Lattice Functions and Lattice Structure.

By

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G. Birkhoff(1) has proved that if in a lattice L a dimension function is defined, then L is a modular lattice. And J. v. Neumann(2) says that if in a complemented continuous lattice L a unique dimension function which has some particular properties is defined, then L is a continuous geometry. These are remarkable facts, which show that the dimension function restricts the structure of the lattice.

In the present paper I investigate this problem in a general way. Let L be a lattice, and a real valued function $\phi(a)$ be defined for all $a \in L$; thus we may say that $\phi(a)$ is a lattice function. The properties of this lattice function $\phi(a)$ may be given in the following way:

(i) $\phi(a)$ is additive when

$$\phi(a \cup b)+\phi(a \cap b)=\phi(a)+\phi(b).$$

(ii) $\phi(a)$ is completely additive when

$$\phi\left(\sum(a_i;\, i=1, 2, \ldots.)\right)=\sum_i \phi(a_i)$$

for any independent system $(a_i;\, i=1, 2, \ldots.)$.

(iii) $\phi(a)$ is non-decreasing when

$$a<b \quad \text{implies} \quad \phi(a) \leqq \phi(b).$$

(iv) $\phi(a)$ is increasing when

$$a<b \quad \text{implies} \quad \phi(a)<\phi(b).$$

I first investigate the relations between the increasing and the non-decreasing properties of the additive function $\phi(a)$ and the struc-

(1) G. Birkhoff [1], 744. The numbers in square brackets refer to the list given at the end of this paper.

(2) J. v. Neumann [1], 99.

ture of L. If $\phi(a)$ is increasing, then L is modular, and L is a metric space with respect to the distance $\delta(a, b)=\phi(a\cup b)-\phi(a\cap b)$. When $\phi(a)$ is non-decreasing, I consider the system of equivalence classes. When $\delta(a, b)=0$, we write $a\equiv b$, and denote by A_a the class of all elements x such that $x\equiv a$. Then the system $(A_a; a\in L)$ is a modular lattice in which an increasing additive lattice function is defined.

Next I investigate the relation between the complete additivity of the lattice function and the structure of the lattice. The following properties of the lattice:

$$\Pi(a_i\cup b; i=1, 2, \dots)=\Pi(a_i; i=1, 2, \dots)\cup b$$

$$\text{when}\quad a_1\geqq a_2\geqq\cdots\geqq a_i\geqq\cdots,$$

and

$$\sum(a_i\cap b; i=1, 2, \dots)=\sum(a_i; i=1, 2, \dots)\cap b$$

$$\text{when}\quad a_1\leqq a_2\leqq\cdots\leqq a_i\leqq\cdots$$

are closely connected with the complete additivity of the lattice function.

Lastly I investigate the relation between the completeness of L with respect to the metric $\delta(a, b)=\phi(a\cup b)-\phi(a\cap b)$ and the structure of L. If L is an irreducible complemented $\aleph_1$-lattice, then L is a continuous geometry when, and only when, L is complete with respect to the metric $\delta(a, b)$.

1. First I shall give axioms and definitions concerning the lattice.(1)

Let a class L of elements $a, b, c, \dots$ be *partially ordered*, that is, a relation $a<b$ (written equivalently $b>a$) holds good for certain pairs of elements of L in such a way that

(I_1) Never is $a<a$;

(I_2) $a<b$, $b<c$ together imply $a<c$.

We write $a\leqq b$ when $a<b$ or $a=b$.

If $\aleph$ is a (finite or infinite) cardinal number, then we say that L is an $\aleph$-*lattice* if the following axiom holds good:

(II_1) For every set $S\leqq L$ of power $<\aleph$ there is an element $\sum(S)$ in L which is a *least upper bound* or *join* of S, i. e.

(a) $\sum(S)\geqq a$ for every $a\in S$,

(b) $x\geqq a$ for every $a\in S$ implies $x\geqq\sum(S)$.

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(1) For details, cf. J. v. Neumann [1], 94–96; [3], 1–3; [5], 5–6.

(II_2) For every set $S \leqq L$ of power $< \aleph$ there is an element $\Pi(S)$ in L which is a *greatest lower bound* or *meet* of S, i. e.

(a) $\Pi(S) \leqq a$ for every $a \in S$,

(b) $x \leqq a$ for every $a \in S$ implies $x \leqq \Pi(S)$.

When $S=(a, b)$, we write $\sum(S)=a \cup b$, $\Pi(S)=a \cap b$. If $\aleph >$ power of L, then the $\aleph$-lattice is called a *continuous lattice*.

In an $\aleph_1$-lattice L, we can introduce a limit of the sequence $(a_i; i=1, 2, \ldots.)$ as follows:

$$\overline{\operatorname*{Lim}_{i\to\infty}} a_i = \Pi\left(\sum(a_i; i=p, p+1, \ldots.); \ p=1, 2, \ldots.\right),$$

$$\underline{\operatorname*{Lim}_{i\to\infty}} a_i = \sum\left(\Pi(a_i; i=p, p+1, \ldots.); \ p=1, 2, \ldots.\right).$$

Of course,
$$\overline{\operatorname*{Lim}_{i\to\infty}} a_i \geqq \underline{\operatorname*{Lim}_{i\to\infty}} a_i .$$

If $\overline{\operatorname*{Lim}_{i\to\infty}} a_i = \underline{\operatorname*{Lim}_{i\to\infty}} a_i = a$, then we say that $(a_i; i=1, 2, \ldots.)$ *converges to* a, and we write

$$\operatorname*{Lim}_{i\to\infty} a_i = a .$$

Especially if $a_1 \geqq a_2 \geqq \cdots \geqq a_i \geqq \cdots$, then

$$\operatorname*{Lim}_{i\to\infty} a_i = \Pi(a_i; i=1, 2, \ldots.),$$

and if $a_1 \leqq a_2 \leqq \cdots \leqq a_i \leqq \cdots$, then

$$\operatorname*{Lim}_{i\to\infty} a_i = \sum(a_i; i=1. 2, \ldots.).$$

If an $\aleph$-lattice L satisfies the following axiom, then we say that L is a *complemented* $\aleph$-lattice.

(III) For any three elements a, b, c, such that $a \leqq b \leqq c$, there exists an element x such that

$$b \cup x = c , \qquad b \cap x = a .$$

If an $\aleph$-lattice L satisfies the following axiom, then we say that L is a *modular* $\aleph$-lattice.

(IV) $\qquad a \leqq c$ implies $(a \cup b) \cap c = a \cup (b \cap c)$.

In an $\aleph$-lattice L, if there exist elements 0 and 1 such that

$$0 \leqq a \leqq 1 \quad \text{for all} \quad a \in L ,$$

then we call 0 and 1 *zero* and *unit elements* respectively. In a continuous lattice the zero and unit elements always exist, i. e.

$$0=\Pi(L), \qquad 1=\sum(L).$$

In a complemented $\aleph$-lattice with the zero element, the element x which satisfies

$$a\cup x=b, \qquad a\cap x=0,$$

is called the *inverse of a in b*. Especially when $b=1$, x is called the *inverse of a*.

If an $\aleph$-lattice L with zero and unit elements satisfies the following axiom, then we say that L is an *irreducible* $\aleph$-lattice.

(V) If a has a unique inverse, then a is either 0 or 1.

