Generalized Geodesic Lines and Equation of Motion in Wave Geometry.

By

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§ 1. Introduction.

In Wave Geometry,[1] on reasonable grounds a four-component vector $u^i\,(=\psi^\dagger A\gamma^i\psi)$ has been taken as a particle momentum-density vector, and from u^i thus defined, the equation of motion of a particle is given by the following equation:

$$\frac{dx^1}{u^1}=\frac{dx^2}{u^2}=\frac{dx^3}{u^3}=\frac{dx^4}{u^4}\,.$$

On the other hand, in the general theory of relativity, the equation of motion of a particle is defined by the variational equation (geodesic lines):

$$\delta\int ds=0\,.$$

Thus there can be two lines of consideration for the definition of equation of motion of a particle. Here the problem arises: Is there any way to unify these two lines of consideration into one? In the present paper we shall inquire into the equation of motion of a particle from this point of view.

According to the principle of Wave Geometry, physical laws must be expressed by operators γ_i and the state function ψ. Therefore, when we take a variational equation as the equation of motion, it is natural to think that the geodesic line (the equation of motion of a free particle in a field) must be given by an expression depending on both γ_i and ψ instead of $\delta\int ds=0$, which depends only on g_{ij}.

By obtaining this generalized geodesic line, we shall show that *the physical laws in the macroscopic world hitherto obtained in Wave-Geometry (Cosmolog, etc.) and the law unifying gravitation and electromagnetism (of the Born type) are unified in the same principle.*

(1) T. Iwatsuki and T. Sibata: This Journal, **10** (1940), 247 (W. G. No. 41).
T. Iwatsuki, Y. Mimura and T. Sibata: Ibid., **8** (1938), 187 (W. G. No. 27).
K. Sakuma: Ibid., **11** (1941), 15 (W. G. No. 42), and other papers entitled "Cosmology in Terms of Wave Geometry."

§ 2. Outline of the theory.

As is well-known, in the general theory of relativity $ds^2(=g_{ij}dx^i dx^j)$ was taken as the interval of events, and the motion of a particle in the space-time determined by g_{ij} was considered to satisfy

$$\delta\int ds=0\,, \tag{1}$$

i. e., the track of a particle moving in the space-time determined by g_{ij} was thought to be given by the equations of geodesic lines:

$$\frac{d^2x^i}{ds^2}+\left\{ {i \atop jk} \right\}\frac{dx^j}{ds}\frac{dx^k}{ds}=0\,. \tag{2}$$

However, Wave Geometry stands upon the fundamental idea that all physical phenomena depend not only on g_{ij}, which gives the geometrical—in a narrow meaning of the word—properties of space-time but also on the state function ψ attached to each point of space-time. And on this fundamental consideration,

$$ds\psi=dx^i\gamma_i\psi \tag{3}$$

is taken as the length in place of ds^2. Therefore, in Wave Geometry, the motion of a particle is also to be determined by an equation depending on both γ_i and ψ. Then what equation is to be taken instead of (I)? This problem has a full analogy in Fermat's principle in geometrical optics.

According to Fermat's principle, the path curve of light is given in homogeneous medium by

$$\delta\int dl=0\,,$$

and in heterohomogeneous medium by

$$\delta\int \mu dl=0\,, \tag{4}$$

μ being the index of refraction.

Thus we get a schema:

$$\delta\int dl=0 \quad : \quad \delta\int \mu dl=0$$

$$= \quad \delta\int ds=0 \quad : \quad ?$$

We shall continue our consideration of this point in the following pages.

Wave Geometry differs from general relativity in the point that besides g_{ij}, ψ too plays a rôle in physical laws; and since ψ is thought to relate to the existence of metric, it seems natural to expect that the equation of motion in Wave Geometry will be given by some equation analogous to

(4), ψ in some way playing the rôle of μ in (4). Therefore we may take some function of ψ in place of μ in (4). Moreover, since μ is not a vectorial quantity, but a scalar one, the quantity to be taken in our theory in place of μ may be some scalar quantity given by ψ.

In Wave Geometry, among sixteen spin-invariant quantities given by ψ, the following are known as scalar(1):

$$\begin{cases} M=\psi^\dagger A\psi, \quad \text{and} \\ N=\psi^\dagger A\gamma_5\psi. \end{cases} \tag{5}$$

It may be most natural, then, for us to take the following as the equation of motion:

$$\delta\int f(M,N)\,ds=0, \tag{6}$$

f being an arbitrary function of the arguments.

On the other hand, the 4-dimensional spin-invariant vector

$$u^i=\psi^\dagger A\gamma^i\psi,$$

A being an hermitian matrix which makes γ^i also hermitian, is taken as a vector proportional to velocity (4-dimensional) for the following reasons(2):

(i) The equation of motion should be included as part of the theory under consideration.

(ii) When we choose such coordinates that

$$ds^2=-\sum_{a,b=1}^{3} g_{ab}dx^a dx^b+g_{44}(dx^4)^2, \qquad g_{44}>0, \tag{7}$$

and identify x^4 with t, the fourth component of u^i becomes $\psi^\dagger\psi$ except for a real factor $\frac{1}{\sqrt{g_{44}}}$, expressing the meaning of density or existence-probability of matter represented by ψ.

(iii) From the relation $g_{ij}u^iu^j\equiv M^2+N^2>0$ (where $M=\psi^\dagger A\psi$, $N=\psi^\dagger A\gamma_5\psi$), if we express by $'u^i$ the component of the vector u^i in a Minkowski local-coordinate system at any point of the space-time whose metric is given by (7), then we have the relation $-('u^1)^2-('u^2)^2-('u^3)^2+('u^4)^2>0$, proving that the above-given relation satisfies the condition that u^i can be taken to represent a momentum-density vector.

(iv) The curve (4-dimensional) generated by u^i is regarded as most appropriate in representing geodesics or trajectories of a particle in both gravitational and non-gravitational fields defined from the angle of Wave Geometry.

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(1) T. Sibata: This Journal **8** (1938), 173 (W. G. No. 26).

(2) T. Iwatsuki, Y. Mimura and T. Sibata: loc. cit.; also T. Iwatsuki and T. Sibata: This Journ., **10** (1940), 247 (W. G. No. 41).

Thus we have

$$u^i = \psi^\dagger A\gamma^i\psi = D\frac{dx^i}{ds} \qquad (D \equiv \sqrt{g_{ij}dx^i dx^j} = \sqrt{M^2+N^2}). \tag{8}$$

On the grounds previously mentioned, we take the following as a fundament to deduce the motion of a particle:

(I) The equation of motion of a particle must satisfy

(A) $$\delta\int f(M, N)\, ds = 0,$$

(II) a 4-dimensional vector $u^i (= \psi^\dagger A\gamma^i\psi)$ must be a tangential vector of the curve given by (I), i.e.,

(B) $$u^i = D\frac{dx^i}{ds}.$$

Now the question arises: When does $u^i = \psi^\dagger A\gamma^i\psi$ become a tangential vector of the curve given by (A)? More exactly, in what form should the fundamental equation for ψ

$$\frac{\partial\psi}{\partial x^i} = (\Gamma_i + \Sigma_i)\psi \tag{9}$$

be? and of what form should the function $f(M, N)$ be? in order that u^i shall become a tangential vector field to the path defined by (A).

In the next section an answer to the question will be given.

§ 3. The equation of motion based on $\delta\int f ds = 0$ and $u^i = D\frac{dx^i}{ds}$.

