_1(U),\ \mathfrak{q}_2(U),\ \ldots,\ \mathfrak{q}_i(U),\ \ldots\}$ *be a complete orthogonal system in* $\mathfrak{H}$. *Then there exists a resolution of identity* $\mathbf{E}(U)$, *which generates* $\mathfrak{q}_i(U)$ $(i = 1, 2, \ldots)$.

Let $\mathfrak{f}$ be any vector, and put

$$\mathfrak{p}_i(U) = \int_U D_{\sigma_i(U)}\, \xi_i(\lambda)\, d\mathfrak{q}_i(U) \qquad (i = 1, 2, \ldots), \tag{1}$$

where $\xi_i(U) = (\mathfrak{f},\ \mathfrak{q}_i(U))$. Then, by sec. 6 $\mathfrak{p}_i(U)$ $(i = 1, 2, \ldots)$ are completely additive vector valued set functions with bases $\rho_i(U) = \|\mathfrak{p}_i(U)\|^2$.

Since $\mathfrak{p}_i(U)$ belongs to $\mathfrak{M}(\mathfrak{q}_i)$,

$$(\mathfrak{p}_i(U),\ \mathfrak{p}_j(U')) = 0 \qquad \text{when} \qquad i \neq j,$$

(1) $\mathfrak{M}(\mathfrak{q}_1) \oplus \mathfrak{M}(\mathfrak{q}_2)$ signifies the closed linear manifold determined by the sets of all vectors which are contained in $\mathfrak{M}(\mathfrak{q}_1)$ or $\mathfrak{M}(\mathfrak{q}_2)$. Cf. Stone, *loc. cit.*, 21.

for all Borel sets U and U'. Hence $\{\mathfrak{p}_1(U), \mathfrak{p}_2(U), \ldots, \mathfrak{p}_i(U), \ldots\}$ is an orthogonal system.

Let $\mathfrak{f}_i$ be the component of $\mathfrak{f}$ which is contained in $\mathfrak{M}(\mathfrak{q}_i)$, then

$$\xi_i(U) = (\mathfrak{f}, \mathfrak{q}_i(U)) = (\mathfrak{f}_i, \mathfrak{q}_i(U)),$$

and by sec. 11, $$\| \xi_i \| = \| \mathfrak{f}_i \| .$$

Hence, from (1), we have

$$\rho_i(U) = \int_U | D_{\sigma_i(U)} \xi_i(\lambda) |^2 d\sigma_i(U) \leqq \| \xi_i \|^2 = \| \mathfrak{f}_i \|^2 .$$

Therefore, $$\rho_1(U) + \rho_2(U) + \ldots + \rho_i(U) + \ldots$$

converges for all Borel sets U. Hence by sec. 13

$$\mathfrak{p}_1(U) + \mathfrak{p}_2(U) + \ldots + \mathfrak{p}_i(U) + \ldots$$

converges strongly to a completely additive vector valued set function, sap $\mathfrak{p}(U)$. And by sec. 12

$$\mathfrak{p}(V) = \mathfrak{f} .$$

Let $\mathrm{E}(U)$ be the transformation which transforms $\mathfrak{f}$ to $\mathfrak{p}(U)$. Then $\mathrm{E}(U)$ is a bounded linear transformation whose domain is $\mathfrak{H}$. Let $\mathfrak{g}$ be any vector, then by sec. 12

$$\mathfrak{g} \,[=]\, \sum_i \int_V D_{\sigma_i(U)} \zeta_i(\lambda)\, d\mathfrak{q}_i(U)$$

where $\zeta_i(U) = (\mathfrak{g}, \mathfrak{q}_i(U))$. But since

$$\mathrm{E}(U)\mathfrak{f} = \mathfrak{p}(U) \,[=]\, \sum_i \int_V D_{\sigma_i(U)} \xi_i(\lambda) h_{(U)}(\lambda)\, d\mathfrak{q}_i(U) , \tag{2}$$

we have $$(\mathrm{E}(U)\mathfrak{f}, \mathfrak{g}) = \sum_i \int_V D_{\sigma_i(U)} \xi_i(\lambda)\, h_{(U)}(\lambda)\, \overline{D_{\sigma_i(U)}\zeta_i(\lambda)}\, d\sigma_i(U) .$$