In an $\aleph$-lattice L with the zero element, let $(a_\sigma;\ \sigma\in I)$ be a subset of L of power $<\aleph$. If

$$\sum(a_\sigma;\ \sigma\in J)\cap\sum(a_\sigma;\ \sigma\in K)=0$$

for every pair of non-intersecting subsets J, K, of I, then we say that $(a_\sigma;\ \sigma\in I)$ is *independent*, and we write $(a_\sigma;\ \sigma\in I)\perp$.

2. If to any element a of an $\aleph$-lattice L there corresponds a real number $\phi(a)$, then we call $\phi(a)$ a *lattice function.*

If $\phi(a)$ satisfies the following relation

$$\phi(a\cup b)+\phi(a\cap b)=\phi(a)+\phi(b),$$

then we say that $\phi(a)$ is *additive.*

If $\qquad a<c$ implies $\phi(a)\leqq\phi(c)$,

then we say that $\phi(a)$ is *non-decreasing;* and if

$$a<c \text{ implies } \phi(a)<\phi(c),$$

then we say that $\phi(a)$ is *increasing.*

THEOREM 2·1. *If an increasing additive function $\phi(a)$ is defined in an $\aleph_0$-lattice L, then L is modular.*

PROOF.[1] When $a\leqq c$, it is evident that

$$(a\cup b)\cap c\geqq a\cup(b\cap c).$$

(1) This proof is due to G. Birkhoff [1], 744,

Now,
$$\begin{aligned}\phi[(a\cup b)\cap c]&=\phi(a\cup b)+\phi(c)-\phi(a\cup b\cup c)\\&=\phi(a)+\phi(b)-\phi(a\cap b)+\phi(c)-\phi(b\cup c)\\&=\phi(a)-\phi(a\cap b)+\phi(b\cap c)\qquad (a\leq c\rightarrow a=a\cap c)\\&=\phi(a)-\phi[a\cap(b\cap c)]+\phi(b\cap c)=\phi[a\cup(b\cap c)]\,.\end{aligned}$$

Hence, by the increasing property of $\phi(a)$, we have

$$(a\cup b)\cap c=a\cup(b\,\underset{\cap}{\cancel{\cup}}\,c).$$

LEMMA 2·1. *When a non-decreasing additive function $\phi(a)$ is defined in an $\aleph_0$-lattice L, put $\delta(a, b)=\phi(a\cup b)-\phi(a\cap b)$. Then*

$$\delta(a, c)\leqq\delta(a, b)+\delta(b, c)\,.$$

PROOF.
$$\begin{aligned}&\delta(a, b)+\delta(b, c)-\delta(a, c)\\&=\phi(a\cup b)-\phi(a\cap b)+\phi(b\cup c)-\phi(b\cap c)\\&\qquad-\phi(a\cup c)+\phi(a\cap c)\\&=2\{\phi(b)+\phi(a\cap c)-\phi(a\cap b)-\phi(b\cap c)\}\\&=2\{\phi[b\cup(a\cap c)]-\phi[(a\cap b)\cup(b\cap c)]\}\geqq 0\,,\end{aligned}$$

for
$$b\cup(a\cap c)\geqq(a\cap b)\cup(b\cap c)\,.$$

THEOREM 2·2. *If an increasing additive function $\phi(a)$ is defined in an $\aleph_0$-lattice L, then L is a metric space with the distance $\delta(a, b)$.*[(1)]

PROOF. By the definition of $\delta(a, b)$ we have

(i) $$\delta(a, b)=\delta(b, a)\,;$$

and by the increasing property of $\phi(a)$ we have

(ii) $\delta(a, b)=0$ when, and only when, $a=b$;

and by Lemma 2·1 we have

(iii) $$\delta(a, c)\geqq\delta(a, b)+\delta(b, c)\,.$$

Hence $\delta(a, b)$ defines a metric in L.

Theorems 2·1 and 2·2 show that when $\phi(a)$ is increasing, then L

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(1) J. v. Neumann introduced the metric $\delta(a, b)=D(a\cup b)-D(a\cap b)$ into continuous geometry. (Cf. J. v. Neumann [2], 106.)

is modular and metric. In the next section I shall investigate the case where $\phi(a)$ is non-decreasing.

3. Let L be a complemented $\aleph_0$-lattice with the zero element. In L a *non-decreasing* additive function $\phi(a)$ is defined.[1] When $\delta(a, b)=0$, we write $a\equiv b$.

LEMMA 3·1. *The relation $\equiv$ is reflexive, symmetric, and transitive.*

PROOF. By definition, $\equiv$ is reflexive and symmetric. By Lemma 2·1, if $a\equiv b$, $b\equiv c$, then $a\equiv c$; that is, $\equiv$ is transitive.

LEMMA 3·2. *$a\equiv b$ when, and only when, $\phi(a)=\phi(b)=\phi(a\cup b)=\phi(a\cap b)$.*

PROOF. Assume that $a\equiv b$. Then $\phi(a\cup b)=\phi(a\cap b)$. Since $a\cup b\geqq a\geqq a\cap b$, we have $\phi(a\cup b)\geqq\phi(a)\geqq\phi(a\cap b)$. Hence $\phi(a\cup b)=\phi(a)=\phi(a\cap b)$. Similarly for $\phi(b)$. The converse assertion is evident from the definition of $\delta(a, b)$.

LEMMA 3·3. *$a\equiv b$ when, and only when, there exist u, v such that $a\cup u=b\cup v$ and $\phi(u)=\phi(v)=\phi(0)$.*

PROOF. Sufficiency.

$$\delta(a\cup u, a)=\phi(a\cup u)-\phi(a)=\phi(u)-\phi(a\cap u).$$

Since $\qquad 0\leqq a\cap u\leqq u \quad\text{and}\quad \phi(0)\leqq\phi(a\cap u)\leqq\phi(u),$

we have $\qquad \phi(0)=\phi(a\cap u)=\phi(u).$

Hence $\qquad \delta(a\cup u, a)=0, \quad$ that is $\quad a\cup u\equiv a.$

Similarly $\qquad b\cup v\equiv b.\quad$ Hence, by Lemma 3·1, $\quad a\equiv b.$

Necessity. Assume that $a\equiv b$, and let u be an inverse of a in $a\cup b$, that is,

$$a\cup u=a\cup b,\qquad a\cap u=0.$$

Then $\qquad \phi(a\cup b)=\phi(a)+\phi(u)-\phi(0).$

Since, by Lemma 3·2, $\phi(a\cup b)=\phi(a)$, we have $\phi(u)=\phi(0)$. Similarly there exists v such that $a\cup v=a\cup b$ and $\phi(v)=\phi(0)$.

LEMMA 3·4. *If $a\equiv b$, $a_1\equiv b_1$, then $a\cup a_1\equiv b\cup b_1$, $a\cap a_1\equiv b\cap b_1$.*

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(1) In this case, $\psi(a)=\phi(a)-\phi(0)$ is also a non-decreasing additive function with the properties: $\psi(0)=0$, $\psi(a)\geqq 0$ for all $a\in L$. We may use $\psi(a)$ instead of $\phi(a)$.

