Logical Structures of Orthogonal Systems in Hilbert Space.

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In order to remove difficulties in the treatment of continuous spectrum I have introduced two kinds of orthogonal systems in Hilbert space, viz. the orthogonal system of closed linear manifolds $\{\mathfrak{M}_U\}$, and the orthogonal system of elements $\{\mathfrak{q}(U)\}$, which both have set indices U.(1) In the present paper I investigate the structures of these orthogonal systems in terms of the lattice theory.

If we consider the manifold implication as the inclusion in the definition of lattice, $\{\mathfrak{M}_U\}$ is a complemented distributive lattice. And in $\{\mathfrak{M}_U\}$ the manifold calculations obey the same laws as in the set calculations; for example,

$$\mathfrak{M}_U \geqq \mathfrak{M}_{U'} \qquad \text{when} \qquad U \geqq U',$$

$$\mathfrak{M}_U \mathfrak{M}_{U'} = \mathfrak{M}_{UU'}, \qquad \mathfrak{M}_U \oplus \mathfrak{M}_{U'} = \mathfrak{M}_{U \dotplus U'}.$$

Let $\mathfrak{f}$ and $\mathfrak{g}$ be any elements in $\mathfrak{H}$, when $(\mathfrak{f}, \mathfrak{g}) = \|\mathfrak{f}\|^2$, we write, as v. Sz. Nagy,(2) $\mathfrak{f} \prec \mathfrak{g}$. If we use this "$\prec$" as the inclusion in the lattice theory, then $\{\mathfrak{q}(U)\}$ is a complemented distributive lattice. And if we denote the meet and join of $\mathfrak{q}(U), \mathfrak{q}(U')$ by $\mathfrak{q}(U) \cdot \mathfrak{q}(U')$ and $\mathfrak{q}(U) \dotplus \mathfrak{q}(U')$ respectively, we have the following relations similar to the set calculations:

$$\mathfrak{q}(U) \succ \mathfrak{q}(U') \qquad \text{when} \qquad U \geqq U',$$

(1) F. Maeda, this Journal, **4** (1934), 57-91; **6** (1936), 115-137; **7** (1937), 103-114, 191-213.

(2) B. v. Sz. Nagy, „Über die Gesamtheit der characteristischen Funktionen im Hilbertischen Funktionenraum", Acta Szeged, **8** (1937), 167. When $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, $\mathfrak{q}(U)$ is represented by $\sigma(EU)$, $\sigma(U)$ being $\|\mathfrak{q}(U)\|^2$. Since $\sigma(EU)$ is the integral of the characteristic function of U with respect to $\sigma(E)$, the conditions obtained by Nagy are nothing but the condition for a system of elements to be an orthogonal system of the form $\{\mathfrak{q}(U)\}$. But on the lattice theory Nagy's conditions cannot be used.

$$\mathfrak{q}(U)\cdot\mathfrak{q}(U')=\mathfrak{q}(UU'), \qquad \mathfrak{q}(U)\dot{+}\mathfrak{q}(U')=\mathfrak{q}(U\dot{+}U').$$

From the fact that $\{\mathfrak{M}_U\}$ and $\{\mathfrak{q}(U)\}$ are complemented distributive lattice we can infer the conditions required in order that any system of closed linear manifolds $\{\mathfrak{M}\}$, and any system of elements $\{\mathfrak{f}\}$, shall be orthogonal systems with set indices. In the present paper I investigate these problems, first where the domain of the set index U is a field (Körper), and next in the more-generalized case.

Definition of Lattice.

1. We shall consider a class L of elements (undefined entities), which will be denoted by $a, b, c, \ldots\ldots$ L satisfies the following properties:

I. L is a *lattice;* that is,

I_1. In L, a relation of inclusion $a\supset b$ is defined, such that

(i) $a\supset a$.

(ii) From $a\supset b$, $b\supset c$ it follows that $a\supset c$.

Remark.—1°. We define equality $a=b$ in L as the simultaneous exsistence of the relations

$$a\supset b, \qquad b\supset a.$$

The class which satisfies I_1 is usually called a *partially ordered system.*

I_2. For every couple of a, b there exists an element $d=a\frown b$, the *meet* or *greatest lower bound* of a and b, having the property that

$$d\subset a,^{(1)} \qquad d\subset b,$$

and $d_1\subset d$ for every other d_1 having the same property.

I_3. For every couple of a, b there exists an element $s=a\smile b$, the *join* or *least upper bound* of a and b, having the property that

$$s\supset a, \qquad s\supset b,$$

and $s_1\supset s$ for every other s_1 with the same property.

Remark.—2°. The definitions of meet and join show that they are uniquely defined and satisfy the following relations:

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(1) $d\subset a$ means $a\supset d$.

$$a \frown b = b \frown a, \qquad a \smile b = b \smile a,$$
$$a \frown a = a, \qquad a \smile a = a,$$
$$a \frown (b \frown c) = (a \frown b) \frown c, \qquad a \smile (b \smile c) = (a \smile b) \smile c,$$
$$a \smile (a \frown b) = a, \qquad a \frown (a \smile b) = a.$$

3° The relations $a \supset b$, $a \frown b = b$ and $a \smile b = a$ are equivalent.

II. L fulfils the *distributive axiom*, that is,

$$a \frown (b \smile c) = (a \frown b) \smile (a \frown c).$$

Remark.—4°. From I and II, we have

$$a \smile (b \frown c) = (a \smile b) \frown (a \smile c).$$

III. L is *complemented*, that is:

III$_1$. There exists an element 0, *zero element*, in L such that $a \supset 0$ for every element a in L; and there exists an element e, *unit element*, in L such that $a \subset e$ for every element a in L.

Remark.—5°. The zero element and the unit element are unique.

III$_2$. For every element a in L, there exists an element a', the *complement* of a, statisfying

$$a \smile a' = e, \qquad a \frown a' = 0.$$

Remark.—6°. The complement a' of a is unique, and $(a')' = a$.

7°. $(a \frown b)' = a' \smile b'$ and $(a \smile b)' = a' \frown b'$.

8°. $a \subset b$ implies $a' \supset b'$.

9°. When $a \subset b$, the simultaneous equations

$$x \smile a = b, \qquad x \frown a = 0$$

have a unique solution $b \frown a'$.

We call the class L which satisfies the axioms I, II, III, the *complemented distributive lattice*, or *Boolean algebra*.[1]

2. For example, let $\mathfrak{K}$ be a field (Körper) of subsets of an abstract space V, which contains V itself. If we consider the set im-

(1) For details, cf. M. H. Stone, "Postulates for Boolean Algebras and Generalized Boolean Algebras," American Journal of Math., **57** (1935), 703–732; and G. Birkhoff and J. v. Neumann, "The Logic of Quantum Mechanics," Annals of Math., **37** (1936), 827–831.

plication as the inclusion in the definition of lattice, then $\mathfrak{K}$ *is a complemented distributive lattice,* in which the meet and join means the product (common part) and sum of sets, the unit element is the space V, the zero element is the empty set, the complement is the complementary set with respect to V.