By the ordinary calculus of variation we have, from (A),

$$\int(\delta f)\, ds + \int f\delta(ds) = 0,$$

or[1]

$$\int\frac{\partial f}{\partial x^\sigma} ds\delta x^\sigma - \int\left[g_{\sigma\epsilon} f\left(\frac{d^2x^\epsilon}{ds^2} + \left\{{\epsilon \atop \mu\nu}\right\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - g_{\sigma\epsilon}\frac{dx^\epsilon}{ds}\frac{df}{ds}\right)\right] ds\delta x^\sigma = 0.$$

So we have

$$\frac{\partial f}{\partial x^\sigma} - \frac{\partial f}{\partial x^\nu}\frac{dx^\nu}{ds}\frac{dx^\epsilon}{ds} g_{\sigma\epsilon} - g_{\sigma\epsilon} f\left[\frac{d^2x^\epsilon}{ds^2} + \left\{{\epsilon \atop \mu\nu}\right\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right] = 0, \tag{10}$$

which gives the equations determining a path.

Since (10) must be satisfied by $\frac{dx^i}{ds}$ defined by (B), we have

(1) Cf., e.g., Eddington: Mathematical Theory of Relativity, p. 60.

$$\frac{\partial f}{\partial x^\nu}\left(\delta^\nu_\sigma - \frac{1}{D}u^\nu u_\sigma\right) - \left(\frac{1}{D}u^\mu \nabla_\mu u_\sigma - \frac{u_\sigma}{D}u^\nu u^\mu \nabla_\mu u_\nu\right) f = 0\,, \qquad (11)$$

because of

$$\frac{d^2x^\epsilon}{ds^2} + \{^{\epsilon}_{\mu\nu}\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \frac{1}{D}u^\nu \nabla_\nu\left(\frac{1}{D}u^\epsilon\right) = \frac{1}{u^l u_l}u^\nu \nabla_\nu u^\epsilon - \frac{u^\epsilon}{(u^l u_l)^2}u^\nu u^\omega \nabla_\omega u_\nu\,.$$

Expanding $\sum_i$ in (9) in sedenion, we have

$$\frac{\partial \Psi}{\partial x^i} = (\Gamma_i + A_i^{\cdot jk}\gamma_j\gamma_k + A_i + A_i^{\cdot 5}\gamma_5 + A_i^{\cdot j}\gamma_j + A_i^{\cdot j5}\gamma_j\gamma_5)\,\Psi\,.$$

With the solution of these differential equations, M, N, and u^i can be calculated; and putting them into (11), and considering the complete integrability of the differential equation (9), we obtain, after some tedious calculations,[1]

$$(\overset{1}{F}_{kl}\cos\theta + \overset{2}{F}_{kl}\sin\theta)\left(1+\rho\frac{\partial \log f}{\partial\rho}\right) - \frac{\partial \log f}{\partial\theta}(\sin\theta\overset{1}{F}_{kl} - \cos\theta\overset{2}{F}_{kl}) = 0\,,$$

$$(\overset{1}{H}_{kl}\cos\theta + \overset{2}{H}_{kl}\sin\theta)\left(1-\rho\frac{\partial \log f}{\partial\rho}\right) + \frac{\partial \log f}{\partial\theta}(\sin\theta\overset{1}{H}_{kl} - \cos\theta\overset{2}{H}_{kl}) = 0\,,$$

$$i\left[\rho\frac{\partial \log f}{\partial\rho}(\cos\theta\overset{2}{F}{}'^{*}_{kl} - \sin\theta\overset{1}{F}{}'^{*}_{kl}) - \frac{\partial \log f}{\partial\theta}(\sin\theta\overset{2}{F}{}'^{*}_{kl} + \cos\theta\overset{1}{F}{}'^{*}_{kl})\right]$$
$$= \cos\theta\overset{1}{F}{}'_{kl} + \sin\theta\overset{2}{F}{}'_{kl}\,,$$

$$i\left[\rho\frac{\partial \log f}{\partial\rho}(\cos\theta\overset{5}{\beta} - \sin\theta\overset{0}{\beta}) - \frac{\partial \log f}{\partial\theta}(\sin\theta\overset{5}{\beta} - \cos\overset{0}{\beta})\right] = i(\sin\theta\overset{0}{\beta} - \cos\theta\overset{5}{\beta})\,,$$

$$-\rho\left(\frac{\partial \log f}{\partial\rho} + 1\right)\varphi_r + i\frac{\partial \log f}{\partial\theta}\tau_r = 0\,,$$

$$-\rho\frac{\partial \log f}{\partial\rho}\tau_r + i\frac{\partial \log f}{\partial\theta}\varphi_r = 0\,,$$

where $\overset{1}{F}_{kl}$, $\overset{2}{F}_{kl}$, $\overset{1}{H}_{kl}$, $\overset{1}{H}_{kl}$, ρ, θ, $\overset{0}{\beta}$, $\overset{5}{\beta}$, τ_r, and φ_r are given in Note I.

Solving A_k, $A_k^{\cdot 5}$, A_{kl}, $A_{kl}^{\cdot 5}$, and $A_k^{\cdot ij}$ from the equations above, we get the following five cases[2]:

(I) $f = F(M, N)$ (F being an arbitrary function),

$$2A_k^{\cdot pq} = \epsilon_k^{\cdot pqr}\chi_r - 2\delta_k^{[p}P^{q]}\,,$$

$$A_k = A_k^{\cdot 5} = 0\,, \qquad A_{kl} = \overset{0}{a}g_{kl}\,, \qquad A_{kl}{}^5 = \overset{5}{a}g_{kl}\,.$$

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(1) Note I.

(2) Note II, IV.

(II) $f=\dfrac{1}{\sqrt{M^2+N^2}}$,

$2A_k^{;pq}=\epsilon_k^{;pqr}\chi_r-2\delta_k^{[p}P^{q]}$,

$A_{kl}=\overset{0}{a}g_{kl}-\overset{1}{F}_{kl}$, $\quad A_{kl}^{;5}=\overset{5}{a}g_{kl}-\overset{2}{F}_{kl}$

(A_k and A_k^5 are arbitrary).

(III) $f=\sqrt{M^2+N^2}$,

$2A_k^{;pq}=\epsilon_k^{;pqr}\chi_r-2\delta_k^{[p}P^{q]}$,

$A_{kl}=\overset{0}{a}g_{kl}+\overset{1}{H}_{kl}$, $\quad A_{kl}^5=\overset{5}{a}g_{kl}+\overset{2}{H}_{kl}$

(A_k and A_k^5 are arbitrary).

(IV) $f=\dfrac{F[(-N\lambda+M\mu)]}{\sqrt{M^2+N^2}}$ (λ, μ being constants),

$\overset{1}{F}_{kl}:\overset{2}{F}_{kl}=-\overset{5}{\beta}:\overset{0}{\beta}=\lambda:\mu$,

$A_{kl}=\overset{0}{a}g_{kl}-\overset{1}{F}_{kl}$, $\quad \overset{5}{A}_{kl}=\overset{5}{a}g_{kl}$,

$2A_k^{;pq}=\epsilon_k^{;pqr}\chi_r-2\delta_k^{[p}P^{q]}$,

$A_k^{;5}=0$, $\quad A_k$ is arbitrary.