PROOF. By Lemma 3·3, there exist u, v, u_1, v_1, such that $a \cup u = b \cup v$, $a_1 \cup u_1 = b_1 \cup v_1$, and $\phi(u) = \phi(v) = \phi(u_1) = \phi(v_1) = \phi(0)$. Then

$$a \cup a_1 \cup u \cup u_1 = b \cup b_1 \cup v \cup v_1 .$$

Since $\phi(u \cup u_1) = \phi(u) + \phi(u_1) - \phi(u \cap u_1) = \phi(0)$, and similarly $\phi(v \cup v_1) = = \phi(0)$, by Lemma 3·3 we have

$$a \cup a_1 \equiv b \cup b_1 .$$

Next, $$a \cap a_1 \leqq (a \cup u) \cap (a_1 \cup u_1) .$$

And $$\phi\{(a \cup u) \cap (a_1 \cup u_1)\} = \phi(a \cup u) + \phi(a_1 \cap u_1) - \phi\{(a \cup a_1) \cup (u \cup u_1)\}$$
$$= \phi(a) + \phi(a_1) - \phi(a \cup a_1) = \phi(a \cap a_1) .$$

Hence $$a \cap a_1 \equiv (a \cup u) \cap (a \cup u_1) .$$

Similarly $$b \cap b_1 \equiv (b \cup v) \cap (b_1 \cup v_1) .$$

Consequently $$a \cap a_1 \equiv b \cap b_1 .$$

Let A_a denote the class of all elements x such that $x \equiv a$, and let $\mathfrak{L}$ denote the class of all A_a, $a \in L$. Then, by Lemma 3·1, the system $(A_a ;\ a \in L)$ is a mutually exclusive and exhaustive partition of L into subclasses.(1) We denote the elements of $\mathfrak{L}$ by $A, B, C, \ldots.$ and give the order of elements in $\mathfrak{L}$ as follows: $A \leqq B$ means the existence of $a \in A$, $b \in B$ with $a \leqq b$. $A < B$ means $A \leqq B$, $A \neq B$.

Since $\phi(a)$ is unique for every $a \in A$, we denote this value by $\phi(A)$. If $A < B$, then $\phi(A) < \phi(B)$. That is, $\phi(A)$ is an *increasing* function defined in $\mathfrak{L}$.

LEMMA 3·5. *When $A \leqq B$, for every $a \in A$, $b \in B$, there exists u such that $a \leqq b \cup u$, $\phi(u) = \phi(0)$ and $b \cup u \in B$.*

PROOF. Since $A \leqq B$, there exist $a_1 \in A$, $b_1 \in B$ such that $a_1 \leqq b_1$. Since $a \equiv a_1$, $b \equiv b_1$, by Lemma 3·3 there exist u_1, u_2, v_1, v_2, such that

$$a \cup u_1 = a_1 \cup u_2 , \quad b \cup v_1 = b_1 \cup v_2 , \quad \phi(u_1) = \phi(u_2) = \phi(v_1) = \phi(v_2) = \phi(0) .$$

Put $u = v_1 \cup u_2$, since $\phi(v_1 \cup u_2) = \phi(0)$, $b \cup u$ belongs to B and

$$a \leqq a_1 \cup u_2 \leqq b_1 \cup u_2 \leqq b \cup v_1 \cup u_2 = b \cup u .$$

(1) J. v. Neumann has investigated similar partitions when L is a Boolean algebra. (Cf. J. v. Neumann [5], 10.)

THEOREM 3·1. *If L is a complemented $\aleph_0$-lattice with the zero element and $\phi(a)$ is a non-decreasing additive function defined in L, then $\mathfrak{L}=(A_a;\ a\in L)$ is a complemented $\aleph_0$-lattice with the zero element, and $\phi(A)$ is an increasing additive function defined in $\mathfrak{L}$.*

PROOF. (i) Since $A<A$ is impossible, by the definition of $<$, we need only prove that the relation $<$ is transitive. Let $A<B$, $B<C$; then $A\leqq B$, $B\leqq C$. Hence, by Lemma 3·5, there exist $a\in A$, $b\in B$, $c\in C$, such that $a\leqq b$, $b\leqq c$. Hence $a\leqq c$. Therefore $A\leqq C$. Since $\phi(A)<\phi(B)<\phi(C)$, we have $A<C$.

(ii) Let A and B be any two elements in $\mathfrak{L}$, with $a\in A$, $b\in B$ such that $A=A_a$, $B=A_b$. Then

$$A_{a\frown b}\leqq A_a\,,\qquad A_{a\frown b}\leqq A_b\,.$$

Suppose now that $C\leqq A_a$, $C\leqq A_b$. And let $c\in C$. Then, by Lemma 3·5, there exist a_1, b_1, such that

$$c\leqq a_1\,,\qquad c\leqq b_1\qquad\text{and}\qquad a_1\equiv a\,,\qquad b_1\equiv b\,.$$

Then $c\leqq a_1\frown b_1$, and, by Lemma 3·4, $a_1\frown b_1\equiv a\frown b$. Hence $C\leqq A_{a\frown b}$.

Consequently $$A_a\frown A_b=A_{a\frown b}\,. \tag{1}$$

Next, $$A_a\leqq A_{a\smile b}\,,\qquad A_b\leqq A_{a\smile b}\,.$$

Suppose that $A_a\leqq C$, $A_b\leqq C$. And let $c\in C$. Then, by Lemma 3·5, there exist u, v, such that $a\leqq c\smile u$, $b\leqq c\smile v$ and $\phi(u)=\phi(v)=\phi(0)$. Put $c_1=c\smile u\smile v$; then $a\leqq c_1$, $b\leqq c_1$; and since $\phi(u\smile v)=\phi(0)$, c_1 belongs to C. Hence $a\smile b\leqq c_1$, and $A_{a\smile b}\leqq C$.

Consequently we have $$A_a\smile A_b=A_{a\smile b}\,. \tag{2}$$

Thus $\mathfrak{L}$ is an $\aleph_0$-lattice.

(iii) In $\mathfrak{L}$, A_0 is the zero element. Let A, B, C, be any elements in $\mathfrak{L}$ such that $A\leqq B\leqq C$. Then, by Lemma 3·5, there exist a, b, c, such that $A=A_a$, $B=A_b$, $C=A_c$, and $a\leqq b\leqq c$. Let x be an element such that

$$b\smile x=c\,,\qquad b\frown x=a\,.$$

Then, by (1) and ~~(2)~~ (ii), $\quad A_b\smile A_x=A_c\,,\qquad A_b\frown A_x=A_a\,.$

Hence $\mathfrak{L}$ is complemented.

(iv) The increasing property of $\phi(A)$ follows from the definition of $\phi(A)$. Next, let A, B be any two elements in $\mathfrak{L}$, and let $A=A_a$, $B=A_b$. Then, by (1) and (2),

$$\phi(A\cup B)+\phi(A\cap B)=\phi(A_{a\cup b})+\phi(A_{a\cap b})=\phi(a\cup b)+\phi(a\cap b)$$
$$=\phi(a)+\phi(b)=\phi(A_a)+\phi(B_b)=\phi(A)+\phi(B)\,.$$

Hence $\phi(A)$ is additive.