Orthogonal Systems of Closed Linear Manifolds.

3. Let $\{\mathfrak{M}_U\}$ be a system of closed linear manifolds in an abstract Hilbert space $\mathfrak{H}$,[1] whose index U is the set in $\mathfrak{K}$, which is defined in sec. 2.[2] When $\{\mathfrak{M}_U\}$ satisfies the following conditions, then, as I have said in a previous paper,[3] $\{\mathfrak{M}_U\}$ is an *orthogonal system of closed linear manifolds.*

(α) $\mathfrak{M}_U \perp \mathfrak{M}_{U'}$ when $UU'=0$,

(β) $\mathfrak{M}_U = \mathfrak{M}_{U_1} \oplus \mathfrak{M}_{U_2} \oplus \cdots\cdots \oplus \mathfrak{M}_{U_n} \oplus \cdots\cdots$[4]

for any decomposition $U = U_1 + U_2 + \cdots\cdots + U_n + \cdots\cdots$[5] in $\mathfrak{K}$.

Let U and U' be any two sets in $\mathfrak{K}$. Put

$$U = UU' + U_1, \qquad U' = UU' + U_2,$$

then by (β) $\quad \mathfrak{M}_U = \mathfrak{M}_{UU'} \oplus \mathfrak{M}_{U_1}, \qquad \mathfrak{M}_{U'} = \mathfrak{M}_{UU'} \oplus \mathfrak{M}_{U_2}.$

Since $U_1 U_2 = 0$, by (α) we have

$$\mathfrak{M}_{UU'} = \mathfrak{M}_U \mathfrak{M}_{U'}.$$

And since $U \dot{+} U' = UU' + U_1 + U_2$, we have

$$\mathfrak{M}_{U \dot{+} U'} = \mathfrak{M}_U \oplus \mathfrak{M}_{U'}.$$

When $U \geqq U'$, we have

$$\mathfrak{M}_U \geqq \mathfrak{M}_{U'}.$$

And when $U' = V - U$, we have

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(1) $\mathfrak{H}$ may be non-separable.

(2) When $\mathfrak{K}$ is a σ-field, we can easily characterize the logical structures of $\{\mathfrak{M}_U\}$, by considering the meet and join of enumerable infinite elements in the definition of the lattice. Similarly for $\{\mathfrak{q}(U)\}$ in sec. 5.

(3) F. Maeda, " Indices of the Orthogonal Systems in Non-Separable Hilbert Space," this Journal, **7** (1937), 111.

(4) This means the closed linear sum.

(5) $U_1 \dot{+} U_2$ means the sum of sets U_1 and U_2. Especially when $U_1 U_2 = 0$, we write $U_1 + U_2$.

$$\mathfrak{M}_{U'} = \mathfrak{M}_V \ominus \mathfrak{M}_U .$$[1]

Thus we see that in $\{\mathfrak{M}_U\}$ the manifold calculations obey the same laws as the set calculations in $\mathfrak{K}$. In the set calculations, the distributive identity $U_1(U_2 \dotplus U_3) = U_1U_2 \dotplus U_1U_3$ holds good; hence in $\{\mathfrak{M}_U\}$ also the distributive identity

$$\mathfrak{M}_{U_1}(\mathfrak{M}_{U_2} \oplus \mathfrak{M}_{U_3}) = \mathfrak{M}_{U_1}\mathfrak{M}_{U_2} \oplus \mathfrak{M}_{U_1}\mathfrak{M}_{U_3}$$

holds good.

Consequently, if we consider the manifold implication as the inclusion in the definition of lattice, $\{\mathfrak{M}_U\}$ *is a complemented distributive lattice*, in which the unit element is $\mathfrak{M}_V$, and it is isomorphic with the field $\mathfrak{K}$ of the set U.[2]

4. Now we proceed to the converse problem. We have the following theorem:

Let $\{\mathfrak{M}\}$ be a system of closed linear manifolds in $\mathfrak{H}$, which satisfy the following conditions:

(i) *$\{\mathfrak{M}\}$ is a complemented distributive lattice, where the inclusion in the lattice theory means the manifold implication.*

(ii) *When $\mathfrak{M}$ belongs to $\{\mathfrak{M}\}$, then $\mathfrak{E} \ominus \mathfrak{M}$ belongs to $\{\mathfrak{M}\}$, $\mathfrak{E}$ being the unit element in $\{\mathfrak{M}\}$.*[3]

Then $\{\mathfrak{M}\}$ is an orthogonal system with set indices.

1°. From (i), we can easily see that the meet and join of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ in the lattice theory are nothing but $\mathfrak{M}_1\mathfrak{M}_2$ and $\mathfrak{M}_1 \oplus \mathfrak{M}_2$.

2°. Denote $\mathfrak{E} \ominus \mathfrak{M}$ by $\mathfrak{M}'$, then

$$\mathfrak{M} \oplus \mathfrak{M}' = \mathfrak{E}, \qquad \mathfrak{M}\mathfrak{M}' = 0 .$$

Since, by Remark 6° in sec. 1, the complement is unique, $\mathfrak{M}'$ is the complement of $\mathfrak{M}$.

3°. By Remark 9° in sec. 1, when $\mathfrak{M}_1 \leqq \mathfrak{M}_2$,

(1) This means the orthogonal complement of $\mathfrak{M}_U$ with respect to $\mathfrak{M}_V$.

(2) Therefore, $\{\mathfrak{M}_U\}$ is isomorphic with the propositional calculus. I have already obtained the relation between $\{\mathfrak{M}_U\}$ and the experimental propositions. Cf. F. Maeda, "Mathematical Foundations of Quantum Mechanics," this Journal, **7** (1937), 200–201.

(3) We cannot eliminate the condition (ii). For example, let $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$ be three elements in $\mathfrak{H}$, which are linearly independent, but no two of which are orthogonal. Denote by $\mathfrak{M}_i, \mathfrak{M}_{ij}, \mathfrak{M}_{ijk}$ the closed linear manifolds determined by $\mathfrak{f}_i, (\mathfrak{f}_i, \mathfrak{f}_j)$ and $(\mathfrak{f}_i, \mathfrak{f}_j, \mathfrak{f}_k)$ respectively. Then the system formed by $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3, \mathfrak{M}_{23}, \mathfrak{M}_{31}, \mathfrak{M}_{12}, \mathfrak{M}_{123}$ and the null-manifold is a complemented distributive lattice. But it is not an orthogonal system.

$$\mathfrak{M} \oplus \mathfrak{M}_1 = \mathfrak{M}_2, \qquad \mathfrak{M}\mathfrak{M}_1 = 0$$

have a unique solution $\mathfrak{M}_2\mathfrak{M}_1' = \mathfrak{M}_2(\mathfrak{E} \ominus \mathfrak{M}_1) = \mathfrak{M}_2 \ominus \mathfrak{M}_1$.