(V) $f=\sqrt{M^2+N^2}F[(N\lambda-M\mu)]$ (λ, μ being constants),

$\overset{1}{H}_{kl}:\overset{2}{H}_{kl}=\lambda:\mu$,

$A_{kl}=\overset{0}{a}g_{kl}+\overset{1}{H}_{kl}$,

$A_{kl}^5=\overset{5}{a}g_{kl}+\overset{2}{H}_{kl}$,

$2A_k^{;pq}=\epsilon_k^{;pqr}\chi_r-2\delta_k^{[p}P^{q]}$,

$A_k^{;5}=0$, $\quad A_k$ is arbitrary.

In these calculations, A_k, $A_k^{;5}$, A_{kl}, $A_{kl}^{;5}$, and $A_k^{;pq}$, the coefficients of expansion of $\sum_i$ in sedenion, were taken to be real.

Moreover, in the cosmology of Wave Geometry, $\sqrt{M^2+N^2}$ has the meaning of invariant density, so we shall restrict ourselves to the less general case when f is an arbitrary function of $\sqrt{M^2+N^2}$, but not an arbitrary function of M and N. Thus cases (IV) and (V) may be left out of consideration, so that we have to treat only the following three cases[(1)]:

(I) f is an arbitrary function of $\sqrt{M^2+N^2}$, and the fundamental equation is reduced to[(2)]

(1) Note IV.

(2) Note V.

$$\nabla_i\Psi=\left(\frac{1}{2}\epsilon_i^{\cdot jkr}\chi_r\gamma_j\gamma_k\right)\Psi,$$

or $\nabla_i\Psi=\alpha\gamma_i\Psi$.

(II) $f=\dfrac{1}{\sqrt{M^2+N^2}}$, and

$$\nabla_i\Psi=\left(\frac{1}{2}\epsilon_i^{\cdot jkr}\chi_r\gamma_j\gamma_k+A_i^{\cdot 5}\gamma_5+\overset{0}{a}\gamma_i+\overset{5}{a}\gamma_i\gamma_5-\overset{1}{F}_{ij}\gamma^j-\overset{2}{F}_{ij}\gamma^j\gamma_5\right)\Psi.$$

(III) $f=\sqrt{M^2+N^2}$, and

$$\nabla_i\Psi=\left(\frac{1}{2}\epsilon_i^{\cdot jkr}\chi_r\gamma_j\gamma_k+A_i^{\cdot 5}\gamma_5+\overset{0}{a}\gamma_i\gamma_5+\overset{1}{H}_{ij}\gamma^j+\overset{2}{H}_{ij}\gamma^j\gamma_5\right)\Psi.$$

For these three cases we have the equation of motion of a particle as follows.[1]

(I) $\lambda^i\nabla_i\lambda^j=0 \qquad \left(\lambda^i\equiv\dfrac{dx^i}{ds}\equiv\dfrac{u^i}{D}\right)$

(II) $\lambda^i\nabla_i\lambda^j=\dfrac{M}{\sqrt{M^2+N^2}}\overset{1}{F}{}_{i}^{\cdot j}\lambda^i+\dfrac{N}{\sqrt{M^2+N^2}}\overset{2}{F}{}_{i}^{\cdot j}\lambda^i,$

where $\overset{1}{F}_{ij}$ and $\overset{2}{F}_{ij}$ are anti-symmetric tensors with respect to i and j.

(III) $\lambda^i\nabla_i\lambda^j=\dfrac{M}{\sqrt{M^2+N^2}}\overset{1}{H}{}_{i}^{\cdot j}\lambda^i+\dfrac{N}{\sqrt{M^2+N^2}}\overset{2}{H}{}_{i}^{\cdot j}\lambda^i,$

where $\overset{1}{H}_{ij}$ and $\overset{2}{H}_{ij}$ are symmetric tensors with respect to i and j.

§ 4. Discussion of the results obtained.

We shall here discuss in detail each case obtained in § 3.

Case (I). This case gives space-time of the Einstein or de-Sitter type, and the result coincides completely with that of the cosmology of Wave Geometry.[2] Moreover, since the equation of motion of a particle in this case is identical with ordinary geodesic lines in Relativity, we see that *the equation derived from (A) is an equation of generalized geodesic lines, including the ordinary geodesic line $\delta\int ds=0$ as a special case.*

Case (II). The equation of motion in this case coincides with the equation of motion of a particle in the field unifying gravitation and electromagnetism of the Born type.

To find the equation of motion of a particle in the field unifying

(1) Note V.

(2) T. Sibata: This Journal, 8 (1938), 199 (W. G. No. 29).

gravitation and electromagnetism has been attempted by several writers, e. g., Weyl, Kalza-Klein, Einstein-Mayer; but in Weyl's theory electromagnetism was not completely fused into the field, and in other theories a five-dimensional space-time (Kalza-Klein), a five-dimensional vector (Einstein-Mayer), etc., were used. In our theory, space-time is four-dimensional; and to each point of this space-time, the state ψ is attached, which seems natural from the point of view of quantum mechanics. Thus we obtained *the equation of motion of a particle in a field unifying gravitation and electromagnetism (of the Born type) under the same principle of motion of a particle in the field of gravitation.*

Case (III). In this case the equation of motion of a particle may be regarded as that in a dissipative medium by comparing this equation with that of classical physics. So that we may say that *the equation of motion in a dissipative medium is also unified under the same principle.*

In conclusion of what has been said above we have the result: The equation of motion of a particle in space of gravitation and electromagnetism or *a* dissipative medium is unified under the same principle: $\delta\int f ds=0$. But in the present stage of our research, the physical meaning of f in (A) is still unclarified, and no satisfactory explanation has been obtained, especially, for the fact that f takes different forms corresponding to the three cases above-obtained. These problem must be left for future research.

Note I.

In order to express (II) in more convenient form for treatment, we shall introduce the following four vectors $\overset{a}{\lambda}{}^j$ defined by[1]

$$\overset{1}{\lambda}{}^j=\frac{u^j}{D},\quad \overset{2}{\lambda}{}^j=\frac{u^j_5}{D},\quad \overset{3}{\lambda}{}^j=\frac{is^i}{D},\quad \overset{4}{\lambda}{}^j=\frac{it^j}{D}\quad (D\equiv\sqrt{M^2+N^2}),\quad \text{(N. 1)}$$

where $u^j_5\equiv\psi^\dagger A\gamma^j\gamma_5\psi$, and s^j and t^j are real and imaginary parts of the vector $\rho^j=\tilde{\psi}C\gamma^j\psi$, i. e.,

$$\rho^j=s^j+it^j,$$

C being the matrix which makes $C\gamma^j$ $(j=1,\ldots,4)$ symmetric. Then since $\overset{a}{\lambda}{}^j$ satisfy the relations[2]

$$\overset{a}{\lambda}{}_l\overset{b}{\lambda}{}^l=\delta^{ab}\qquad (a, b=1,\ldots,4)$$

(11) is expressed in the form:

(1) T. Sibata: This Journal, 8 (1938),
(2) T. Sibata: loc. cit., 185.

$$\overset{(a)}{\lambda^\nu}\frac{\partial f}{\partial x^\nu}-\frac{1}{\sqrt{M^2+N^2}}\overset{(a)}{\lambda^\sigma}\overset{1}{\lambda^\mu}\nabla_\mu u_\sigma=0 \qquad (a=2,3,4) \qquad \text{(N. 2)}$$

which is the condition that u^l shall generate the path for which $\delta\int f ds=0$.(1)

In general f may be a function of x and ψ, but, as a first step, we shall assume that f involves as a function of ψ, $M\equiv\psi^\dagger A\psi$ and $N\equiv\psi^\dagger A\gamma_5\psi$ only, so that f is a function of M, N, and x. Under this assumption (N. 2) is expressed as

$$\overset{(a)}{\lambda^\nu}\left(\frac{\partial f}{\partial M}\frac{\partial M}{\partial x^\nu}+\frac{\partial f}{\partial N}\frac{\partial N}{\partial x^\nu}+\left[\frac{\partial f}{\partial x^\nu}\right]\right)-\frac{f}{\sqrt{M^2+N^2}}\overset{(a)}{\lambda^\sigma}\overset{1}{\lambda^\mu}\nabla_\mu u_\sigma=0 \quad (a=2,3,4) \qquad \text{(N. 3)}$$

where $\left[\frac{\partial f}{\partial x^\nu}\right]$ means partial differentiation with respect to x^ν regarding M and N as constants.