Similarly $$(\mathfrak{f}, \mathrm{E}(U)\mathfrak{g}) = \sum_i \int_V D_{\sigma_i(U)} \xi_i(\lambda)\, \overline{D_{\sigma_i(U)}\zeta_i(\lambda)}\, h_{(U)}(\lambda) d\sigma_i(U) .$$

Hence $$(\mathrm{E}(U)\mathfrak{f}, \mathfrak{g}) = (\mathfrak{f}, \mathrm{E}(U)\mathfrak{g})$$

for any vectors $\mathfrak{f}$ and $\mathfrak{g}$. Therefore, $\mathrm{E}(U)$ is a self-adjoint transformation. Now, by (2)

$$\mathrm{E}(U)\mathrm{E}(U')\mathfrak{f} \,[=]\, \sum_i \int_V D_{\sigma_i(U)} \xi_i'(\lambda)\, h_{(U)}(\lambda)\, d\mathfrak{q}_i(U) \tag{3}$$

where $$\xi_i'(U) = (\mathrm{E}(U')\,\mathfrak{f},\ \mathfrak{q}_i(U))\,.$$

But, since $$\mathfrak{q}_i(U) = \int_V h_{(U)}(\lambda)\, d\mathfrak{q}_i(U)\,,$$

we have, by (2),

$$\xi'(U) = \int_V D_{\sigma_i(U)}\, \xi_i(\lambda)\, h_{(U')}(\lambda)\, h_{(U)}(\lambda)\, d\sigma_i(U)$$

$$= \int_U D_{\sigma_i(U)}\, \xi_i(\lambda)\, h_{(U')}(\lambda)\, d\sigma_i(U)\,.$$

That is $$D_{\sigma_i(U)}\, \xi_i'(\lambda) = D_{\sigma_i(U)}\, \xi_i(\lambda)\, h_{(U')}(\lambda)$$

almost everywhere (σ_i). Hence, from (3),

$$\mathrm{E}(U)\,\mathrm{E}(U')\,\mathfrak{f}\,[=]\sum_i \int_V D_{\sigma_i(U)}\, \xi_i(\lambda)\, h_{(U')}(\lambda)\, h_{(U)}(\lambda)\, d\mathfrak{q}_i(U)$$

$$[=]\sum_i \int_V D_{\sigma_i(U)}\, \xi_i(\lambda)\, h_{(UU')}(\lambda)\, d\mathfrak{q}_i(U)\,.$$

Hence, from (2), $$\mathrm{E}(U)\,\mathrm{E}(U') = \mathrm{E}(UU')\,\mathfrak{f}\,.$$

Thus, $\mathrm{E}(U)$ satisfies the condition (α) of the resolution of identity.[1] But, since $\mathfrak{p}(U)$ is completely additive, and $\mathfrak{p}(V) = \mathfrak{f}$, it is evident that $\mathrm{E}(U)$ satisfies the conditions (β) and (γ) of the resolution of identity.