4°. Hence in $\{\mathfrak{M}\}$, when $\mathfrak{M}_1 \leqq \mathfrak{M}_2$, we can always write

$$\mathfrak{M}_2 = \mathfrak{M}_1 \oplus \mathfrak{M}_3, \qquad \mathfrak{M}_1 \perp \mathfrak{M}_3,$$

and $\mathfrak{M}_3 \in \{\mathfrak{M}\}$.

5°. When $\mathfrak{M}_1\mathfrak{M}_2 = 0$, then $\mathfrak{M}_1 \perp \mathfrak{M}_2$. For, put $\mathfrak{M}_3 = \mathfrak{M}_1 \oplus \mathfrak{M}_2$; then, since $\mathfrak{M}_3 \geqq \mathfrak{M}_1$, by 3° it must follow that $\mathfrak{M}_2 = \mathfrak{M}_3 \ominus \mathfrak{M}_1$. Hence $\mathfrak{M}_1 \perp \mathfrak{M}_2$.

6°. For any two manifolds $\mathfrak{M}_1, \mathfrak{M}_2$ in $\{\mathfrak{M}\}$, by 4°, we can write

$$\mathfrak{M}_1 = \mathfrak{M}_1\mathfrak{M}_2 \oplus \mathfrak{M}_3, \qquad \mathfrak{M}_1\mathfrak{M}_2 \perp \mathfrak{M}_3,$$

$$\mathfrak{M}_2 = \mathfrak{M}_1\mathfrak{M}_2 \oplus \mathfrak{M}_4, \qquad \mathfrak{M}_1\mathfrak{M}_2 \perp \mathfrak{M}_4.$$

Since $\mathfrak{M}_3\mathfrak{M}_4 = 0$, by 5° $\mathfrak{M}_3 \perp \mathfrak{M}_4$.

7°. Any two manifolds $\mathfrak{M}_1, \mathfrak{M}_2$ in $\{\mathfrak{M}\}$ are permutable; that is, if we denote the projections on $\mathfrak{M}_1$ and $\mathfrak{M}_2$ by $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ respectively, then $\boldsymbol{P}_1\boldsymbol{P}_2 = \boldsymbol{P}_2\boldsymbol{P}_1$. For, by 6°, we have

$$\boldsymbol{P}_1\mathfrak{f} = \boldsymbol{P}_{12}\mathfrak{f} + \boldsymbol{P}_3\mathfrak{f}$$

for any element $\mathfrak{f}$ in $\mathfrak{H}$, where $\boldsymbol{P}_{12}$ is the projection on $\mathfrak{M}_1\mathfrak{M}_2$. Since $\boldsymbol{P}_{12}\mathfrak{f} \in \mathfrak{M}_2$, $\boldsymbol{P}_3\mathfrak{f} \perp \mathfrak{M}_2$, we have $\boldsymbol{P}_2\boldsymbol{P}_1\mathfrak{f} = \boldsymbol{P}_{12}\mathfrak{f}$. Similarly $\boldsymbol{P}_1\boldsymbol{P}_2\mathfrak{f} = \boldsymbol{P}_{12}\mathfrak{f}$. Hence $\boldsymbol{P}_2\boldsymbol{P}_1 = \boldsymbol{P}_1\boldsymbol{P}_2$.

8°. Let $\boldsymbol{P}$ be the projection on $\mathfrak{M}$. Then we have a system of projections $\{\boldsymbol{P}\}$ corresponding to $\{\mathfrak{M}\}$. And $\{\boldsymbol{P}\}$ is a subset of an Abel's ring. Hence, by the theorem of J. v. Neumann, any projection $\boldsymbol{P}$ in $\{\boldsymbol{P}\}$ is expressed in the form

$$\boldsymbol{P} = \int_V f(\lambda)\boldsymbol{E}(dU),$$

where $\boldsymbol{E}(U)$ is a resolution of identity corresponding to a self-adjoint operator $\boldsymbol{A}$;[1] that is, $\boldsymbol{E}(U)$ is a projection which satisfies the following conditions:[2]

(α') $\boldsymbol{E}(U)\boldsymbol{E}(U') = 0$ when $UU' = 0$,

(β') $\boldsymbol{E}(U) = \boldsymbol{E}(U_1) + \boldsymbol{E}(U_2) + \cdots\cdots$ when $U = U_1 + U_2 + \cdots\cdots$,

(γ') $\boldsymbol{E}(V) = \boldsymbol{1}$ where V is a finite interval in $(-\infty, +\infty)$,

and
$$\boldsymbol{A} = \int_V \lambda \boldsymbol{E}(dU).$$

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(1) J. v. Neumann, „Über Funktionen von Funktionaloperatoren", Annals of Math., 32 (1931), 214.

(2) F. Maeda, this Journal, **4** (1934), 78, 91; **7** (1937), 111.

Since $\boldsymbol{P}$ is a projection, the functional value of $f(\lambda)$ is 1 or 0;[1] that is, $f(\lambda)$ is the characteristic function of a set U in V. Consequently,

$$\boldsymbol{P}=\boldsymbol{E}(U).$$

Denote the range of $\boldsymbol{P}=\boldsymbol{E}(U)$ by $\mathfrak{M}_U$; then we can attribute to any closed linear manifold in $\{\mathfrak{M}\}$ the set index U, such that $\{\mathfrak{M}_U\}$ satisfy (α), (β) in sec. 3.

Orthogonal Systems of Elements

5. Let $\{\mathfrak{q}(U)\}$ be a system of elements in $\mathfrak{H}$, whose index is the set U in $\mathfrak{K}$, which is defined in sec. 2. When $\{\mathfrak{q}(U)\}$ satisfies the following conditions, then, as I have said in previous papers,[2] $\{\mathfrak{q}(U)\}$ is an *orthogonal system of elements.*

(α) $\big(\mathfrak{q}(U), \mathfrak{q}(U')\big)=0$ when $UU'=0$,

(β) $\mathfrak{q}(U)=\mathfrak{q}(U_1)+\mathfrak{q}(U_2)+\cdots\cdots+\mathfrak{q}(U_n)+\cdots\cdots$[3]

for any decomposition $U=U_1+U_2+\cdots\cdots+U_n+\cdots\cdots$ in $\mathfrak{K}$.