On the other hand, ψ is the solution of the fundamental equation of the form

$$\nabla_i\psi=(A_i^{\cdot jk}\gamma_j\gamma_k+A_i+A_i^{\cdot 5}\gamma_5+A_i^{\cdot j}\gamma_j+A_i^{\cdot j5}\gamma_j\gamma_5)\psi \qquad \text{(N. 4)}$$

hence(2)

$$\left.\begin{aligned}\nabla_k u_j&=4a_{kj}^{\cdot\cdot q}u_q+2a_k u_j+2(a_{kj}M+a_{kj}^{\cdot\cdot 5}N)\\&\quad+2i(b_k^{\cdot pq}u_{jpq}+b_k^{\cdot 5}u_{j5}+b_k^{\cdot l}u_{jl}+b_k^{\cdot l5}u_{jl5}),\\ \nabla_k M&=2(ib_k^{\cdot pq}u_{pq}+a_k M+a_k^{\cdot 5}N+a_k^{\cdot l}u_l+ib_k^{\cdot l5}u_{l5}),\\ \nabla_k N&=2(ib_k^{\cdot pq}u_{pq5}+a_k N-a_k^{\cdot 5}M-ib_k^{\cdot l}u_{l5}+a_k^{\cdot l5}u_l),\end{aligned}\right\} \qquad \text{(N. 5)}$$

where $a_k^{\cdot pq}$, $b_k^{\cdot pq}$ etc., are real and imaginary parts of $A_k^{\cdot pq}$ etc., i. e.,

$$\left.\begin{aligned}&A_k^{\cdot pq}=a_k^{\cdot pq}+ib_k^{\cdot pq}, \quad A_k^{\cdot 5}=a_k^{\cdot 5}+ib_k^{\cdot 5}, \quad A_k=a_k+ib_k\\&A_k^{\cdot l}=a_k^{\cdot l}+ib_k^{\cdot l}, \qquad A_k^{\cdot l5}=a_k^{\cdot l5}+ib_k^{\cdot l5}.\end{aligned}\right\} \qquad \text{(N. 6)}$$

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(1) This result is also obtained by considering the geodesic line regarding $G_{ij}\equiv f^2 g_{ij}$ as the fundamental tensor. For, the condition that u^l shall give the path for which $\delta\int f ds=0$ is equivalent to that under which u^l is tangential to the geodesic line determined by G_{lm}, i. e.,

$$u^i\left[\frac{\partial u^j}{\partial x^i}-\left\{{j \atop ik}\right\}_{G_{lm}}u^k\right]=Qu^j \qquad (※)$$

where

$$\left\{{j \atop ik}\right\}_{G_{lm}}=\frac{1}{2}\frac{1}{f^2}g^{jp}\left[\frac{\partial(f^2 g_{ip})}{\partial x^k}+\frac{\partial(f^2 g_{kp})}{\partial x^i}-\frac{\partial(f^2 g_{ik})}{\partial x^p}\right]$$

$$=\left\{{j \atop ik}\right\}_{g_{lm}}+\frac{1}{2f^2}\left[\frac{\partial f^2}{\partial x^k}\delta_i^j+\frac{\partial f^2}{\partial x^i}\delta_k^j-\frac{\partial f^2}{\partial x^p}g^{jp}g_{ik}\right].$$

Therefore (※) is rewritten as

$$u^i\nabla_i u^j+\frac{2}{f}\frac{\partial f}{\partial x^k}u^k u^j-\frac{1}{f}\frac{\partial f}{\partial x^p}g^{jp}(u^i u_i)=Qu^j,$$

from which, multiplying by $\overset{a}{\lambda_j}$, we have (4).

(2) T. Sibata: This Journal, **9** (1939), 184.

Therefore, substituting (N. 5) into (N. 3), we have

$$\left.\begin{aligned}&\overset{(a)}{\lambda^k}\frac{\partial\log f}{\partial M}(ib_{k}^{\cdot pq}u_{pq}+a_kM+a_{k}^{\cdot 5}N+a_{k}^{\cdot l}u_l+ib_{k}^{\cdot l5}u_{l5})\\&+\overset{(a)}{\lambda^k}\frac{\partial\log f}{\partial N}(ib_{k}^{\cdot pq}u_{pq5}+a_kN-a_{k}^{\cdot 5}M-ib_{k}^{\cdot l}u_{l5}+a_{k}^{\cdot l5}u_l)+\frac{1}{2}\overset{(a)}{\lambda^k}\left[\frac{\partial\log f}{\partial x^k}\right]\\&-\frac{\overset{1}{\lambda^k}\overset{(a)}{\lambda^j}}{\sqrt{M^2+N^2}}[2a_{kj}^{\cdot\cdot q}u_q+a_ku_j+a_{kj}M+a_{kj}^{\cdot\cdot 5}N\\&\qquad+i(b_{k}^{\cdot pq}u_{jpq}+b_{k}^{\cdot 5}u_{j5}+b_{k}^{\cdot l}u_{jl}+b_{k}^{\cdot l5}u_{jl5})]\qquad(a=2,3,4)\end{aligned}\right\}\quad\text{(N. 7)}$$

But since[1]

$$\left.\begin{aligned}&u_{kj}=2(-N\lambda_{[k}^{1}\lambda_{j]}^{2}+iM\lambda_{[k}^{3}\lambda_{j]}^{4})\\&u_{kj5}=-\frac{i}{2}\epsilon_{kjpq}u^{pq}=2(iN\lambda_{[k}^{3}\lambda_{j]}^{4}+M\lambda_{[k}^{1}\lambda_{j]}^{2}),\\&u_{jkl}=i\epsilon_{jklm}u^{m}_{\cdot 5}\end{aligned}\right\}\quad\text{(N. 8)}$$

from equation (N. 7), corresponding to $a=2,3,4$, we have the following three equations:

$$\left.\begin{aligned}&\overset{2}{\lambda^{k}}(P_{k}^{\cdot pq}\overset{1}{\lambda}_p\overset{2}{\lambda}_q+P_k+P_{kl}\overset{1}{\lambda^l}+P'_{kl}\overset{2}{\lambda^l})\\&-\overset{1}{\lambda^k}(2a_{kjq}\overset{2}{\lambda^j}\overset{1}{\lambda^q}+\overset{2}{\lambda^j}Q_{kj}+ib_{k}^{\cdot 5}+Q'_{kl}\overset{1}{\lambda^l}=0\,,\\&\overset{3}{\lambda^k}(P_{k}^{\cdot pq}\overset{1}{\lambda}_p\overset{2}{\lambda}_q+P_k+P_{kl}\overset{1}{\lambda^l}+P'_{kl}\overset{2}{\lambda^l})\\&-\overset{1}{\lambda^k}(2a_{kjq}\overset{3}{\lambda^j}\overset{1}{\lambda^q}+\overset{3}{\lambda^j}Q_{kj}-2b_{k}^{\cdot pq}\overset{4}{\lambda}_p\overset{1}{\lambda^q}-\overset{4}{\lambda^p}Q''_{kp})=0\,,\\&\overset{4}{\lambda^k}(A_{k}^{\cdot pq}\overset{1}{\lambda}_p\overset{2}{\lambda}_q+P_k+P_{kl}\overset{1}{\lambda^l}+P'_{kl}\overset{2}{\lambda^l})\\&-\overset{1}{\lambda^k}(2a_{kjq}\overset{4}{\lambda^j}\overset{1}{\lambda^q}+\overset{4}{\lambda^j}Q_{kj}-2b_{k}^{\cdot pq}\overset{1}{\lambda}_p\overset{3}{\lambda}_q+\overset{3}{\lambda^p}Q''_{kp})=0\,,\end{aligned}\right\}\quad\text{(N. 9)}$$

where $P_{k}^{\cdot pq}$, P_k, etc., are defined by

$$P_{k}^{\cdot pq}\equiv2\frac{\partial\log f}{\partial M}(-iNb_{k}^{\cdot pq*}-Mb_{k}^{\cdot pq*})+2\frac{\partial\log f}{\partial N}(-Nb_{k}^{\cdot pq*}+iMb_{k}^{\cdot pq}),$$

$$P_k\equiv\frac{\partial\log f}{\partial M}(a_kM+a_{k}^{\cdot 5}N)+\frac{\partial\log f}{\partial N}(a_kN-a_{k}^{\cdot 5}M)+\frac{1}{2}\left[\frac{\partial\log f}{\partial x^k}\right],$$

$$P_{kl}\equiv\sqrt{M^2+N^2}\left(\frac{\partial\log f}{\partial N}a_{kl}+\frac{\partial\log f}{\partial N}a_{kl}^{\cdot\cdot 5}\right),$$

$$P'_{kl}\equiv i\sqrt{M^2+N^2}\left(\frac{\partial\log f}{\partial M}b_{kl}^{\cdot\cdot 5}-\frac{\partial\log f}{\partial N}b_{kl}\right),$$

(1) loc. cit., 186.

$$Q_{kl} \equiv \frac{1}{\sqrt{M^2+N^2}}(a_{kl}M+a_{kl}{}^5N),$$

$$Q'_{kl} \equiv \frac{i}{\sqrt{M^2+N^2}}(b_{kl}N-b_{kl}^{\cdot\,5}M),$$

$$Q''_{kl} \equiv \frac{1}{\sqrt{M^2+N^2}}(b_{kl}M+b_{kl}{}^5N).$$

Here we take the case in which the fundamental equation for ψ is completely integrable, i. e., ψ may be considered as taking any assigned initial value. In this case, since M, N, and $\overset{a}{\lambda}_i$ take any assigned initial values within the limits of $\overset{a}{\lambda}_i\overset{b}{\lambda}{}^i=\delta^{ab}$ $(a, b=1, \ldots\ldots, 4)$, (N. 9) must hold good identically for all values of $\overset{a}{\lambda}{}^i$ satisfying the relations $\overset{a}{\lambda}_i\overset{b}{\lambda}{}^i=\delta^{ab}$. Hence it must be true that

$$\left.\begin{array}{ll}
P_k^{\cdot pq} = \epsilon_k^{\cdot pqr}\varphi_r+2\delta_k^{[p}\gamma^{q]}, & \\
2a_k^{\cdot pq} = \epsilon_k^{\cdot pqr}\chi_r+2\delta_k^{[p}\omega^{q]}, & \\
P_{kl}=ag_{kl}+F_{kl}+H_{kl} & \begin{pmatrix}F_{kl} \text{ is antisymmetric} \\ H_{kl} \text{ is symmetric}\end{pmatrix} \\
P'_{kl}=a'g_{kl}+F'_{kl}, & (F'_{kl} \text{ is antisymmetric}) \\
Q_{kl}=\beta g_{kl}-F_{kl}+H_{kl}, & \\
Q'_{kl}=\beta' g_{kl}+G'_{kl}, & (G'_{kl} \text{ is antisymmetric}) \\
Q''_{kl}=\beta' g_{kl}+F'^{*}_{kl} & \\
2b_k^{\cdot pq}=\epsilon_k^{\cdot pqr}\tau_r+2\delta_k^{[p}\varkappa^{q]}, &
\end{array}\right\} \quad \text{(N. 10)}$$

with the relations:

$$\gamma^k=-ib^{k5}, \quad \omega^k=-p^k, \quad a'=\beta', \quad \varphi^k=\varkappa^k.$$

Therefore, it follows that

$$\left.\begin{array}{ll}
2b_k^{\cdot pq}=\epsilon_k^{\cdot pqr}\tau_r+2\delta_k^{[p}\varphi^{q]}, & \\
b_{kl}=\overset{0}{\beta}g_{kl}+\overset{0}{f}{}'^{*}_{kl}, \quad b_{kl}^{\cdot\,5}=\overset{5}{\beta}g_{kl}+\overset{5}{f}{}'^{*}_{kl}, & (\overset{0}{f}_{kl} \text{ and } \overset{5}{f}{}'_{kl} \text{ are antisymmetric}) \\
a' = \frac{i}{\sqrt{M^2+N^2}}(N\overset{0}{\beta}-M\overset{5}{\beta}), & \\
a_{kl}=\overset{0}{a}g_{kl}-\overset{0}{f}_{kl}+\overset{0}{h}_{kl} & (\overset{0}{f}_{kl}, \overset{5}{f}_{kl} \text{ are antisymmetric}) \\
a_{kl}{}^5=\overset{5}{a}g_{kl}-\overset{5}{f}_{kl}+h_{kl} & (\overset{0}{h}_{kl}, \overset{5}{h}_{kl} \text{ are symmetric}) \\
2a_k^{\cdot pq}=\epsilon_k^{\cdot pqr}\chi_r-2\delta_k^{[p}P^{q]} &
\end{array}\right\} \quad \text{(N. 11)}$$

$$\left.\begin{aligned}
&\sqrt{M^2+N^2}\left(\frac{\partial \log f}{\partial M}q_{kl}+\frac{\partial \log f}{\partial N}a_{kl}{}^5\right)=ag_{kl}\\
&\qquad\qquad+\frac{1}{\sqrt{M^2+N^2}}\{M(\overset{0}{f}_{kl}+\overset{0}{h}_{kl})+N(\overset{5}{f}_{kl}+\overset{5}{h}_{kl})\}\,,\\
&i\sqrt{M^2+N^2}\left(\frac{\partial \log f}{\partial M}b_{kl}{}^5-\frac{\partial \log f}{\partial N}b_{kl}\right)=\alpha' g_{kl}\\
&\qquad\qquad+\frac{1}{\sqrt{M^2+N^2}}\{M\overset{0}{f}{}'_{kl}+N\overset{5}{f}{}'_{kl}\}\\
&2\frac{\partial \log f}{\partial M}(-iNb_k^{;pq}-Mb_k^{;pq*})+2\frac{\partial \log f}{\partial N}(-Nb_k^{;pq*}+iMb_k^{;pq})\\
&\qquad\qquad=\epsilon_k^{;pqr}\varphi_r-2i\delta_k^{[p}b^{q]5}\,,\\
&P_k\equiv\frac{\partial \log f}{\partial M}(a_kM+a_k^{;5}N)+\frac{\partial \log f}{\partial N}(a_kN-a_k^{;5}M)\\
&\qquad\qquad+\frac{1}{2}\left[\frac{\partial \log f}{\partial x^k}\right]\quad \text{is independent of } M \text{ and } N.
\end{aligned}\right\}\quad (\text{N. }12)$$