From Theorem 3·1, when L is an $\aleph_0$-lattice with a *non-decreasing* additive function, we can convert L into an $\aleph_0$-lattice $\mathfrak{L}$ with an *increasing* additive function. From Theorem 2·1 and 2·2, $\mathfrak{L}$ is modular and metric. But especially when L is *distributive*, that is,

$$a\cap(b\cup c)=(a\cap b)\cup(a\cap c)\,,$$

then $\mathfrak{L}$ is also *distributive.*[1] For, let $A=A_a$, $B=A_b$, $C=A_c$. Then, by (1) and $(\cancel{2})$,
(ii)

$$A_a\cap(A_b\cup A_c)=A_a\cap A_{b\cup c}=A_{a\cap(b\cup c)}=A_{(a\cap b)\cup(a\cap c)}=A_{a\cap b}\cup A_{a\cap c}$$
$$=(A_a\cap A_b)\cup(A_a\cap A_c)\,.$$

4. Let L be an $\aleph_1$-lattice with the zero element, and $\phi(a)$ a lattice function defined in L. If, for every finite or infinite independent system $(a_i;\, i=1, 2, \ldots.)$,

$$\phi\big(\textstyle\sum(a_i;\, 1, 2, \ldots.)\big)=\sum\limits_i\phi(a_i)\,, \qquad (1)$$

then we say that $\phi(a)$ is *completely additive.*

THEOREM 4·1. *Let a lattice function $\phi(a)$ be defined in a complemented $\aleph_1$-lattice L with the zero element. Then the following conditions (α) and (β) are equivalent.*

(α) *$\phi(a)$ is completely additive.*

(β) *$\phi(a)$ is additive, that is $\phi(a\cup b)+\phi(a\cap b)=\phi(a)+\phi(b)$, and $\phi(0)=0$, and $\phi(\operatorname{Lim}_{i\to\infty} a_i)=\lim_{i\to\infty}\phi(a_i)$ when $a_1\leqq a_2\leqq\cdots\leqq a_i\leqq\cdots$.*

PROOF. (i) First assume that (α) holds good. Since $(a, 0)\perp$, we have

$$\phi(a)=\phi(a\cup 0)=\phi(a)+\phi(0)\,.$$

Hence

$$\phi(0)=0\,.$$

(1) Lebesgue measure defined for measurable sets with finite measure belongs to this case.

To prove the additivity, let c be an inverse of $a \frown b$ in a; that is,

$$(a \frown b) \smile c = a, \qquad (a \frown b) \frown c = 0.$$

Then
$$c \smile b = c \smile [(a \frown b) \smile b] = a \smile b,$$

and
$$c \frown b = (a \frown b) \frown c = 0.$$

Hence, by (1), we have $[(a\cdot b) \vee c]\cdot b = ab \rightarrow ab \vee c\cdot b = ab \rightarrow cb \leq a\cdot b \rightarrow$

$$\phi(a) = \phi(a \frown b) + \phi(c) \quad \text{and} \quad \phi(a \smile b) = \phi(c) + \phi(b).$$

Consequently
$$\phi(a \smile b) + \phi(a \frown b) = \phi(a) + \phi(b).$$

Next, when
$$a_1 \leqq a_2 \leqq \cdots \leqq a_i \leqq \cdots,$$

let a_i' be an inverse of a_i in a_{i+1}; that is,

$$a_i \smile a_i' = a_{i+1}, \qquad a_i \frown a_i' = 0.$$

Then
$$(a_1 \smile a_1' \smile a_2' \smile \cdots \smile a_i') \frown a_{i+1}' = a_{i+1} \frown a_{i+1}' = 0.$$

Hence, for any i, $(a_1, a_1', a_2', \cdots, a_i') \perp$.(1) Consequently $(a_1, a_1', a_2', \ldots, a_i', \ldots) \perp$.(2)

And
$$\operatorname*{Lim}_{i \to \infty} a_i = a_1 \smile a_1' \smile a_2' \smile \cdots \smile a_i' \smile \cdots.$$

Hence
$$\phi(\operatorname*{Lim}_{i \to \infty} a_i) = \phi(a_1) + \phi(a_1') + \cdots + \phi(a_i') + \cdots = \lim_{i \to \infty} \phi(a_i).$$

(ii) Next assume that (β) holds good. And let $(a_i; i = 1, 2, \ldots) \perp$. Put $b_n = \sum(a_i; i = 1, 2, \ldots, n)$. Then $b_1 \leqq b_2 \leqq \cdots \leqq b_n \leqq \cdots$,

and
$$\phi(b_n) = \sum_{i=1}^{n} \phi(a_i).$$

Hence
$$\phi\big(\sum(a_i; i = 1, 2, \ldots)\big) = \phi(\operatorname*{Lim}_{n \to \infty} b_n) = \lim_{n \to \infty} \phi(b_n) = \sum_{i=1}^{\infty} \phi(a_i).$$

Consequently $\phi(a)$ is completely additive.

THEOREM 4·2. *Let a complemented $\aleph_1$-lattice L with the zero element satisfy the following condition:*

$$\operatorname*{Lim}_{i \to \infty} (a_i \smile b) = (\operatorname*{Lim}_{i \to \infty} a_i) \smile b \quad \textit{when } a_1 \geqq a_2 \geqq \cdots \geqq a_i \geqq \cdots. \tag{2}$$

(1) J. v. Neumann [3], 11.
(2) J. v. Neumann [3], 12.

If $\phi(a)$ is completely additive function defined in L, then

$$\phi(\operatorname*{Lim}_{i\to\infty} a_i)=\lim_{i\to\infty}\phi(a_i) \qquad \textit{when } a_1 \geqq a_2 \geqq \cdots \geqq a_i \geqq \cdots. \quad (3)$$

Especially when $\phi(a)$ is completely additive and non-decreasing, then, for any sequence $(a_i; i=1, 2, \ldots.)$,

$$\phi(\overline{\operatorname*{Lim}_{i\to\infty}}\, a_i) \geqq \overline{\lim_{i\to\infty}}\, \phi(a_i)\,, \qquad \phi(\underline{\operatorname*{Lim}_{i\to\infty}}\, a_i) \leqq \underline{\lim_{i\to\infty}}\, \phi(a_i)\,;$$

and if $(a_i; i=1, 2, \ldots.)$ converges, then

$$\phi(\operatorname*{Lim}_{i\to\infty} a_i)=\lim_{i\to\infty}\phi(a_i)\,. \qquad (4)$$

PROOF. (i) When $a_1 \geqq a_2 \geqq \cdots \geqq a_i \geqq \cdots$, put $c=\operatorname*{Lim}_{i\to\infty} a_i$, and let a_i' be an inverse of a_i in a_{i-1}. That is,

$$a_i \cup a_i' = a_{i-1}\,, \qquad a_i \cap a_i' = 0\,.$$

Then $\quad (c \cup a_n' \cup a_{n-1}' \cup \cdots \cup a_{n-i+1}') \cap a_{n-1}' \leqq a_{n-i} \cap a_{n-1}' = 0\,.$

Hence $(c, a_n', a_{n-1}', \ldots., a_{n-i}') \perp$ for any n and i.(1) Therefore $(c, a_2', a_3', \ldots., a_n', \ldots.) \perp$.(2)

$$\begin{aligned} c \cup \sum(a_n'; n=2, 3, \ldots.) &= (\operatorname*{Lim}_{i\to\infty} a_i) \cup \sum(a_n'; n=2, 3, \ldots.) \\ &= \operatorname*{Lim}_{i\to\infty} \{a_i \cup \sum(a_n'; n=2, 3, \ldots.)\} \qquad \text{by } (2) \\ &= \operatorname*{Lim}_{i\to\infty} \{a_i \cup \sum(a_n'; n=2, 3, \ldots., i)\} = a_1\,, \end{aligned}$$

for $\qquad a_i \cup \sum(a_n'; n=2, 3, \ldots., i) = a_1 \qquad$ for all i.