Let U and U' be any two sets in $\mathfrak{K}$. Put

$$U=UU'+U_1, \qquad U'=UU'+U_2;$$

then, by (β) $\quad \mathfrak{q}(U)=\mathfrak{q}(UU')+\mathfrak{q}(U_1), \qquad \mathfrak{q}(U')=\mathfrak{q}(UU')+\mathfrak{q}(U_2),$

Since $U_1U_2=0$, by (α) we have

$$\big(\mathfrak{q}(U), \mathfrak{q}(U')\big)=\sigma(UU'),$$

where $\sigma(U)=\|\mathfrak{q}(U)\|^2$.

6. In order to investigate the logical structure of $\{\mathfrak{q}(U)\}$ we introduce the following conception, already used by v. Sz. Nagy.[4] Let $\mathfrak{f}, \mathfrak{g}$ be any elements in $\mathfrak{H}$. When $(\mathfrak{f}, \mathfrak{g})=\|\mathfrak{f}\|^2$, we write

$$\mathfrak{f}<\mathfrak{g}.$$

Of course,

$$\mathfrak{f}<\mathfrak{f}.$$

When $\mathfrak{f}<\mathfrak{g}$, $\qquad (\mathfrak{f}, \mathfrak{f}-\mathfrak{g})=\|\mathfrak{f}\|^2-(\mathfrak{f}, \mathfrak{g})=0.$

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(1) Almost everywhere in some sense. Cf. M. H. Stone, *Linear Transformations in Hilbert Space*, (1932), 232.

(2) F. Maeda, this Journal, **4** (1934), 70; **6** (1936), 118–119; **7** (1937), 107–108.

(3) This means strong convergence.

(4) B. v. Sz. Nagy, loc. cit.

Hence $$\mathfrak{g}=\mathfrak{f}+(\mathfrak{g}-\mathfrak{f}) \quad \text{and} \quad \mathfrak{f}\perp\mathfrak{g}-\mathfrak{f}.$$

Therefore $$\|\mathfrak{g}\|^2=\|\mathfrak{f}\|^2+\|\mathfrak{g}-\mathfrak{f}\|^2. \tag{1}$$

When $\mathfrak{f}\prec\mathfrak{g}$ and $\mathfrak{g}\prec\mathfrak{f}$ simultaneously, since $\|\mathfrak{f}\|^2=\|\mathfrak{g}\|^2$, from (1) we have $\mathfrak{f}=\mathfrak{g}$.

In general, $\mathfrak{f}\prec\mathfrak{g}, \mathfrak{g}\prec\mathfrak{h}$ do not imply $\mathfrak{f}\prec\mathfrak{h}$. For example, $\mathfrak{f}\not\perp\mathfrak{g}$, $\mathfrak{g}\not\perp\mathfrak{h}$ and $\mathfrak{h}=a\mathfrak{f}+b\mathfrak{g}$. Consequently, we cannot say that every system of elements in $\mathfrak{H}$ is a partially ordered system.

But we can easily see that $\{\mathfrak{q}(U)\}$ is a partially ordered system. For since $\big(\mathfrak{q}(U), \mathfrak{q}(U')\big)=\sigma(UU'), \mathfrak{q}(U)\succ\mathfrak{q}(U')$ when, and only when, $U\geqq U'$ almost everywhere (σ).

Hence if we denote the meet and join of $\mathfrak{q}(U), \mathfrak{q}(U')$ by $\mathfrak{q}(U)\cdot\mathfrak{q}(U')$ and $\mathfrak{q}(U)\dot{+}\mathfrak{q}(U')$ respectively, then it is evident that

$$\mathfrak{q}(U)\cdot\mathfrak{q}(U')=\mathfrak{q}(UU'), \qquad \mathfrak{q}(U)\dot{+}\mathfrak{q}(U')=\mathfrak{q}(U\dot{+}U').$$

Thus the calculations of $\mathfrak{q}(U)$ obey the same laws as the set calculations. Hence $\{\mathfrak{q}(U)\}$ *is a complemented distributive lattice*, where the unit element is $\mathfrak{q}(V)$, and the complement of $\mathfrak{q}(U)$ is $\mathfrak{q}(V-U)$.

7. Next we proceed to the converse problem. We have the following theorem:

Let $\{\mathfrak{f}\}$ *be a system of elements in* $\mathfrak{H}$ *which satisfies the following conditions:*

(i) $\{\mathfrak{f}\}$ *is a complemented distributive lattice, where the inclusion in the lattice theory means "* $\prec$ *" defined in sec. 6.*

(ii) *When* $\mathfrak{f}$ *belongs to* $\{\mathfrak{f}\}$, *then* $\mathfrak{e}-\mathfrak{f}$ *belongs to* $\{\mathfrak{f}\}$, $\mathfrak{e}$ *being the unit element in* $\{\mathfrak{f}\}$.

Then $\{\mathfrak{f}\}$ *is an orthogonal system with set indices.*

1°. In what follows, $\mathfrak{f}, \mathfrak{g}$ means the elements in $\{\mathfrak{f}\}$. As in sec. 6, we denote the meet and join of $\mathfrak{f}$ and $\mathfrak{g}$ by $\mathfrak{f}\cdot\mathfrak{g}$ and $\mathfrak{f}\dot{+}\mathfrak{g}$.

2°. $\mathfrak{f}\dot{+}\mathfrak{g}=\mathfrak{f}+\mathfrak{g}$ when, and only when, $\mathfrak{f}\perp\mathfrak{g}$. For, when $\mathfrak{f}\perp\mathfrak{g}$, $(\mathfrak{f}+\mathfrak{g}, \mathfrak{f})=\|\mathfrak{f}\|^2$. Hence $\mathfrak{f}\prec\mathfrak{f}+\mathfrak{g}$. Similarly $\mathfrak{g}\prec\mathfrak{f}+\mathfrak{g}$. Therefore $\mathfrak{f}\dot{+}\mathfrak{g}\prec\mathfrak{f}+\mathfrak{g}$. And

$$(\mathfrak{f}+\mathfrak{g}, \mathfrak{f}\dot{+}\mathfrak{g})=(\mathfrak{f}, \mathfrak{f}\dot{+}\mathfrak{g})+(\mathfrak{g}, \mathfrak{f}\dot{+}\mathfrak{g})=\|\mathfrak{f}\|^2+\|\mathfrak{g}\|^2=\|\mathfrak{f}+\mathfrak{g}\|^2.$$

Hence $\mathfrak{f}\dot{+}\mathfrak{g}\succ\mathfrak{f}+\mathfrak{g}$. Consequently we have $\mathfrak{f}\dot{+}\mathfrak{g}=\mathfrak{f}+\mathfrak{g}$.