Now, put $$\mathfrak{f}_j = \mathfrak{q}_j(V) \qquad (j = 1, 2, \ldots\ldots)\,.$$

Then $$\mathrm{E}(U)\,\mathfrak{f}_j\,[=]\sum_i \int_U D_{\sigma_i(U)}\, \xi_i^{(j)}(\lambda)\, d\mathfrak{q}_i(U)$$

where $$\xi_i^{(j)}(U) = (\mathfrak{f}_j,\ \mathfrak{q}_i(U)) = (\mathfrak{q}_j(V),\ \mathfrak{q}_i(U))$$

$$\begin{cases} = \|\,\mathfrak{q}_i(U)\,\|^2 = \sigma_i(U) & \text{when} \quad i = j\,, \\ = 0 & \text{when} \quad i \neq j\,. \end{cases}$$

Hence $$\mathrm{E}(U)\,\mathfrak{f}_j = \int_U d\mathfrak{q}_j(U) = \mathfrak{q}_j(U)\,.$$

Therefore, $\mathfrak{q}_j(U)$ ($j = 1, 2, \ldots\ldots$) are generated by $\mathrm{E}(U)$. Consequently, $\mathrm{E}(U)$ is the required resolution of identity.

(1) Cf. sec. 15.

20. *Let* $\{\mathfrak{q}_1(U), \mathfrak{q}_2(U), \ldots, \mathfrak{q}_i(U), \ldots\}$ *be an orthogonal system, generated by a resolution of identity* $\mathrm{E}(U)$; *and let* $\xi_i(U)$ *be set functions in* $\mathfrak{L}_2(\sigma_i)$ $(i = 1, 2, \ldots)$. *If*

$$\sum_i \int_U | D_{\sigma_i(U)} \xi_i(\lambda) |^2 d\sigma_i(U)$$

converges to a finite value, say $\rho(U)$, *for all Borel subsets* U *of* V, *then*

$$\sum_i \int_U D_{\sigma_i(U)} \xi_i(\lambda)\, d\mathfrak{q}_i(U)$$

converges strongly to a completely additive vector valued set function, say $\mathfrak{p}(U)$, *with base* $\rho(U)$. *And* $\mathfrak{p}(U)$ *is generated by* $\mathrm{E}(U)$.[1]

Put $$\mathfrak{p}_i(U) = \int_U D_{\sigma_i(U)} \xi_i(\lambda)\, d\mathfrak{q}_i(U) \qquad (i = 1, 2, \ldots),$$

then $$\{\mathfrak{p}_1(U), \mathfrak{p}_2(U), \ldots, \mathfrak{p}_i(U), \ldots\}$$

is also an orthogonal system with base

$$\{\rho_1(U), \rho_2(U), \ldots, \rho_i(U), \ldots\},$$

where $$\rho_i(U) = \int_U | D_{\sigma_i(U)} \xi_i(\lambda) |^2 d\sigma_i(U) \qquad (i = 1, 2, \ldots).$$

Hence, by sec. 13, $\sum_i \mathfrak{p}_i(U)$ converges strongly to a completely additive vector valued set function, say $\mathfrak{p}(U)$, with base $\rho(U)$.

Put $$\mathfrak{b} = \mathfrak{p}(V), \qquad \mathfrak{b}_i = \mathfrak{p}_i(V) \qquad (i = 1, 2, \ldots).$$

Then $$\mathfrak{b} [=] \mathfrak{b}_1 + \mathfrak{b}_2 + \ldots + \mathfrak{b}_i + \ldots.$$

But, by sec. 16

$$\mathrm{E}(U)\, \mathfrak{b}_i [=] \int_U D_{\sigma_i(U)} \xi_i(\lambda)\, d\mathfrak{q}_i(U) \qquad (i = 1, 2, \ldots).$$

Hence $$\mathrm{E}(U)\, \mathfrak{b} [=] \sum_i \mathrm{E}(U)\, \mathfrak{b}_i [=] \sum_i \int_U D_{\sigma_i(U)} \xi_i(\lambda)\, d\mathfrak{q}_i(U).$$

That is, $$\mathrm{E}(U)\, \mathfrak{b} = \mathfrak{p}(U).$$

(1) Cf. F. Maeda, this journal, **3** (1933), 270.

Consequently, $\mathfrak{p}(U)$ is generated by $\mathbf{E}(U)$.