Now from (N. 12) we shall solve $\log f$ as a function of M, N, and x. Putting

$$M=\rho\cos\theta\,,\qquad N=P\sin\varphi\,,\qquad (\text{N. }13)$$

since $\dfrac{\partial f}{\partial M}=\cos\theta\dfrac{\partial f}{\partial\rho}-\dfrac{1}{\rho}\sin\theta\dfrac{\partial f}{\partial\theta}$, $\quad\dfrac{\partial f}{\partial N}=\sin\theta\dfrac{\partial f}{\partial\rho}+\dfrac{1}{\rho}\cos\theta\dfrac{\partial f}{\partial\theta}$,

by taking antisymmetric and symmetric parts of the equations of (N. 12), we have

$$(\overset{0}{f}_{kl}\cos\theta+\overset{5}{f}_{kl}\sin\theta)\left(1+\frac{\partial \log f}{\partial\rho}\right)-\frac{\partial \log f}{\partial\theta}(\overset{0}{f}_{kl}\sin\theta-\overset{5}{f}_{kl}\cos\theta)=0\,,\quad (\text{N. }14)$$

$$(\overset{0}{h}_{kl}\cos\theta+\overset{5}{h}_{kl}\sin\theta)\left(1-\rho\frac{\partial \log f}{\partial\rho}\right)+\frac{\partial \log f}{\partial\theta}(\overset{0}{h}_{kl}\sin\theta-\overset{5}{h}_{kl}\cos\theta)=0\,,\quad (\text{N. }15)$$

$$i\left[\rho\frac{\partial \log f}{\partial\rho}(\overset{5}{f}{}'_{kl}\cos\theta-\overset{0}{f}{}'^{*}_{kl}\sin\theta)-\frac{\partial \log f}{\partial\theta}(\overset{5}{f}{}'^{*}_{kl}\sin\theta+\overset{0}{f}{}'^{*}_{kl}\cos\theta)\right.$$
$$=\overset{0}{f}{}'_{kl}\cos\theta+\overset{5}{f}{}'_{kl}\sin\theta\,,\quad (\text{N. }16)$$

$$i\left[\rho\frac{\partial \log f}{\partial\rho}(\overset{5}{\beta}\cos\theta-\overset{0}{\beta}\sin\theta)-\frac{\partial \log f}{\partial\theta}(\overset{5}{\beta}\sin\theta+\overset{0}{\beta}\cos\theta)\right.$$
$$=i(\overset{0}{\beta}\sin\theta-\overset{5}{\beta}\cos\theta)\quad (\text{N. }17)$$

$$-\left(\rho\frac{\partial \log f}{\partial\rho}+1\right)\varphi_r+i\frac{\partial \log f}{\partial\theta}\tau_r=0\,,\qquad (\text{N. }18)$$

$$-\rho\frac{\partial \log f}{\partial\rho}\tau_r+i\frac{\partial \log f}{\partial\theta}\varphi_r=-ib_r^{;5}\,,\qquad (\text{N. }19)$$

and $P_k \equiv \rho \frac{\partial \log f}{\partial \rho} a_k - \frac{\partial \log f}{\partial \theta} a_k^{;5} + \frac{1}{2}\left[\frac{\partial \log f}{\partial x^k}\right]$ is independent of ρ and θ.

Note II.

Solving the above-given equations (N. 14)–(N. 19) we have the following results: Equation (N. 14) gives, unless $\overset{0}{f}_{kl} = \overset{5}{f}_{kl}$,

$$\log f = -\log \rho + F[\log \rho(-\lambda \sin \theta + \mu \cos \theta); x]$$

with $$\overset{0}{f}_{kl} : \overset{5}{f}_{kl} = \lambda : \mu \qquad \text{for all } k, l;$$

otherwise $$\log f = -\log \rho + X(x),$$

where F and X are arbitrary functions of their arguments.

Equations (N. 15) provides, unless $\overset{0}{h}_{kl} = \overset{5}{h}_{kl} = 0$,

$$\log f = \log \rho + F[\log \rho(\lambda \sin \theta - \mu \cos \theta); x],$$

with $$\overset{0}{h}_{kl} : \overset{5}{h}_{kl} = \lambda : \mu \qquad \text{for all } k, l;$$

otherwise $$\log f = \log \rho.$$

Equation (N. 16) provides, unless $\overset{0}{f'}_{kl} = \overset{5}{f'}_{kl} = 0$,

$$\log f = \pm \log \rho + X(x)$$

with $$\overset{0}{f'}_{kl} = \pm i \overset{5}{f'}{}^{*}_{kl}$$

(the double signs $\pm$ correspond to each other). The proof of this result will be given later.(1)

(N. 14) gives, unless $\overset{0}{\beta} = \overset{5}{\beta} = 0$,

$$\log f = -\log \rho + F[\log \rho(\overset{5}{\beta} \sin \theta + \overset{0}{\beta} \cos \theta); x]$$

(N. 18) and (N. 19) are treated similarly.

By combining the equations above, all the possible cases are obtained as follows:

(I) $f = e^{X(x)} F(M, N)$ (X and F being arbitrary functions of their arguments);

$$2a_k^{;pq} = \dot{\epsilon}_k^{;pqr} \chi_r - 2\delta_k^{[p} P^{q]} \qquad \left(2P_k \equiv \frac{\partial X}{\partial x^k}\right),$$

$$b_k^{;pq} = 0, \quad b_k^{;5} = 0, \quad a_k = a_k^{;5} = 0 \quad (b_k \text{ is arbitrary}),$$

$$a_{kl} = \overset{0}{a} g_{kl}, \quad a_{kl}^{;5} = \overset{5}{a} g_{kl}, \quad b_{kl} = b_{kl}^{;5} = 0.$$

(1) Note III.

(I)′ $f=e^{X(x)}F(\sqrt{M^2+N^2}$ (F being an arbitrary function of $\sqrt{M^2+N^2}$);

$$2a_{k}^{\cdot pq}=\epsilon_{k}^{\cdot pqr}\chi_r-2\delta_k^{[p}P^{q]} \qquad \left(2P_k\equiv\frac{\partial X}{\partial x^k}\right),$$

$$b_k^{\cdot pq}=0\,,\quad b_k^{\cdot 5}=a_k=0 \qquad (a_k^{\cdot 5} \text{ and } b_k \text{ are arbitrary}),$$

$$a_{kl}=\overset{0}{a}g_{kl}\,,\quad a_{kl}^{\cdot\cdot 5}=\overset{5}{a}g_{kl}\,,\quad b_{kl}=b_{kl}^{\cdot\cdot 5}=0\,.$$