Next, when $\mathfrak{f}\dotplus\mathfrak{g}=\mathfrak{f}+\mathfrak{g}$, we have $(\mathfrak{f},\mathfrak{f}\dotplus\mathfrak{g})=(\mathfrak{f},\mathfrak{f}+\mathfrak{g})$; that is $\|\mathfrak{f}\|^2=\|\mathfrak{f}\|^2+(\mathfrak{f},\mathfrak{g})$. Consequently $(\mathfrak{f},\mathfrak{g})=0$.

3°. When $\mathfrak{f}\perp\mathfrak{g}$, then $\mathfrak{f}\cdot\mathfrak{g}=0$. For, let $\mathfrak{h}$ be any element in $\{\mathfrak{f}\}$, such that $\mathfrak{f}>\mathfrak{h}, \mathfrak{g}>\mathfrak{h}$. Put $\mathfrak{k}=\mathfrak{f}+\mathfrak{g}$; then, by 2°, $\mathfrak{k}=\mathfrak{f}\dotplus\mathfrak{g}$. Hence $\mathfrak{k}>\mathfrak{f}>\mathfrak{h}$. Since $\|\mathfrak{h}\|^2=(\mathfrak{k},\mathfrak{h})=(\mathfrak{f},\mathfrak{h})+(\mathfrak{g},\mathfrak{h})=\|\mathfrak{h}\|^2+\|\mathfrak{h}\|^2$, we have $\mathfrak{h}=0$. Consequently $\mathfrak{f}\cdot\mathfrak{g}=0$.

4°. Denote $\mathfrak{e}-\mathfrak{f}$ by $\mathfrak{f}'$. Then, since $\mathfrak{f}\perp\mathfrak{f}'$, by 2° and 3° we have

$$\mathfrak{f}'\dotplus\mathfrak{f}=\mathfrak{e}, \qquad \mathfrak{f}'\cdot\mathfrak{f}=0.$$

But, by Remark 6° in sec 1, the complement is unique, $\mathfrak{f}'$ is the complement of $\mathfrak{f}$.

5°. When $\mathfrak{f}>\mathfrak{h}_1, \mathfrak{f}'>\mathfrak{h}_2$, then $\mathfrak{h}_1\perp\mathfrak{h}_2$. For, since

$$(\mathfrak{h}_1,\mathfrak{f}')=(\mathfrak{h}_1,\mathfrak{e}-\mathfrak{f})=(\mathfrak{h}_1,\mathfrak{e})-(\mathfrak{h}_1,\mathfrak{f})=\|\mathfrak{h}_1\|^2-\|\mathfrak{h}_1\|^2=0,$$

by 2°, $\mathfrak{h}_1+\mathfrak{f}'=\mathfrak{h}_1\dotplus\mathfrak{f}'$ and $\mathfrak{h}_1+\mathfrak{f}'>\mathfrak{f}'>\mathfrak{h}_2$. Hence

$$(\mathfrak{h}_1,\mathfrak{h}_2)=(\mathfrak{h}_1+\mathfrak{f}',\mathfrak{h}_2)-(\mathfrak{f}',\mathfrak{h}_2)=\|\mathfrak{h}_2\|^2-\|\mathfrak{h}_2\|^2=0.$$

6°. Take an element $\mathfrak{f}$ in $\{\mathfrak{f}\}$. Denote by $\mathfrak{M}_\mathfrak{f}$ the closed linear manifold determined by all the elements $\mathfrak{g}$ in $\{\mathfrak{f}\}$ such that $\mathfrak{g}<\mathfrak{f}$. It is evident that if $\mathfrak{f}_1<\mathfrak{f}_2$, then $\mathfrak{M}_{\mathfrak{f}_1}\leqq\mathfrak{M}_{\mathfrak{f}_2}$.

7°. Since $\mathfrak{f}\cdot\mathfrak{g}<\mathfrak{f}$, we have $\mathfrak{M}_{\mathfrak{f}\cdot\mathfrak{g}}\leqq\mathfrak{M}_\mathfrak{f}$. Similarly $\mathfrak{M}_{\mathfrak{f}\cdot\mathfrak{g}}\leqq\mathfrak{M}_\mathfrak{g}$. Hence $\mathfrak{M}_{\mathfrak{f}\cdot\mathfrak{g}}\leqq\mathfrak{M}_\mathfrak{f}\mathfrak{M}_\mathfrak{g}$. Next, let $\mathfrak{h}$ be any element in $\{\mathfrak{f}\}$ which belongs to $\mathfrak{M}_\mathfrak{f}\mathfrak{M}_\mathfrak{g}$. Then, since $\mathfrak{h}\in\mathfrak{M}_\mathfrak{f}$, we have $\mathfrak{h}<\mathfrak{f}$. Similarly $\mathfrak{h}<\mathfrak{g}$. Hence $\mathfrak{h}<\mathfrak{f}\cdot\mathfrak{g}$. That is, $\mathfrak{h}\in\mathfrak{M}_{\mathfrak{f}\cdot\mathfrak{g}}$. Therefore, $\mathfrak{M}_\mathfrak{f}\mathfrak{M}_\mathfrak{g}\leqq\mathfrak{M}_{\mathfrak{f}\cdot\mathfrak{g}}$. Combining this with the above result, we have $\mathfrak{M}_{\mathfrak{f}\cdot\mathfrak{g}}=\mathfrak{M}_\mathfrak{f}\mathfrak{M}_\mathfrak{g}$.

8°. From 5°, we have $\mathfrak{M}_\mathfrak{f}\perp\mathfrak{M}_{\mathfrak{f}'}$. Let $\mathfrak{h}$ be any element in $\{\mathfrak{f}\}$. Since, from 5°, $\mathfrak{h}\cdot\mathfrak{f}\perp\mathfrak{h}\cdot\mathfrak{f}'$, we have $\mathfrak{h}=\mathfrak{h}\cdot(\mathfrak{f}+\mathfrak{f}')=\mathfrak{h}\cdot\mathfrak{f}+\mathfrak{h}\cdot\mathfrak{f}'$. Hence $\mathfrak{M}_\mathfrak{e}=\mathfrak{M}_\mathfrak{f}\oplus\mathfrak{M}_{\mathfrak{f}'}$. That is, $\mathfrak{M}_{\mathfrak{f}'}$ is the orthogonal complement of $\mathfrak{M}_\mathfrak{f}$ with respect to $\mathfrak{M}_\mathfrak{e}$.