Hence $\qquad \phi(a_1) = \phi(c) + \sum_{n=2}^{\infty} \phi(a_n')\,.$

Since $\qquad \phi(a_i) + \phi(a_i') = \phi(a_{i-1}) \qquad (i=2, 3, \ldots.)\,,$

we have $\quad \phi(\operatorname*{Lim}_{i\to\infty} a_i) = \phi(c) = \lim_{i\to\infty} \{\phi(a_1) - \sum_{n=2}^{i} \phi(a_n')\} = \lim_{i\to\infty} \phi(a_i)\,.$

(ii) Next, assume that $\phi(a)$ is non-decreasing, and let $(a_i; i=1, 2, \ldots.)$ be any sequence. Put

$$b_i = \sum(a_p; p=i, i+1, \ldots.)\,.$$

Then $\qquad b_1 \geqq b_2 \geqq \cdots \geqq b_i \geqq \cdots$ and $\overline{\operatorname*{Lim}_{i\to\infty}}\, a_i = \operatorname{Lim} b_i\,.$

(1) J. v. Neumann [3], 11.
(2) J. v. Neumann [3], 12.

Hence, by (3), $\phi(\overline{\operatorname*{Lim}_{i\to\infty}} a_i)=\lim_{i\to\infty}\phi(b_i)$.

By the non-decreasing property of $\phi(a)$, since $b_i \geqq a_i$, we have

$$\phi(b_i)\geqq\phi(a_i).$$

Hence $$\lim_{i\to\infty}\phi(b_i)\geqq\overline{\lim_{i\to\infty}}\phi(a_i).$$

Consequently, $$\phi(\overline{\operatorname*{Lim}_{i\to\infty}} a_i)\geqq\overline{\lim_{i\to\infty}}\phi(a_i).$$

Similarly, from (β) of Theorem 4·1, we have

$$\phi(\underline{\operatorname*{Lim}_{i\to\infty}} a_i)\leqq\underline{\lim_{i\to\infty}}\phi(a_i).$$

When $(a_i; i=1, 2, \ldots.)$ converges,

$$\overline{\lim_{i\to\infty}}\phi(a_i)\leqq\phi(\operatorname*{Lim}_{i\to\infty} a_i)\leqq\underline{\lim_{i\to\infty}}\phi(a_i).$$

Hence $$\phi(\operatorname*{Lim}_{i\to\infty} a_i)=\lim_{i\to\infty}\phi(a_i).$$

Since (3) and (4) are essential properties of the completely additive function, in order to treat such a function $\phi(a)$ in a complemented $\aleph_1$-lattice L we must assume the condition:

$$\operatorname*{Lim}_{i\to\infty}(a_i\cup b)=(\operatorname*{Lim}_{i\to\infty} a_i)\cup b \qquad \text{when} \quad a_1\geqq a_2\geqq\cdots\geqq a_i\geqq\cdots.$$

With respect to the dual condition:

$$\operatorname*{Lim}_{i\to\infty}(a_i\cap b)=(\operatorname*{Lim}_{i\to\infty} a_i)\cap b \qquad \text{when} \quad a_1\leqq a_2\leqq\cdots\leqq a_i\leqq\cdots,$$

we have the following theorem:

THEOREM 4·3. *When a completely additive increasing function $\phi(a)$ is defined in a complemented $\aleph_1$-lattice L with the zero element, then in L there obtains the following relation:*

$$\operatorname*{Lim}_{i\to\infty}(a_i\cap b)=(\operatorname*{Lim}_{i\to\infty} a_i)\cap b \qquad \textit{when} \quad a_1\leqq a_2\leqq\cdots\leqq a_i\leqq\cdots.$$

PROOF. When $a_1\leqq a_2\leqq\cdots\leqq a_i\leqq\cdots$, since $a_i\cap b\leqq(\operatorname*{Lim}_{i\to\infty} a_i)\cap b$, we have

$$\operatorname*{Lim}_{i\to\infty}(a_i\cap b)\leqq(\operatorname*{Lim}_{i\to\infty} a_i)\cap b. \tag{5}$$

From Theorem 4·1, we have

$$\phi(\operatorname*{Lim}_{i\to\infty} a_i)=\lim_{i\to\infty}\phi(a_i), \qquad \phi\big(\operatorname*{Lim}_{i\to\infty}(a_i\cap b)\big)=\lim_{i\to\infty}\phi(a_i\cap b).$$

Hence
$$\phi\big(\operatorname*{Lim}_{i\to\infty}(a_i\frown b)\big)=\lim_{i\to\infty}\{\phi(a_i)+\phi(b)+\phi(a_i\smile b)\}$$
$$=\lim_{i\to\infty}\phi(a_i)+\phi(b)+\lim_{i\to\infty}\phi(a_i\smile b)$$
$$=\phi(\operatorname*{Lim}_{i\to\infty}a_i)+\phi(b)+\phi\big(\operatorname*{Lim}_{i\to\infty}(a_i\smile b)\big)$$
$$=\phi(\operatorname*{Lim}_{i\to\infty}a_i)+\phi(b)+\phi\big((\operatorname*{Lim}_{i\to\infty}a_i)\smile b\big)=\phi\big((\operatorname*{Lim}_{i\to\infty}a_i)\frown b\big).$$

Hence, from the increasing property of $\phi(a)$, by (5) we have
$$\operatorname*{Lim}_{i\to\infty}(a_i\frown b)=(\operatorname*{Lim}_{i\to\infty}a_i)\frown b.$$

5. THEOREM 5·1.(1) *If in an $\aleph_1$-lattice L an increasing additive function $\phi(a)$ is defined, then the following two conditions* (α), (β) *are equivalent.*

(α) *L is complete with respect to the metric $\delta(a,b)=\phi(a\smile b)-\phi(a\frown b)$.*

(β) *If $a_1\geqq a_2\geqq\cdots\geqq a_i\geqq\cdots$ or $a_1\leqq a_2\leqq\cdots\leqq a_i\leqq\cdots$, then $\phi(\operatorname*{Lim}_{i\to\infty}a_i)=\lim_{i\to\infty}\phi(a_i)$.*

Especially when L is a complemented $\aleph_1$-lattice with the zero element, and $\phi(0)=0$, then (α) *and* (β) *are equivalent to the following* (γ):

(γ) *$\phi(a)$ is completely additive in L, and*
$$\operatorname*{Lim}_{i\to\infty}(a_i\smile b)=(\operatorname*{Lim}_{i\to\infty}a_i)\smile b \qquad \textit{when}\quad a_1\geqq a_2\geqq\cdots\geqq a_i\geqq\cdots.$$

PROOF. (i)(2) First assume that (α) holds good, and let
$$a_1\geqq a_2\geqq\cdots\geqq a_i\geqq\cdots.$$

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(1) From this theorem we find that the continuous geometry is complete with respect to the metric $\delta(a,b)=D(a\smile b)-D(a\frown b)$. This fact is mentioned by J. v. Neumann. (Cf. J. v. Neumann [2], 107.)

(2) The inference $(\alpha)\to(\beta)$ may be stated in a slightly different form:

THEOREM. *If, in an $\aleph_0$-lattice L with zero and unit elements, an increasing additive function $\phi(a)$ is defined, and L is complete with respect to the metric $\delta(a,b)$, then L is an $\aleph_1$-lattice; and if $a_1\geqq a_2\geqq\cdots\geqq a_i\geqq\cdots$, or $a_1\leqq a_2\leqq\cdots\leqq a_i\leqq\cdots$, then $\phi(\operatorname*{Lim}_{i\to\infty}a_i)=\lim_{i\to\infty}\phi(a_i)$.*

PROOF. Let $a_1\geqq a_2\geqq\cdots\geqq a_i\geqq\cdots$; then, as above, there exists an element a such that (3) and (4) hold good. Next, let b be any element such that $b\leqq a_i$ for all i. Then, by (3), $a_i\geqq a\smile b$ for all i. Hence $\phi(a_i)\geqq\phi(a\smile b)\geqq\phi(a)$. Consequently, by (4), $a\smile b=a$; that is, $a\geqq b$. Therefore a is effective as $\Pi(a_i;i=1,2,\ldots.)=\operatorname*{Lim}_{i\to\infty}a_i$, and, by (4), $\phi(\operatorname*{Lim}_{i\to\infty}a_i)=\lim_{i\to\infty}\phi(a_i)$. Since $\Pi(a_i;i=1,2,\ldots.)$ exists when $a_1\geqq a_2\geqq\cdots\geqq a_i\geqq\cdots$, $\Pi(a_i;i=1,2,\ldots.)$ exists for any set $(a_i;i=1,2,\ldots.)$. Similarly for the dual case.