(I)″ Specially, when F does not contain $\sqrt{M^2+N^2}$, a_k becomes arbitray and

$$2b_k^{\cdot pq}=\epsilon_k^{\cdot pqr}\tau_r\,.$$

(II) $f=e^{X(x)}/\sqrt{M^2+N^2}$;

$$2b_k^{\cdot pq}=-i\epsilon_k^{\cdot pqr}b_r^{\cdot 5}+2\delta_k^{[p}\varphi^{q]}\,,$$

$$2a_k^{\cdot pq}=\epsilon_k^{\cdot pqr}\chi_r-2\delta_k^{[p}P^{q]} \qquad \left(P_k\equiv-a_k+\frac{1}{2}\frac{\partial X}{\partial x^k}\right),$$

$$b_{kl}=\overset{0}{\beta}g_{kl}+\overset{0}{f}{}'^{*}_{kl}\,,\qquad b_{kl}^{\cdot\cdot 5}=\overset{5}{\beta}g_{kl}+i\overset{0}{f}{}'_{kl}\,,$$

$$a_{kl}=\overset{0}{a}g_{hl}-\overset{0}{f}_{kl}\,,\qquad a_{kl}^{\cdot\cdot 5}=\overset{5}{a}g_{kl}-\overset{5}{f}_{kl}$$

(a_k, $a_k^{\cdot 5}$, b_k and $b_k^{\cdot 5}$ are arbitrary).

(III) $f=e^{X(x)}\sqrt{M^2+N^2}$;

$$2b_k^{\cdot pq}=i\epsilon_k^{\cdot pqr}b_r^{\cdot 6}\,,$$

$$2a_k^{\cdot pq}=\epsilon_k^{\cdot pqr}\chi_r-2\delta_k^{[p}P^{q]} \qquad \left(P_k\equiv a_k+\frac{1}{2}\frac{\partial X}{\partial x^k}\right),$$

$$b_{kl}=\overset{0}{f}{}'^{*}_{kl}\,,\qquad b_{kl}^{\cdot\cdot 5}=-i\overset{0}{f}{}'_{kl}\,,$$

$$a_{kl}=\overset{0}{a}g_{kl}+\overset{0}{h}_{kl}\,,\qquad a_{kl}^{\cdot\cdot 5}=\overset{5}{a}g_{kl}+\overset{5}{h}_{kl}\,.$$

(IV) $f=\dfrac{e^X}{\sqrt{M^2+N^2}}F[(-N\lambda\times M\mu)]$;

$$\overset{0}{f}_{kl}:\overset{5}{f}_{kl}=-\overset{5}{\beta}:\overset{0}{\beta}=\lambda:\mu \qquad (\lambda \text{ and } \mu \text{ are constants}),$$

$$b_k^{\cdot pq}=0\,,$$

$$b_{kl}=\overset{0}{\beta}g_{kl}\,,\qquad b_{kl}^{\cdot\cdot 5}=\overset{5}{\beta}g_{kl}\,,$$

$$a_{kl}=\overset{0}{a}g_{kl}-\overset{0}{f}_{kl}\,,\qquad a_{kl}^{\cdot\cdot 5}=\overset{5}{a}g_{kl}-\overset{5}{f}_{kl}\,,$$

$$2a_k^{\cdot pq}=\epsilon_k^{\cdot pqr}\chi_r-2\delta_k^{[p}P^{q]} \qquad \left(P_k=-a_k+\frac{1}{2}\frac{\partial X}{\partial x^k}\right),$$

$$a_k^{\cdot 5}=b_k^{\cdot 5}=0 \qquad (a_k \text{ and } b_k \text{ are arbitrary}).$$

(V) $f=e^{X(x)}\sqrt{M^2+N^2}\,F[(N\lambda-M\mu)]$;

$$\overset{0}{h}_{kl} : \overset{5}{h}_{kl} = \lambda : \mu \qquad (\lambda \text{ and } \mu \text{ are constants}),$$

$$b_k^{\cdot pq} = 0\,, \qquad b_k^{\cdot 5} = 0\,,$$

$$\overset{0}{b}_{kl} = \beta g_{kl}\,, \qquad b_{kl}^{\cdot\cdot 5} = \beta g_{kl}\,,$$

$$a_{kl} = \overset{0}{a} g_{kl} + \overset{0}{h}_{kl}\,, \qquad a_{kl}^{\cdot\cdot 5} = \overset{5}{a} g_{kl} + \overset{5}{h}_{kl}\,,$$

$$2a_k^{\cdot pq} = \epsilon_k^{\cdot pqr}\chi_r - 2\delta_k^{[p} P^{q]} \qquad \left(P_k = a_k + \frac{1}{2}\frac{\partial X}{\partial x^k}\right),$$

$$a_k^{\cdot 5} = 0 \quad (a_k \text{ and } b_k \text{ are arbitrary}).$$

Beside these cases there exists also a function f which satisfies (N. 18) and (N. 19) only. But it has no important meaning, so we neglect it here.

Note III.

The equations

$$i\left[\rho\frac{\partial \log f}{\partial \rho}(\overset{5}{f}{}'^{*}_{kl} \cos\theta - \overset{0}{f}{}'^{*}_{kl} \sin\theta) - \frac{\partial \log f}{\partial \theta}(\overset{5}{f}{}'^{*}_{kl} \sin\theta + \overset{0}{f}{}'^{*}_{kl} \cos\theta)\right] = \overset{0}{f}{}'_{kl} \cos\theta + \overset{5}{f}{}'_{kl} \sin\theta\,,$$

are expressed as follows:

$$i\left[\rho\frac{\partial \log f}{\partial \rho}\lambda^*_{kl} - \frac{\partial \log f}{\partial \theta}\mu^*_{kl}\right] = \mu_{kl}\,, \tag{N. 21}$$

with

$$i\left[\rho\frac{\partial \log f}{\partial \rho}\lambda_{kl} - \frac{\partial \log f}{\partial \theta}\mu_{kl}\right] = \mu^*_{kl}\,, \tag{N. 22}$$

where $\lambda_{kl} \equiv \overset{5}{f}{}'_{kl} \cos\theta - \overset{0}{f}{}'_{kl} \sin\theta\,, \qquad \mu_{kl} \equiv \overset{5}{f}{}'_{kl} \sin\theta + \overset{0}{f}{}'_{kl} \cos\theta$

From (N. 2) and (N. 22), we have

$$-i\frac{\partial \log f}{\partial \theta}(\mu^*_{kl}\lambda_{kl} - \mu_{kl}\lambda^*_{kl}) = \mu_{kl}\lambda_{kl} - \mu^*_{kl}\lambda^*_{kl}\,, \qquad (k, l \text{ not summed}) \tag{N. 23}$$

hence if $\mu^*_{kl}\lambda_{kl} - \mu_{kl}\lambda^*_{kl} \neq 0$, $\log f$ must be of the form:

$$\log f = \Theta(\theta) + R(\rho, x)\,, \tag{N. 24}$$

where $\Theta(\theta)$ is a function of θ alone and R does not contain θ. Therefore, substituting (N. 24) into (N. 21) and (N. 22), and eliminating $\frac{\partial \log f}{\partial \theta}$, we have

$$i\rho\frac{\partial R}{\partial \rho}(\lambda^*_{kl}\mu_{kl} - \lambda_{kl}\mu^*_{kl}) = (\mu_{kl})^2 - (\mu^*_{kl})^2\,,$$