9°. Then
$$\mathfrak{M}_{\mathfrak{f}\dotplus\mathfrak{g}}=\mathfrak{M}_\mathfrak{e}\ominus\mathfrak{M}_{(\mathfrak{f}\dotplus\mathfrak{g})'}=\mathfrak{M}_\mathfrak{e}\ominus\mathfrak{M}_{\mathfrak{f}'\cdot\mathfrak{g}'}$$
$$=\mathfrak{M}_\mathfrak{e}\ominus\mathfrak{M}_{\mathfrak{f}'}\mathfrak{M}_{\mathfrak{g}'} \qquad \text{by } 7°.$$

Consequently we have $\mathfrak{M}_{\mathfrak{f}\dotplus\mathfrak{g}}=\mathfrak{M}_\mathfrak{f}\oplus\mathfrak{M}_\mathfrak{g}$.

10°. From 6°–9°, we can easily see that the calculation of $\mathfrak{M}_\mathfrak{f}$ obeys the same laws as that of $\mathfrak{f}$ in $\{\mathfrak{f}\}$. Hence $\{\mathfrak{M}_\mathfrak{f}\}$ is a complemented distributive lattice, in which $\mathfrak{M}_\mathfrak{e}$ is the unit element, and

$\mathfrak{M}_{\mathfrak{e}} \ominus \mathfrak{M}_{\mathfrak{f}}$ belongs to $\{\mathfrak{M}_{\mathfrak{f}}\}$ with $\mathfrak{M}_{\mathfrak{f}}$. Hence, by the theorem of sec. 4, we can find a resolution of identity $E(U)$ defined for all U in a field $\mathfrak{K}$ which is isomorph to $\{\mathfrak{f}\}$, such that $E(U)$ is the projection on $\mathfrak{M}_{\mathfrak{f}} = \mathfrak{M}_U$. Put $E(U)\mathfrak{e} = \mathfrak{q}(U)$. Since $\mathfrak{f} \in \mathfrak{M}_{\mathfrak{f}}$ and $\mathfrak{f}' \perp \mathfrak{M}_{\mathfrak{f}}$, we have $E(U)\mathfrak{f} = \mathfrak{f}$, $E(U)\mathfrak{f}' = 0$. Hence

$$\mathfrak{q}(U) = E(U)\mathfrak{e} = E(U)\mathfrak{f} + E(U)\mathfrak{f}' = \mathfrak{f}.$$

Thus $\{\mathfrak{f}\}$ is the orthogonal system $\{\mathfrak{q}(U)\}$ with set indices.

Generalized Case.

8. Hitherto I have investigated the orthogonal systems $\{\mathfrak{M}_U\}$ or $\{\mathfrak{q}(U)\}$ where the index system is the field $\mathfrak{K}$ of sets containing the unit element. Next consider the case where the index system is the differential set system $\mathfrak{K}\mathfrak{D}V$.(1) The differential set system $\mathfrak{K}\mathfrak{D}V$ is composed of $\mathfrak{K}V_\alpha(\alpha \in \mathfrak{A})$, where $\mathfrak{K}V_\alpha$ are fields with unit elements V_α, and $V = \sum_{\alpha \in \mathfrak{A}} V_\alpha$. Hence, in order that a system of closed linear manifolds $\{\mathfrak{M}\}$ shall be an orthogonal system with indices U in the differential set system $\mathfrak{K}\mathfrak{D}V$, it is necessary and sufficient that $\{\mathfrak{M}\}$ be decomposed in the form

$$\{\mathfrak{M}\} = \sum_{\alpha \in \mathfrak{A}} \{\mathfrak{M}\}_{(\alpha)},$$

such that when $\alpha \neq \beta$ any manifold in $\{\mathfrak{M}\}_{(\alpha)}$ is orthogonal to each manifold in $\{\mathfrak{M}\}_{(\beta)}$, and each $\{\mathfrak{M}\}_{(\alpha)}$ is a lattice satisfying the conditions of sec. 4.

Similarly for the orthogonal system of elements.

9. But there is another case, where the index system is the field of sets having no unit element,—for instance, the system of all Lebsgue measurable sets with finite measure. In this case we may use the *generalized Boolean algebra* discussed by Stone.(2) This is a system which satisfies the postulates I and II in sec. 1, and following postulate III′.

III′. L is *sub-complemented;* that is,

(1) For the details of the differential set aystem, cf. F. Maeda, this Journal, **6** (1936), 20–21; **7** (1937), 104–105. When $\mathfrak{H}$ is not separable, $\mathfrak{K}\mathfrak{D}V$ may be a non-enumerable differential set system.

(2) M. H. Stone, loc. cit. American Journal of Math., **57** (1935), 721–728.

III_1'. There exists an element 0, *zero element*, in L such that $a \supset 0$ for every element a in L.

III_2'. For any elements a and b in L, such that $a \subset b$, the simultaneous equations

$$x \cup a = b, \qquad x \cap a = 0$$

have a solution.

Remark.—The solution of III_2' is unique.

Hence we may call the generalized Boolean algebra the *sub-complemented distributive lattice.*

10. Now assume that $\mathfrak{H}$ is separable. We have the following theorem:

Let $\{\mathfrak{M}\}$ be a system of closed linear manifolds in $\mathfrak{H}$, which satisfies the following conditions:

(i) *$\{\mathfrak{M}\}$ is a sub-complemented distributive lattice, where the inclusion in the lattice theory means the manifold implication.*

(ii) *When $\mathfrak{M}_1, \mathfrak{M}_2$ belong to $\{\mathfrak{M}\}$, and $\mathfrak{M}_1 \leqq \mathfrak{M}_2$, then $\mathfrak{M}_2 \ominus \mathfrak{M}_1$ belongs to $\{\mathfrak{M}\}$.*

Then $\{\mathfrak{M}\}$ is an orthogonal system with set indices.

Let $\mathfrak{E}$ be the closed linear manifold determined by the elements of all closed linear manifolds in $\{\mathfrak{M}\}$. Since $\mathfrak{H}$ is separable, from $\{\mathfrak{M}\}$ we can pick up a finite or infinite sequence $\{\mathfrak{M}_\nu\}$ so that

$$\mathfrak{E} = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \cdots\cdots \oplus \mathfrak{M}_\nu \oplus \cdots\cdots$$

Define another sequence of closed linear manifolds by

$$\mathfrak{M}^{(1)} = \mathfrak{M}_1, \quad \mathfrak{M}^{(2)} = \mathfrak{M}_2 \ominus \mathfrak{M}^{(1)}\mathfrak{M}_2, \cdots\cdots, \quad \mathfrak{M}^{(\nu)} = \mathfrak{M}_\nu \ominus (\sum_{\mu<\nu} \mathfrak{M}^{(\mu)})\mathfrak{M}_\nu, \cdots\cdots.$$

We can easily see that $\mathfrak{M}^{(\nu)}$ belongs to $\{\mathfrak{M}\}$ for all ν.