Since $\phi(a_i)\geqq\phi(a_j)\geqq\phi(\operatorname*{Lim}_{i\to\infty}a_i)$ when $i<j$, the sequence $\big(\phi(a_i);\,i=1, 2,\dots.\big)$ converges. Hence

$$\lim_{i,j\to\infty}[\phi(a_i)-\phi(a_j)]=0\,.$$

Since $\delta(a_i, a_j)=\phi(a_i)-\phi(a_j)$ $(i<j)$, and L is complete, there exists $a\in L$, such that

$$\lim_{i\to\infty}\delta(a_i, a)=0\,. \tag{1}$$

Now, $\quad \delta(a_i, a)=\phi(a_i\cup a)-\phi(a_i\frown a)\geqq\phi(a)-\phi(a_i\frown a)\geqq 0\,,$

and we have $$\lim_{i\to\infty}\phi(a_i\frown a)=\phi(a)\,. \tag{2$'$}$$

When $i<j$, $a_i\frown a\geqq a_j\frown a$. Consequently $\big(\phi(a_i\frown a);\,i=1, 2,\dots.\big)$ is a monotone non-increasing sequence, and

$$\phi(a_i\frown a)\leqq\phi(a)\qquad\text{for all } i.$$

Hence (2) is absurd unless $\phi(a_i\frown a)=\phi(a)$ for all i, that is $a_i\frown a=a$,

and $$a_i\geqq a\qquad\text{for all } i. \tag{3}$$

Then, since $$\delta(a_i, a)=\phi(a_i)-\phi(a)\,,$$

by (1) we have $$\lim_{i\to\infty}\phi(a_i)=\phi(a)\,. \tag{4}$$

Since, from (3), $a_i\geqq\operatorname*{Lim}_{i\to\infty}a_i\geqq a$, we have $\phi(a_i)\geqq\phi(\operatorname*{Lim}_{i\to\infty}a_i)\geqq\phi(a)$.

Hence, by (4), $\quad \operatorname*{Lim}_{i\to\infty}a_i=a\,,$ and $\lim_{i\to\infty}\phi(a_i)=\phi(\operatorname*{Lim}_{i\to\infty}a_i)\,.$

Similarly we can prove the case where $a_1\leqq a_2\leqq\cdots\leqq a_i\leqq\cdots.$ Consequently (β) holds good.

(ii) Next, assume that (β) holds good. Let $(a_i;\,i=1, 2,\dots.)$ be any sequence such that $\lim_{i,j\to\infty}\delta(a_i, a_j)=0$. First I shall show that there exists a partial sequence $(a_{n_\nu};\,\nu=1, 2,\dots.)$ which converges to an element a. Next I shall prove that $\lim_{i\to\infty}\delta(a_i, a)=0$.

Since $$\delta(a_i, a_j)=\phi(a_i\cup a_j)-\phi(a_i\frown a_j)\,,$$

we have $$0\leqq\phi(a_i\cup a_j)-\phi(a_i)\leqq\delta(a_i, a_j)\,, \tag{5}$$

$$0\leqq\phi(a_i)-\phi(a_i\frown a_j)\leqq\delta(a_i, a_j)\,. \tag{6}$$

Take n_1, n_2, such that

$$n_1 < n_2 \quad \text{and} \quad \delta(a_{n_1}, a_{n_2}) \leqq \frac{1}{2}.$$

After taking $n_1, n_2, \ldots, n_p$, take n_{p+1} such that

$$n_p < n_{p+1} \quad \text{and} \quad \delta(a_{n_p}, a_{n_{p+1}}) \leqq \frac{1}{2^p}. \tag{7}$$

Now $\phi(a_{n_\nu} \cup a_{n_{\nu+1}} \cup \cdots \cup a_{n_{\nu+\mu+1}}) - \phi(a_{n_\nu} \cup a_{n_{\nu+1}} \cup \cdots \cup a_{n_{\nu+\mu}})$

$$= \phi(a_{n_{\nu+\mu+1}}) - \phi\{(a_{n_\nu} \cup a_{n_{\nu+1}} \cup \cdots \cup a_{n_{\nu+\mu}}) \cap a_{n_{\nu+\mu+1}}\}$$

$$\leqq \phi(a_{n_{\nu+\mu+1}}) - \phi\{(a_{n_{\nu+1}} \cup a_{n_{\nu+2}} \cup \cdots \cup a_{n_{\nu+\mu}}) \cap a_{n_{\nu+\mu+1}}\}$$

$$= \phi(a_{n_{\nu+1}} \cup a_{n_{\nu+2}} \cup \cdots \cup a_{n_{\nu+\mu+1}}) - \phi(a_{n_{\nu+1}} \cup a_{n_{\nu+2}} \cup \cdots \cup a_{n_{\nu+\mu}}).$$

Proceeding in this way, we have, by (5) and (7),

$$\phi(a_{n_\nu} \cup a_{n_{\nu+1}} \cup \cdots \cup a_{n_{\nu+\mu+1}}) - \phi(a_{n_\nu} \cup a_{n_{\nu+1}} \cup \cdots \cup a_{n_{\nu+\mu}})$$

$$\leqq \phi(a_{n_{\nu+\mu}} \cup a_{n_{\nu+\mu+1}}) - \phi(a_{n_{\nu+\mu}}) \leqq \frac{1}{2^{\nu+\mu}}.$$

Add these inequalities for $\mu = 0, 1, 2, \ldots, \eta$; then

$$\phi(a_{n_\nu} \cup a_{n_{\nu+1}} \cup \cdots \cup a_{n_{\nu+\eta+1}}) - \phi(a_{n_\nu}) \leqq \sum_{\mu=0}^{\eta} \frac{1}{2^{\nu+\mu}}.$$

Let $\eta \to \infty$; then, by condition (β), we have

$$\phi(a^{(\nu)}) - \phi(a_{n_\nu}) \leqq \frac{1}{2^{\nu-1}}, \tag{8}$$

where $$a^{(\nu)} = \sum(a_{n_p};\ p = \nu, \nu+1, \ldots).$$

Similarly, using (6) and (7), we have

$$\phi(a_{n_\nu}) - \phi(a_{(\nu)}) \leqq \frac{1}{2^{\nu-1}}, \tag{9}$$

where $$a_{(\nu)} = \Pi(a_{n_p};\ p = \nu, \nu+1, \ldots).$$

By (8) and (9), we have

$$\phi(a^{(\nu)}) - \phi(a_{(\nu)}) \leqq \frac{1}{2^{\nu-2}}. \tag{10}$$

Hence, by condition (β),

$$\operatorname*{Lim}_{\nu\to\infty} a^{(\nu)} = \operatorname*{Lim}_{\nu\to\infty} a_{(\nu)},$$

that is, $\operatorname{Lim}_{\nu\to\infty} a_{n_\nu}$ exists. Put $a=\operatorname{Lim}_{\nu\to\infty} a_{n_\nu}$. Since $a_{(\nu)}\leqq a\leqq a^{(\nu)}$, by (10) we have
$$\delta(a^{(\nu)}, a)=\phi(a^{(\nu)})-\phi(a)\leqq\phi(a^{(\nu)})-\phi(a_{(\nu)})\leqq\frac{1}{2^{\nu-2}}. \qquad (11)$$