21. Conversely, *when* $\{\mathfrak{q}_1(U), \mathfrak{q}_2(U), \ldots, \mathfrak{q}_i(U), \ldots\}$ *is a complete orthogonal system in* $\mathfrak{H}$, *generated by a resolution of identity* $\mathbf{E}(U)$, *any completely additive vector valued set function* $\mathfrak{p}(U)$ *generated by* $\mathbf{E}(U)$, *is expressed as follows:*

$$\mathfrak{p}(U) [=] \sum_i \int_U D_{\sigma_i(U)}\, \xi_i(\lambda)\, d\mathfrak{q}_i(U),$$

where $\xi_i(U)$ *are set functions in* $\mathfrak{L}_2(\sigma_i)$ $(i = 1, 2, \ldots)$. (1)

Let $\mathfrak{b}$ be the vector such that

$$\mathbf{E}(U)\,\mathfrak{b} = \mathfrak{p}(U).$$

By sec. 12 $\mathfrak{b}$ is expressed as follows:

$$\mathfrak{b} [=] \sum_i \int_V D_{\sigma_i(U)}\, \xi_i(\lambda)\, d\mathfrak{q}_i(U)$$

where $\qquad \xi_i(U) = (\mathfrak{f}, \mathfrak{q}_i(U)) \qquad (i = 1, 2, \ldots).$

And $\xi_i(U)$ belongs to $\mathfrak{L}_2(\sigma_i)$ for each value of i. Now, by sec. 16

$$\mathbf{E}(U)\,\mathfrak{b} [=] \sum_i \mathbf{E}(U) \int_V D_{\sigma_i(U)}\, \xi_i(\lambda)\, d\mathfrak{q}_i(U)$$

$$[=] \sum_i \int_U D_{\sigma_i(U)}\, \xi_i(\lambda)\, d\mathfrak{q}_i(U).$$

Hence, we have the required result.

Linear Transformations expressed in the Integral Forms.

22. *Let* $\mathbf{E}(U)$ *be a resolution of identity, and let* $f(\lambda)$ *be a finite Baire's function defined in* V *almost everywhere* (E).(2) *If we denote the set of vectors* $\mathfrak{f}$ *such that*

$$\int_V |f(\lambda)|^2\, d\, \|\mathbf{E}(U)\,\mathfrak{f}\|^2$$

---

(1) Cf. F. Maeda, *ibid.*, 270.

(2) That is, $f(\lambda)$ is defined at all points of V except the points of the set H where $\mathbf{E}(H) = 0$.

is finite, by $\mathfrak{D}(f)$ then $\mathfrak{D}(f)$ is a linear manifold which is dense in $\mathfrak{H}$.

Since $\displaystyle\int_V |f(\lambda)|^2 d\,\|\mathbf{E}(U)\,a\mathfrak{f}\|^2 = a^2 \int_V |f(\lambda)|^2 d\,\|\mathbf{E}(U)\,\mathfrak{f}\,\|^2,$

$a\mathfrak{f}$ belongs to $\mathfrak{D}(f)$ with $\mathfrak{f}$. Next, let $\mathfrak{f}$ and $\mathfrak{g}$ belong to $\mathfrak{D}(f)$. Since

$$\|\mathbf{E}(U)(\mathfrak{f}+\mathfrak{g})\|^2 = \|\mathbf{E}(U)\mathfrak{f}\|^2 + (\mathbf{E}(U)\,\mathfrak{f}, \mathfrak{g}) + (\mathbf{E}(U)\,\mathfrak{g}, \mathfrak{f}) + \|\mathbf{E}(U)\,\mathfrak{g}\,\|^2,$$

and by sec. 17 $\displaystyle\int_V |f(\lambda)|^2 d(\mathbf{E}(U)\,\mathfrak{f}, \mathfrak{g})$, $\displaystyle\int_V |f(\lambda)|^2 d(\mathbf{E}(U)\,\mathfrak{g}, \mathfrak{f})$ are finite, $\displaystyle\int_V |f(\lambda)|^2 d\,\|\mathbf{E}(U)(\mathfrak{f}+\mathfrak{g})\,\|^2$ is finite. Therefore, $\mathfrak{f}+\mathfrak{g}$ belongs to $\mathfrak{D}(f)$. Consequently, $\mathfrak{D}(f)$ is a linear manifold.