Since $\mathfrak{M}^{(1)}\mathfrak{M}^{(2)} = 0$, we have $\mathfrak{M}^{(1)} \perp \mathfrak{M}^{(2)}$.(1) Similarly $\mathfrak{M}^{(1)} \oplus \mathfrak{M}^{(2)} \perp \mathfrak{M}^{(3)}$. In general $\sum_{\mu<\nu} \mathfrak{M}^{(\mu)} \perp \mathfrak{M}^{(\nu)}$. Hence $\{\mathfrak{M}^{(\nu)}\}$ is an orthogonal system such that

$$\mathfrak{E} = \mathfrak{M}^{(1)} \oplus \mathfrak{M}^{(2)} \oplus \cdots\cdots \oplus \mathfrak{M}^{(\nu)} \oplus \cdots\cdots.$$

Denote the system of closed linear manifolds $\mathfrak{M}$ in $\{\mathfrak{M}\}$ satisfying $\mathfrak{M}^{(\nu)} \geqq \mathfrak{M}$ by $\{\mathfrak{M}\}_{(\nu)}$. Then $\{\mathfrak{M}\}_{(\nu)}$ is a lattice satisfying the conditions in sec. 4. Hence $\sum_\nu \{\mathfrak{M}\}_{(\nu)}$ is a system of closed linear manifolds satis-

(1) 5° in sec. 4 holds good also in this section.

fying the conditions of sec. 8. Hence $\sum_\nu\{\mathfrak{M}\}_{(\nu)}$ is an orthogonal system $\{\mathfrak{M}_U\}$ whose indices are sets in a differential set system $\mathfrak{K}\mathfrak{D}V$. $\mathfrak{K}\mathfrak{D}V$ is composed of fields $\mathfrak{K}V_\nu$ with unit elements V_ν, and $V=\sum_\nu V_\nu$.

When $\mathfrak{M}$ is a manifold in $\{\mathfrak{M}\}$, but does not belong to $\sum_\nu\{\mathfrak{M}\}_{(\nu)}$, we have

$$\mathfrak{M}=\mathfrak{E}\mathfrak{M}=\mathfrak{M}^{(1)}\mathfrak{M}\oplus\mathfrak{M}^{(2)}\mathfrak{M}\oplus\cdots\cdots\oplus\mathfrak{M}^{(\nu)}\mathfrak{M}\oplus\cdots\cdots.$$

Since $\mathfrak{M}^{(\nu)}\mathfrak{M}\in\{\mathfrak{M}\}_{(\nu)}$, $\mathfrak{M}^{(\nu)}\mathfrak{M}$ is expressed in the form $\mathfrak{M}_{U_\nu}$ where $U_\nu\in\mathfrak{K}V_\nu$. Thus to $\mathfrak{M}$ we can attribute the index U, where

$$U=U_1+U_2+\cdots\cdots+U_\nu+\cdots\cdots.$$

Consequently $\{\mathfrak{M}\}$ is an orthogonal system $\{\mathfrak{M}_U\}$ with set indices, the system of the sets U being the field containing $\mathfrak{K}\mathfrak{D}V$.

11. As in the preceding section, $\mathfrak{H}$ being separable, for the system of elements we have the following theorem:

Let $\{\mathfrak{f}\}$ be a system of elements in $\mathfrak{H}$ satisfying the following conditions:

(i) *$\{\mathfrak{f}\}$ is a sub-complemented distributive lattice, where the inclusion in the lattice theory means "$\prec$" defined in sec. 6.*

(ii) *When $\mathfrak{f}, \mathfrak{g}$ belong to $\{\mathfrak{f}\}$ and $\mathfrak{f}\prec\mathfrak{g}$, then $\mathfrak{g}-\mathfrak{f}$ belongs $\{\mathfrak{f}\}$.*

Then $\{\mathfrak{f}\}$ is an orthogonal system with set indices.

1°, 2°, 3°, 6°, 7°, in sec. 7, hold good also in this section. Instead of 4°, 5°, 8°, 9°, 10°, in sec 7, we have the following: 4′°, 5′°, 8′°, 9′°, 10′°.

4′°. When $\mathfrak{f}\prec\mathfrak{g}$, by (ii) $\mathfrak{g}-\mathfrak{f}$ belongs to $\{\mathfrak{f}\}$, and by sec. 6 $\mathfrak{g}-\mathfrak{f}\perp\mathfrak{f}$. Hence

$$x\dot{+}\mathfrak{f}=\mathfrak{g}\,,\qquad x\cdot\mathfrak{f}=0$$

has a unique solution $\mathfrak{g}-\mathfrak{f}$.

5′°. Let $\mathfrak{f}$ and $\mathfrak{f}_1$ be any two elements in $\{\mathfrak{f}\}$ such that $\mathfrak{f}\perp\mathfrak{f}_1$. If $\mathfrak{h}$ and $\mathfrak{h}_1$ are elements in $\{\mathfrak{f}\}$ such that $\mathfrak{f}\succ\mathfrak{h}, \mathfrak{f}_1\succ\mathfrak{h}_1$, then $\mathfrak{h}\perp\mathfrak{h}_1$. For, since $\mathfrak{f}+\mathfrak{f}_1\succ\mathfrak{f}\succ\mathfrak{h}$, we have

$$(\mathfrak{h}, \mathfrak{f}_1)=(\mathfrak{h}, \mathfrak{f}+\mathfrak{f}_1)-(\mathfrak{h}, \mathfrak{f})=\|\mathfrak{h}\|^2-\|\mathfrak{h}\|^2=0\,.$$

And, since $\mathfrak{h}+\mathfrak{f}_1\succ\mathfrak{f}_1\succ\mathfrak{h}_1$, we have

$$(\mathfrak{h}, \mathfrak{h}_1)=(\mathfrak{h}+\mathfrak{f}_1, \mathfrak{h}_1)-(\mathfrak{f}_1, \mathfrak{h}_1)=\|\mathfrak{h}_1\|^2-\|\mathfrak{h}_1\|^2=0\,.$$

8′°. When $\mathfrak{f}\prec\mathfrak{g}$, since $\mathfrak{f}\perp\mathfrak{g}-\mathfrak{f}$. from 5′°, we have $\mathfrak{M}_\mathfrak{f}\perp\mathfrak{M}_{\mathfrak{g}-\mathfrak{f}}$. Let $\mathfrak{h}$ be any element in $\mathfrak{M}_\mathfrak{g}$. Since from 5′°, $\mathfrak{h}\cdot\mathfrak{f}\perp\mathfrak{h}\cdot(\mathfrak{g}-\mathfrak{f})$, we have

$$\mathfrak{h}=\mathfrak{h}\cdot\{\mathfrak{f}+(\mathfrak{g}-\mathfrak{f})\}=\mathfrak{h}\cdot\mathfrak{f}+\mathfrak{h}\cdot(\mathfrak{g}-\mathfrak{f}).$$

Hence $\mathfrak{M}_{\mathfrak{g}}=\mathfrak{M}_{\mathfrak{f}}\oplus\mathfrak{M}_{\mathfrak{g}-\mathfrak{f}}$. That is, $\mathfrak{M}_{\mathfrak{g}-\mathfrak{f}}$ is the orthogonal complement of $\mathfrak{M}_{\mathfrak{f}}$ with respect to $\mathfrak{M}_{\mathfrak{g}}$.