For any given positive number ε, there exists an integer I such that
$$\delta(a_i, a_j)<\varepsilon \qquad \text{for} \quad i, j\geqq I. \qquad (12)$$

Let ν be such that $\frac{1}{2^{\nu-2}}\leqq\varepsilon$. If we take n_p such that $n_p\geqq I$, $p\geqq\nu$, then
$$a_{(\nu)}\leqq a_{n_p}\leqq a^{(\nu)}.$$
Hence, by (10),
$$\delta(a^{(\nu)}, a_{n_p})=\phi(a^{(\nu)})-\phi(a_{n_p})\leqq\phi(a^{(\nu)})-\phi(a_{(\nu)})\leqq\frac{1}{2^{\nu-2}}. \qquad (13)$$
By (11), (12), and (13), we have
$$\delta(a_i, a)\leqq\delta(a_i, a_{n_p})+\delta(a_{n_p}, a^{(\nu)})+\delta(a^{(\nu)}, a)<3\varepsilon,$$
when $i\geqq I$. Consequently $\lim_{i\to\infty}\delta(a_i, a)=0$.

Thus condition (α) holds good.

From (i) and (ii), (α) and (β) are equivalent.

Next we shall prove the equivalency of (β) and (γ).

(iii) Assume that (β) holds good. By Theorem 4·1, $\phi(a)$ is completely additive. When $a_1\geqq a_2\geqq\cdots\geqq a_i\geqq\cdots$, from the relation $\phi(\operatorname{Lim}_{i\to\infty} a_i)=\lim_{i\to\infty}\phi(a_i)$ we can obtain
$$\operatorname{Lim}_{i\to\infty}(a_i\cup b)=(\operatorname{Lim}_{i\to\infty} a_i)\cup b$$
by the method dual to the proof of Theorem 4.3.

Consequently, from (β), (γ) holds good.

(iv) Next assume that (γ) holds good. Then, by Theorems 4·1 and 4·2, (β) holds good.

From (iii) and (iv), (β) and (γ) are equivalent.

THEOREM 5·2. *If, in an $\aleph_1$-lattice*[1] *L, an increasing additive function $\phi(a)$ is defined, and L is complete with respect to the metric $\delta(a, b)=\phi(a\cup b)-\phi(a\cap b)$, then*

(1) We may replace "$\aleph_1$-lattice" by "$\aleph_0$-lattice with zero and unit elements"; cf. p. 97, footnote (2).

(i) *L is modular;*

(ii) *L is a continuous lattice;*

(iii) *let Ω be any Cantor ordinal number; then in a system $(a_\alpha;\ \alpha<\Omega)$,*

(a) *if $\alpha<\beta$ implies $a_\alpha \geqq a_\beta$, then*

$$\Pi\big((a_\alpha;\ \alpha<\Omega)\big)\cup b=\Pi(a_\alpha\cup b;\ \alpha<\Omega),$$

(b) *if $\alpha<\beta$ implies $a_\alpha\leqq a_\beta$, then*

$$\big(\sum(a_\alpha;\ \alpha<\Omega)\big)\cap b=\sum(a_\alpha\cap b;\ \alpha<\Omega).$$

(iv) *$\phi(a)$ is completely additive when $\phi(0)=0$.*(1)

PROOF. The modularity of L is proved in Theorem 2·1. From Theorem 5·1, if $a_1\geqq a_2\geqq\cdots\cdots\geqq a_i\geqq\cdots\cdots$, or $a_1\leqq a_2\leqq\cdots\cdots\leqq a_i\leqq\cdots\cdots$, then $\phi(\operatorname{Lim}_{i\to\infty} a_i)=\lim_{i\to\infty}\phi(a_i)$. Starting from this property, we can prove (ii) and (iii) as J. v. Neumann(2) has done, or in a dual way. Complete additivity follows from Theorems 4·1 and 5·1.

THEOREM 5·3. *If, in an irreducible*(3) *complemented $\aleph_1$-lattice*(4) *L, an increasing additive function $\phi(a)$ is defined, and L is complete with respect to the metric $\delta(a,b)=\phi(a\cup b)-\phi(a\cap b)$, then*

(i) *L is a continuous geometry;*

(ii) *$\phi(a)$ is expressed by the dimension fuction $D(a)$ of L as follows:* $\phi(0)=0$, $\phi(1)=1$

$$\phi(a)=\alpha D(a)+\beta,$$

α, β [illegible] uniquely

where α and β are real numbers;

(iii) *$\phi(a)$ has a discrete bounded range or a continuous bounded range, according as L satisfies the chain condition or not.*

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(1) Or we may say that $\phi(a)-\phi(0)$ is completely additive.

(2) J. v. Neumann [4], 164–166, Lemmas 18·5 and 18·6.

(3) Here we add a theorem which shows the relation between the irreducibility of the lattice and the uniqueness of the lattice function.

~~THEOREM.~~ *If, in an complemented $\aleph_0$-lattice L with zero and unit elements, we can define only one additive function $\phi(a)$ such that $\phi(0)=\alpha$ and $\phi(1)=\beta$, α,β being given real numbers, then L is irreducible.*

PROOF. Since $\psi(a)=\phi(a)-\phi(0)$ is additive, and $\psi(0)=0$, we can assume, without loss of generality, that $\alpha=0$. If ~~L is reducible~~, then there exist c, d such that $c\cup d=1$, $c\cap d=0$, and $x=(x\cap c)\cup(x\cap d)$ for all $x\in L$. Let p, q be any real numbers such that $p\phi(c)+q\phi(d)=\beta$. And put $\phi_1(x)=p\phi(x\cap c)+q\phi(x\cap d)$. Then $\phi_1(0)=0$, $\phi_1(1)=\beta$, and $\phi_1(a)$ is additive. Hence there are many additive functions $\phi(a)$ such that $\phi(0)=0$, $\phi(1)=\beta$. And this fact contradicts the assumpti on.

(4) We may replace "$\aleph_1$-lattice" by "$\aleph_0$-lattice with zero and unit elements"; cf. p. 97, footnote (2).

PROOF. (i) Complementariness and irreducibility are assumed. Hence, by Theorem 5·2, L satisfies all the axioms of continuous geometry.(1)

(ii) By the increasingness of $\phi(a)$,

$$\phi(0) \leqq \phi(a) \leqq \phi(1) \quad \text{for all} \quad a \in L.$$

Hence the range of $\phi(a)$ has either an upper bound or a lower bound. Therefore, by a theorem proved by J. v. Neumann,(2)

$$\phi(a) = \alpha D(a) + \beta.$$

(iii) This is evident from (ii), since the range of $D(a)$ is $\left(0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right)$, or all real numbers between 0 and 1.(3)

THEOREM 5·4. *If L is a complemented $\aleph_1$-lattice with the zero element in which a non-decreasing additive function $\phi(a)$*(4) *is defined, and*

$$\phi(\operatorname*{Lim}_{i\to\infty} a) = \lim_{i\to\infty} \phi(a_i) \tag{1}$$

when $a_1 \leqq a_2 \leqq \cdots\cdots \leqq a_i \leqq \cdots\cdots$ *or* $a_1 \geqq a_2 \geqq \cdots\cdots \geqq a_i \geqq \cdots\cdots$,

then $\mathfrak{L} = (A_a;\ a \in L)$(5) *is a complemented modular continuous lattice which satisfies* (iii) *of Theorem* 5·2, *and $\phi(A)$ is completely additive when $\phi(0) = 0$.*

PROOF. (i) From Theorem 3·1, $\mathfrak{L}$ is a complemented $\aleph_0$-lattice with the zero element, and $\phi(A)$ is an increasing additive function defined in $\mathfrak{L}$.