Next, I will shew that $\mathfrak{D}(f)$ is dense in $\mathfrak{H}$. Let V_N be the set of points where $|f(\lambda)| < N$, N being a positive integer. Then, if we put

$$V' = \operatorname*{Lim}_{N\to\infty} V_N\,,$$

then
$$\mathbf{E}(V') = \mathbf{E}(V)\,.$$

Let $\mathfrak{f}$ be any vector in $\mathfrak{H}$, then by sec. 15

$$\operatorname*{[lim]}_{N\to\infty} \mathbf{E}(V_N)\,\mathfrak{f} = \mathbf{E}(V)\,\mathfrak{f} = \mathfrak{f}\,. \tag{1}$$

But, since

$$\int_V |f(\lambda)|^2 d\,\|\mathbf{E}(U)\,\mathbf{E}(V_N)\,\mathfrak{f}\,\|^2 = \int_{V_N} |f(\lambda)|^2 d\,\|\mathbf{E}(U)\,\mathfrak{f}\,\|^2,$$

$\mathbf{E}(V_N)\,\mathfrak{f}$ belongs to $\mathfrak{D}(f)$ for any value of N. Therefore, by (1), $\mathfrak{D}(f)$ is dense in $\mathfrak{H}$.

23. *Let $\{\mathfrak{f}_n\}$ be a sequence of vectors belonging to $\mathfrak{D}(f)$, such that there exists a constant number K satisfying*

$$\int_V |f(\lambda)|^2 d\,\|\mathbf{E}(U)\,\mathfrak{f}_n\,\|^2 < K \qquad (n = 1, 2, \ldots\ldots). \tag{1}$$

If
$$\operatorname*{[lim]}_{n\to\infty} \mathfrak{f}_n = \mathfrak{f}\,,$$

then $\mathfrak{f}$ belongs to $\mathfrak{D}(f)$.

Let $f^N(\lambda)$ be the bounded Baire's function defined in sec. 5. Then

$$\left\| \int_V f^N(\lambda)\, dE(U)(\mathfrak{f}-\mathfrak{f}_n) \right\|^2 = \int_V |f^N(\lambda)|^2 d\| E(U)(\mathfrak{f}-\mathfrak{f}_n)\|^2 \leqq N^2 \|\mathfrak{f}-\mathfrak{f}_n\|^2 .$$

Hence $$\underset{n\to\infty}{[\lim]} \int_V f^N(\lambda)\, dE(U)\, \mathfrak{f}_n = \int_V f^N(\lambda)\, dE(U)\, \mathfrak{f} . \qquad (2)$$

But, from (1) $$\int_V |f^N(\lambda)|^2 d\| E(U)\, \mathfrak{f}_n\|^2 < K \qquad (n = 1, 2, \dots .)$$

for any N. Hence, from (2)

$$\int_V |f^N(\lambda)|^2 d\| E(U)\, \mathfrak{f}\|^2 \leqq K$$

for any N. That is, $\int_V |f(\lambda)|^2 d\| E(U)\, \mathfrak{f}\|^2$ is finite, and $\mathfrak{f}$ belongs to $\mathfrak{D}(f)$.

24. *If we put*

$$T_f\, \mathfrak{f} = \int_V f(\lambda)\, dE(U)\, \mathfrak{f} ,$$

then T_f *is a closed linear transformation with domain* $\mathfrak{D}(f)$. (1)

Since it is evident that T_f is a linear transformation with domain $\mathfrak{D}(f)$, I need only show that T_f is closed.