9′°. Let $\mathfrak{f},\mathfrak{g}$ be any elements in $\{\mathfrak{f}\}$; and put $\mathfrak{h}=\mathfrak{f}\dot{+}\mathfrak{g}$. Then $\mathfrak{h}-\mathfrak{g}\perp\mathfrak{g}$ and $\mathfrak{h}-\mathfrak{f}\perp\mathfrak{f}$. Since

$$\mathfrak{h}-\mathfrak{g}=(\mathfrak{h}-\mathfrak{g})\cdot\mathfrak{h}=(\mathfrak{h}-\mathfrak{g})\cdot(\mathfrak{f}\dot{+}\mathfrak{g})=(\mathfrak{h}-\mathfrak{g})\cdot\mathfrak{f}\dot{+}(\mathfrak{h}-\mathfrak{g})\cdot\mathfrak{g}=(\mathfrak{h}-\mathfrak{g})\cdot\mathfrak{f},\text{[1]}$$

and

$$\mathfrak{h}-\mathfrak{g}=(\mathfrak{h}-\mathfrak{g})\cdot\mathfrak{h}=(\mathfrak{h}-\mathfrak{g})\cdot\{\mathfrak{f}+(\mathfrak{h}-\mathfrak{f})\}=(\mathfrak{h}-\mathfrak{g})\cdot\mathfrak{f}+(\mathfrak{h}-\mathfrak{g})\cdot(\mathfrak{h}-\mathfrak{f}),$$

we have $(\mathfrak{h}-\mathfrak{g})\cdot(\mathfrak{h}-\mathfrak{f})=0$. Hence, by 7°,

$$\mathfrak{M}_{\mathfrak{h}-\mathfrak{f}}\mathfrak{M}_{\mathfrak{h}-\mathfrak{g}}=\mathfrak{M}_{(\mathfrak{h}-\mathfrak{f})\cdot(\mathfrak{h}-\mathfrak{g})}=0.$$

But by 8′° $\quad \mathfrak{M}_{\mathfrak{h}-\mathfrak{f}}=\mathfrak{M}_{\mathfrak{h}}\ominus\mathfrak{M}_{\mathfrak{f}},\qquad \mathfrak{M}_{\mathfrak{h}-\mathfrak{g}}=\mathfrak{M}_{\mathfrak{h}}\ominus\mathfrak{M}_{\mathfrak{g}}.$

Consequently we have $\mathfrak{M}_{\mathfrak{h}}=\mathfrak{M}_{\mathfrak{f}}\oplus\mathfrak{M}_{\mathfrak{g}}$.

10′°. From 6°, 7°, 8′°, 9′° we can easily see that the calculation of $\mathfrak{M}_{\mathfrak{f}}$ obeys the same laws as that of $\mathfrak{f}$ in $\{\mathfrak{f}\}$. Hence $\{\mathfrak{M}_{\mathfrak{f}}\}$ is a lattice satisfying the conditions of sec. 10. Hence, by sec. 10, we have an orthogonal system $\{\mathfrak{M}_U\}$ whose indices are sets in a differential set system $\mathfrak{K}\mathfrak{D}V$. $\mathfrak{K}\mathfrak{D}V$ is composed of fields $\mathfrak{K}V_\nu$ with unit elements V_ν.

Let $\boldsymbol{E}(U)$ be the projection on $\mathfrak{M}_U$. Then, by sec. 10,

$$\boldsymbol{E}(U)=\boldsymbol{E}(U_1)+\boldsymbol{E}(U_2)+\cdots\cdots+\boldsymbol{E}(U_\nu)+\cdots\cdots$$

where $\qquad U_\nu\in\mathfrak{K}V_\nu\qquad(\nu=1,2,\ldots\ldots).$

And let $\mathfrak{f}_\nu$ be the element of $\{\mathfrak{f}\}$ such that $\mathfrak{M}_{\mathfrak{f}_\nu}=\mathfrak{M}_{V_\nu}$. And put

$$\mathfrak{q}(U)=\boldsymbol{E}(U_1)\mathfrak{f}_1+\boldsymbol{E}(U_2)\mathfrak{f}_2+\cdots\cdots+\boldsymbol{E}(U_\nu)\mathfrak{f}_\nu+\cdots\cdots.$$

When $\mathfrak{f}$ is the element in $\{\mathfrak{f}\}$ such that $\mathfrak{M}_{\mathfrak{f}}=\mathfrak{M}_U$, denote $\mathfrak{g}_\nu=\boldsymbol{E}(U_\nu)\mathfrak{f}$.

Then
$$\mathfrak{f}=\boldsymbol{E}(U)\mathfrak{f}=\boldsymbol{E}(U_1)\mathfrak{f}+\boldsymbol{E}(U_2)\mathfrak{f}+\cdots\cdots+\boldsymbol{E}(U_\nu)\mathfrak{f}+\cdots\cdots$$
$$=\mathfrak{g}_1+\mathfrak{g}_2+\cdots\cdots+\mathfrak{g}_\nu+\cdots\cdots.$$

Since $\mathfrak{g}_\nu\in\mathfrak{M}_{U_\nu}$, $\mathfrak{f}_\nu-\mathfrak{g}_\nu\perp\mathfrak{M}_{U_\nu}$, we have

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(1) By 3°.

$$E(U_\nu)\mathfrak{g}_\nu=\mathfrak{g}_\nu \quad \text{and} \quad E(U_\nu)(\mathfrak{f}_\nu-\mathfrak{g}_\nu)=0\,.$$

Hence $$E(U_\nu)\mathfrak{f}_\nu=E(U_\nu)(\mathfrak{f}_\nu-\mathfrak{g}_\nu)+E(U_\nu)\mathfrak{g}_\nu=\mathfrak{g}_\nu\,.$$

Consequently we have $$\mathfrak{f}=\mathfrak{q}(U).$$

Thus $\{\mathfrak{f}\}$ is the orthogonal system $\{\mathfrak{q}(U)\}$ with set indices.

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