(ii) Now I shall prove that $\mathfrak{L}$ is an $\aleph_1$-lattice and

$$\phi(\operatorname*{Lim}_{i\to\infty} A_i) = \lim_{i\to\infty} \phi(A_i) \tag{2}$$

when $A_1 \leqq A_2 \leqq \cdots\cdots \leqq A_i \leqq \cdots\cdots$ or $A_1 \geqq A_2 \geqq \cdots\cdots \geqq A_i \geqq \cdots\cdots$.

(1) J. v. Neumann [1], 94–96; [3], 1–3.

(2) J. v. Neumann [3], 70.

(3) J. v. Neumann [1], 99; [3], 69.

(4) If we use a *completely additive* function $\phi(a)$ instead of the *additive* function, then we must assume, instead of (1), the following condition:

$$\operatorname*{Lim}_{i\to\infty} (a_i \smile b) = (\operatorname*{Lim}_{i\to\infty} a_i) \smile b \quad \text{when} \quad a_1 \geqq a_2 \geqq \cdots\cdots \geqq a_i \geqq \cdots\cdots.$$

(Cf. Theorems 4·1 and 4·2).

(5) Cf. sec. 3.

Let $A_1 \leqq A_2 \leqq \cdots \leqq A_i \leqq \cdots$. Applying Lemma 3·5 successively, we have $a_1 \leqq a_2 \leqq \cdots \leqq a_i \leqq \cdots$ such that $A_i = A_{a_i}$ for all i. Put $\operatorname*{Lim}_{i\to\infty} a_i = a$. Then $A_i \leqq A_a$ for all i. Next, let A_b be any element in L such that $A_i \leqq A_b$ for all i. Then, by Lemma 3·5, there exist $u_i (i = 1, 2, \ldots.)$, such that $a_i \leqq b \cup u_i$ and $\phi(u_i) = 0$. Hence

$$a = \operatorname*{Lim}_{i\to\infty} a_i \leqq \sum(b \cup u_i\,;\ i = 1, 2, \ldots.)\,. \tag{3}$$

If we put $b_n = \sum(b \cup u_i\,;\ i = 1, 2, \ldots., n) = b \cup \sum(u_i\,;\ i = 1, \ldots., n)$, then $b_1 \leqq b_2 \leqq \cdots \leqq b_n \leqq \cdots$ and $\operatorname*{Lim}_{i\to\infty} b_n = \sum(b \cup u_i\,;\ i = 1, 2, \ldots.)$. Since $\phi\big(\sum(u_i\,;\ i = 1, 2, \ldots., n)\big) = \phi(0)$ we have $\phi(b_n) = \phi(b)$ and $\phi(\operatorname*{Lim}_{i\to\infty} b_n) = \lim_{i\to\infty} \phi(b_n) = \phi(b)$. Since $b \leqq \operatorname*{Lim}_{i\to\infty} b_n$, we have $b \equiv \operatorname*{Lim}_{i\to\infty} b_n$, and $\operatorname*{Lim}_{i\to\infty} b_n \in A_b$. Hence, by (3), $A_a \leqq A_b$. Consequently A_a is effective as $\sum(A_i\,;\ i = 1, 2, \ldots.) = \operatorname*{Lim}_{i\to\infty} A_i$, and

$$\phi(\operatorname*{Lim}_{i\to\infty} A_i) = \phi(a) = \lim_{i\to\infty} \phi(a_i) = \lim_{i\to\infty} \phi(A_i)\,.$$

Next let $A_1 \geqq A_2 \geqq \cdots \geqq A_i \geqq \cdots$, and let $c_i (i = 1, 2, \ldots.)$ be such that $A_i = A_{c_i}$. By Lemma 3·5, there exist $u_i (i = 1, 2, \ldots.)$ such that $c_{i+1} \leqq c_i \cup u_i$ and $\phi(u_i) = \phi(0)$. Put $a_n = c_n \cup \sum(u_i\,; i = n, n+1, \ldots.)$; then, since $\phi\big(\sum(u_i\,;\ i = n, n+1, \ldots., n+m)\big) = 0$, we have, by (1), $\phi\big(\sum(u_i\,;\ i = n,\ n+1, \ldots.)\big) = 0$. Hence $a_n \in A_n$ and $a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq \cdots$. Put $\operatorname*{Lim}_{i\to\infty} a_i = a$. Then $A_i \geqq A_a$ for all i. Next, let A_b be any element in $\mathfrak{L}$ such that $A_i \geqq A_b$ for all i. By Lemma 3·5, there exist u_i $(i = 1, 2, \ldots.)$ snch that $b \leqq a_i \cup u_i$ and $\phi(u_i) = \phi(0)$. Put

$$a'_n = a_n \cup \sum(u_i\,;\ i = n, n+1, \ldots.)\,.$$

Then, as above,
$$a'_n \equiv a_n\,. \tag{4}$$

Now, $a'_1 \geqq a'_2 \geqq \cdots \geqq a'_i \geqq \cdots$; put $a' = \operatorname*{Lim}_{i\to\infty} a'_i$. Since $a'_i \geqq a_i$, we have
$$a' = \operatorname*{Lim}_{i\to\infty} a'_i \geqq \operatorname*{Lim}_{i\to\infty} a_i = a\,.$$

Since, by (1) and (4), $\phi(\operatorname*{Lim}_{i\to\infty} a'_i) = \lim_{i\to\infty} \phi(a'_i) = \lim_{i\to\infty} \phi(a_i) = \phi(\operatorname*{Lim}_{i\to\infty} a_i)$,

we have
$$a' \equiv a\,.$$

Since $a'_i \geqq b$, we have $a' = \operatorname*{Lim}_{i\to\infty} a'_i \geqq b$.

Hence $$A_a = A_{a'} \geqq A_b .$$

Consequently A_a is effective as $\Pi(A_i\,;\ i=1, 2, \ldots.) = \operatorname*{Lim}_{i\to\infty} A_i$, and

$$\phi(\operatorname*{Lim}_{i\to\infty} A_i) = \phi(\lim_{i\to\infty} a_i) = \lim_{i\to\infty} \phi(a_i) = \lim_{i\to\infty} \phi(A_i) .$$

Since $\mathfrak{L}$ is an $\aleph_0$-lattice, and $\operatorname*{Lim}_{i\to\infty} A_i$ exists when $A_1 \leqq A_2 \leqq \cdots\cdot \leqq A_i \leqq \cdots\cdot$ or $A_1 \geqq A_2 \geqq \cdots\cdot \geqq A_i \geqq \cdots\cdot$, $\mathfrak{L}$ is an $\aleph_1$-lattice.

(iii) Since (2) holds good, by Theorem 5·1 $\mathfrak{L}$ is complete with respect to the metric $\delta(A, B) = \phi(A \cup B) - \phi(A \cap B)$. Hence, by Theorem 5·2, $\mathfrak{L}$ is a modular continuous lattice which satisfies (iii) of Theorem 5·2, and $\phi(A)$ is completely additive when $\phi(A_0) = 0$, that is, $\phi(0) = 0$.

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