Let $\{\mathfrak{f}_n\}$ be a sequence of vectors such that there exist vectors $\mathfrak{f}$ and $\mathfrak{g}$ satisfying

$$\underset{n\to\infty}{[\lim]}\, \mathfrak{f}_n = \mathfrak{f} , \qquad (1)$$

$$\underset{n\to\infty}{[\lim]}\, T_f\, \mathfrak{f}_n = \mathfrak{g} . \qquad (2)$$

From (2), there exists a constant number K such that

$$\| T_f\, \mathfrak{f}_n\|^2 < K \qquad (n = 1, 2, \dots .) .$$

Hence, by the preceding section, $\mathfrak{f}$ belongs to $\mathfrak{D}(f)$.

Let $\mathfrak{h}$ be any vector in $\mathfrak{D}(f)$. Then, from (2),

$$\lim_{n\to\infty} (T_f\, \mathfrak{f}_n , \mathfrak{h}) = (\mathfrak{g} , \mathfrak{h}) . \qquad (3)$$

(1) Cf. Stone, *loc. cit.*, 229.

But

$$(\mathbf{T}_f \mathfrak{f}_n, \mathfrak{h}) = \int_V f(\lambda)\, d\,(\mathbf{E}(U)\,\mathfrak{f}_n, \mathfrak{h}) = \int_V f(\lambda)\, d(\mathfrak{f}_n, \mathbf{E}(U)\,\mathfrak{h}) = (\mathfrak{f}_n, \mathbf{T}_{\bar{f}}\mathfrak{h}).$$

Hence, by (1)

$$\lim_{n\to\infty} (\mathbf{T}_f \mathfrak{f}_n, \mathfrak{h}) = \lim_{n\to\infty} (\mathfrak{f}_n, \mathbf{T}_{\bar{f}}\mathfrak{h}) = (\mathfrak{f}, \mathbf{T}_{\bar{f}}\mathfrak{h}) = (\mathbf{T}_f \mathfrak{f}, \mathfrak{h}). \tag{4}$$

From (3) and (4) $\qquad (\mathfrak{g}, \mathfrak{h}) = (\mathbf{T}_f \mathfrak{f}, \mathfrak{h})$

for any $\mathfrak{h}$ in $\mathfrak{D}(f)$. Therefore, since $\mathfrak{D}(f)$ is dense in $\mathfrak{H}$, we have

$$\mathfrak{g} = \mathbf{T}_f \mathfrak{f}.$$

That is, $\mathbf{T}_f$ is closed.

Let U' be any Borel subset of V. Then, if $\mathfrak{f}$ belongs to $\mathfrak{D}(f)$, then $\mathbf{E}(U')\,\mathfrak{f}$ belongs to $\mathfrak{D}(f)$, and

$$\mathbf{T}_f \mathbf{E}(U')\,\mathfrak{f} = \mathbf{E}(U')\,\mathbf{T}_f \mathfrak{f}.\text{(1)} \tag{5}$$

Since $\quad \int_V |f(\lambda)|^2 d\,\|\mathbf{E}(U)\,\mathbf{E}(U')\,\mathfrak{f}\,\|^2 = \int_{U'} |f(\lambda)|^2 d\,\|\mathbf{E}(U)\,\mathfrak{f}\,\|^2$

is finite, $\mathbf{E}(U')\,\mathfrak{f}$ belongs to $\mathfrak{D}(f)$. And

$$\mathbf{T}_f \mathbf{E}(U')\,\mathfrak{f} = \int_V f(\lambda)\, d\mathbf{E}(U)\,\mathbf{E}(U')\,\mathfrak{f} = \int_{U'} f(\lambda)\, d\mathbf{E}(U)\,\mathfrak{f}.$$

But, by sec. 16 $\qquad \mathbf{E}(U')\,\mathbf{T}_f \mathfrak{f} = \int_{U'} f(\lambda)\, d\mathbf{E}(U)\,\mathfrak{f}.$

Hence, we have (5).

As Stone did, we can prove that $\mathbf{T}_{\bar{f}}$ is the adjoint transformation of $\mathbf{T}_f$,[(2)] and other properties of $\mathbf{T}_f$ which arise from the properties of $f(\lambda)$.[(3)]

25. Let $\mathbf{E}(U)$ be a resolution of identity, and

$$\mathbf{T}_f \mathfrak{f} = \int_V f(\lambda)\, d\mathbf{E}(U)\,\mathfrak{f}, \tag{1}$$

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(1) Cf. Stone, *loc. cit.*, 222.
(2) *Ibid.*, 229.
(3) *Ibid.*, 222, 230, 232.

then, as I have noted at the end of the preceding section, T_f is a self-adjoint transformation when $f(\lambda)$ is real almost everywhere (E), and a unitary transformation when $|f(\lambda)| = 1$ almost everywhere (E).

Now the converse problem is: Any self-adjoint or unitary transformation can be expressed in the integral form (1).

First consider the bounded self-adjoint transformation H. Since

$$(H\mathfrak{f}, \mathfrak{f}) = (\mathfrak{f}, H\mathfrak{f}),$$

$(H\mathfrak{f}, \mathfrak{f})$ is real. Let the upper and lower bounds of $(H\mathfrak{g}, \mathfrak{g})$ for all normalized vectors $\mathfrak{g}$ be M and m respectively. Then, as M. H. Stone proved,[1] corresponding to any real Baire's function $f(\lambda)$ such that

$$|f(\lambda)| \leqq K$$

for any point λ in the interval $m \leqq \lambda \leqq M$, K being a constant number, a bounded self-adjoint transformation $f(H)$ exists, and

$$||f(H)\mathfrak{f}|| \leqq K||\mathfrak{f}|| \tag{1}$$

for any vector $\mathfrak{f}$.

Denote the closed interval $[m, M]$ by I. Let $h_{(U)}(\lambda)$ be da la Vallée Poussin's characteristic function of Borel subset U of I.[2] Then $h_{(U)}(\lambda)$ is a bounded Baire's function. Hence there exists a bounded self-adjoint transformation $h_{(U)}(H)$. But, since

$$h_{(U)}(\lambda)\,h_{(U')}(\lambda) = h_{(UU')}(\lambda),$$

we have
$$h_{(U)}(H)\,h_{(U')}(H)\mathfrak{f} = h_{(UU')}(H)\mathfrak{f}.$$

When
$$U = U_1 + U_2 + \dots + U_i + \dots$$

put
$$M_i = U_1 + U_2 + \dots + U_i,$$

then, since $\{h_{(M_i)}(\lambda)\}$ is a sequence of bounded Baire's functions which converges to $h_{(U)}(\lambda)$, we have

$$[\lim_{i\to\infty}]\, h_{(M_i)}(H)\mathfrak{f} = h_{(U)}(H)\mathfrak{f}.$$ [3]

(1) Stone, *loc. cit.*, 230.
(2) Cf. Sec. 6.
(3) Stone, *loc. cit.*, 226.

That is, $$h_{(U)}(\mathbf{H})\,\mathfrak{f}\,[=]\sum_i h_{(U_i)}(\mathbf{H})\,\mathfrak{f}\,.$$

And, since $$h_{(I)}(\lambda)=1\,,$$

we have $$h_{(I)}(\mathbf{H})\,\mathfrak{f}=\mathfrak{f}\,.$$

Therefore, by sec. 15, $h_{(U)}(\mathbf{H})$ is a resolution of identity defined in the Euclidean space of one dimension R_1. Denote this by $\mathrm{E}(U)$.

Decompose I into the sum of intervals

$$I=U_1+U_2+\ldots\ldots+U_i+\ldots\ldots$$

such that the length of U_i is less than a given positive number ε. Hence, if λ_i is a point in U_i, then

$$|\lambda_i-\lambda|\leqq\varepsilon$$

for any point λ in U_i. Therefore, if we put

$$f(\lambda)=\sum_i \lambda_i h_{(U_i)}(\lambda)\,,$$

then $$|f(\lambda)-\lambda|\leqq\varepsilon$$

for any point λ in I. Hence, by (1)

$$\|\{f(\mathbf{H})-\mathbf{H}\}\,\mathfrak{f}\|\leqq\varepsilon\,\|f\|\,.$$

Hence $$\sum_i \lambda_i\, h_{(U_i)}(\mathbf{H})\,\mathfrak{f}=\sum_i \lambda_i\,\mathrm{E}(U_i)\,\mathfrak{f}$$

converges strongly to $\mathbf{H}\mathfrak{f}$. That is,

$$\mathbf{H}\,\mathfrak{f}=\int_I \lambda d\mathrm{E}(U)\,\mathfrak{f}\,.$$

Thus, we find the resolution of identity $\mathrm{E}(U)$, defined in R_1, which corresponds to the bounded self-adjoint transformation $\mathbf{H}$, and

$$\mathbf{H}\,\mathfrak{f}=\int_I \lambda d\mathrm{E}(U)\,\mathfrak{f}\,.$$

where I is the closed interval $[m, M]$.

Then by conveniently modifying the method of J. v. Neumann,[1] we find the resolution of identity $\mathrm{E}(U)$, defined in R_1, which corresponds to the unitary transformation $\mathbf{U}$, and

(1) J. v. Neumann, *Math. Ann.* **102** (1929), 115–122, 91–96, 410–416.

$$U\mathfrak{f} = \int_I e^{2\pi i\lambda}\, d\mathrm{E}(U)\,\mathfrak{f}\,,$$

where I is the semi-closed interval (0,1]. And corresponding to any self-adjoint transformation $\boldsymbol{H}$, we can find a resolution of identity $\mathrm{E}(U)$, defind in R_1, and

$$\boldsymbol{H}\mathfrak{f} = \int_{R_1} \lambda d\mathrm{E}(U)\,\mathfrak{f}$$

for all $\mathfrak{f}$, such that $\int_{R_1} \lambda^2 d\,||\,\mathrm{E}\ U)\,\mathfrak{f}\,||^2$ are finite.

Similarly, corresponding to any maximal normal transformation T, we can find a resolution of identity $\mathrm{E}(U)$, defined in R_2, and

$$\boldsymbol{T}\mathfrak{f} = \int_{R_2} \zeta d\mathrm{E}(U)\,\mathfrak{f}\ ^{(1)}$$

for all $\mathfrak{f}$, such that $\int_{R_2} |\,\zeta\,|^2 d\,||\,\mathrm{E}(U)\,\mathfrak{f}\,||^2$ are finite.

By the preceding section, when $\mathrm{E}(U)$ corresponds to $\boldsymbol{H}$,

$$\boldsymbol{H}\mathrm{E}(U)\,\mathfrak{f} = \mathrm{E}(U)\,\boldsymbol{H}\mathfrak{f}\,.$$

Put $$\mathrm{E}(U)\,\mathfrak{f} = \mathfrak{q}(U)\,,$$

then, by sec. 16 $$\boldsymbol{H}\mathfrak{q}(U) = \int_U \lambda d\mathfrak{q}(U)\,.$$

Therefore, $\mathfrak{q}(U)$ corresponds to what is called by Hellinger "a differential solution".[2]

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(1) ζ is a point function whose value is a complex number ζ at the point ζ in R_2.

(2) Cf. F. Riesz, *Les systèmes d'équations linéaires a une infinité d'inconnues*, (1913), 149; and E. Hellinger, *Crelle's Journal*, **136** (1909), 240.