or

$$i\rho\frac{\partial R}{\partial \rho}(\overset{5}{f}{}'^{*}_{kl}\overset{0}{f}{}'_{kl} - \overset{0}{f}{}'^{*}_{kl}\overset{5}{f}{}'_{kl}) = \cos^2\theta(\overset{0}{f}{}'^{2}_{kl} - \overset{0}{f}{}'^{*2}_{kl}) + \sin^2\theta(\overset{5}{f}{}'^{2}_{kl} - \overset{5}{f}{}'^{*2}_{kl})$$
$$= 2\sin\theta\cos\theta(\overset{0}{f}{}'_{kl}\overset{5}{f}{}'_{kl} - \overset{0}{f}{}'^{*}_{kl}\overset{5}{f}{}'^{*}_{kl})\,,$$

from which it must follow that

$$\text{(coefficient of } \cos^2\theta) \qquad \overset{0}{f}{}'^2_{kl}-\overset{0}{f}{}'^{*2}_{kl}-\overset{5}{f}{}'^2_{kl}+\overset{5}{f}{}'^{*2}_{kl}=0\,, \tag{N. 25}$$

$$\text{(coefficient of } \sin\theta\cos\theta) \qquad \overset{0}{f}{}'_{kl}\overset{5}{f}{}'_{kl}-\overset{0}{f}{}'^{*}_{kl}\overset{5}{f}{}'^{*}_{kl}=0\,. \tag{N. 26}$$

From the last equation, we have

$$\left.\begin{aligned} \overset{0}{f}{}'_{kl}&=\omega_{kl}\overset{5}{f}{}'^{*}_{kl}\,, \qquad (k, l \text{ not summed}) \\ \overset{0}{f}{}'^{*}_{kl}&=\omega_{kl}\overset{5}{f}{}'_{kl}\,. \qquad (k, l \text{ not summed}) \end{aligned}\right\} \tag{N. 27}$$

Substituting (N. 27) into (N. 25), we see that

$$(1+\omega_{kl}^2)(\overset{5}{f}{}'^2_{kl}-\overset{5}{f}{}'^{*2}_{kl})=0\,,$$

from which, since $\overset{5}{f}{}'_{kl}$ is real and $\overset{5}{f}{}'^{*}_{kl}$ is purely imaginary, it follows that $\omega_{kl}=\pm i$, unless $\overset{5}{f}_{kl}=\overset{0}{f}{}'_{kl}=0$.

So that we have $\qquad \overset{0}{f}{}'_{kl}=\pm i\overset{5}{f}{}'^{*}_{kl}\,.$

Then $\log f$ is solved as follows:

$$\log f=\pm\log\rho+X(x)\,.$$

Note IV.

We here consider the special case of Note II when f involves M and N as a function of $\sqrt{M^2+N^2}$ (invariant density). So we exclude cases (IV) and (V) from consideration. Further, if we consider the conditions of complete integrability for ψ of the fundamental equation (N. 4), the relations above become simpler. For from the conditions of complete integrability of (N. 14) it must be true that

$$\nabla_{[i}A_{j]}=0\,,$$

i. e., A_j must be a gradient vector: $A_j=\dfrac{\partial A}{\partial x^j}$; hence by the transformation $\psi=e^A\psi'$ we have the fundamental equation for ψ' for which $A_j=0$ and the vector $u^j/\sqrt{u^lu_l}$ is unaltered. So from the beginning we assume that $A_j=0$ in (N. 4). Under this assumption we consider the case in which f is a function of $\sqrt{M^2+N^2}$ only and does not contain x. Then we have

(1) When $\mu^*_{kl}\lambda_{kl}-\mu_{kl}\lambda^*_{kl}=0$, from (3) it must follow that

$$\mu_{kl}\lambda_{kl}-\mu^*_{kl}\lambda_{kl}=0\,.$$

From these equations it follows that $\overset{0}{f}{}'_{kl}=\overset{5}{f}{}'_{kl}=0$.

the result: When f is a function of $\sqrt{M^2+N^2}$, in order that $u^l=\psi^\dagger A\gamma^l\psi$, made from ψ, the solution of the fundamental equation for ψ (being completely integrable)

$$\nabla_i\psi=(A_i^{\cdot jk}\gamma_{jk}+A_i^{\cdot 5}\gamma_5+A_i^{\cdot j}\gamma_j+A_i^{\cdot j5}\gamma_j\gamma_5)\psi\ , \qquad \text{(N. 28)}$$

shall generate the path for which $\delta\int f ds=0$, f must be one of the following four kinds:

(I)′ f=arbitrary function of $\sqrt{M^2+N^2}$,

(I)″ f=constant,

(II) $f=\dfrac{1}{\sqrt{M^2+N^2}}$,

(III) $f=\sqrt{M^2+N^2}$,

with the corresponding fundamental equation (N. 28) determined by the following relation respectively: ($A_k^{\cdot pq}=a_k^{\cdot pq}+ib_k^{\cdot pq}$, $A_k^{\cdot 5}=a_k^{\cdot 5}+ib_k^{\cdot 5}$, $A_{kl}=a_{kl}+ib_{kl}$, $A_{kl}^{\cdot\cdot 5}=a_{kl}+ib_{kl}^{\cdot\cdot 5}$)

(I)′ $b_k^{\cdot pq}=0\ ,\qquad b_{kl}=b_{kl}^{\cdot\cdot 5}=0\ ,$

$a_{kl}=\overset{0}{a}g_{kl}\ ,\qquad a_{kl}^{\cdot\cdot 5}=\overset{5}{a}g_{kl}\ ,$

$2a_k^{\cdot pq}=\epsilon_k^{\cdot pqr}\chi_r\ ,\qquad b_k^{\cdot 5}=0\ ,$

(I)″ $2b_k^{\cdot pq}=\epsilon_k^{\cdot pqr}\tau_r$; the others being the same as in (I)′,

(II) $2b_k^{\cdot pq}=-i\epsilon_k^{\cdot pqr}b_r^{\cdot 5}+2\delta_k^{[p}\varphi^{q]}\ ,$

$2a_k^{\cdot pq}=\epsilon_k^{\cdot pqr}\chi_r\ ,$

$b_{kl}=\overset{0}{\beta}g_{kl}-\overset{0}{f}{}'^{*}_{kl}\ ,\qquad b_{kl}^{\cdot\cdot 5}=\overset{5}{\beta}g_{kl}+i\overset{0}{f}{}'_{kl}\ ,$

$a_{kl}=\overset{0}{a}g_{kl}-\overset{0}{f}_{kl}\ ,\qquad a_{kl}^{\cdot\cdot 5}=\overset{5}{a}g_{kl}-\overset{5}{f}_{kl}\ ,$

($\overset{0}{f}_{kl}$ and f'_{kl} are antisymmetric tensors),

(III) $2b_k^{\cdot pq}=i\epsilon_k^{\cdot pqr}b_r^{\cdot 5}\ ,$

$2a_k^{\cdot pq}=\epsilon_k^{\cdot pqr}\chi_r\ ,$

$b_{kl}=\overset{0}{f}{}'^{*}_{kl}\ ,\qquad b_{kl}^{\cdot\cdot 5}=-i\overset{0}{f}{}'_{kl}\ ,$

$a_{kl}=\overset{0}{a}g_{kl}+\overset{0}{h}_{kl}\ ,\qquad a_{kl}^{\cdot\cdot 5}=\overset{5}{a}g_{kl}+\overset{5}{h}_{kl}\ ,$

($\overset{0}{h}_{kl}$ and $\overset{5}{h}_{kl}$ are symmetric tensors),

$a_k^{\cdot 5}$ being arbitrary in all cases.

Specially, when $A_i^{\cdot jk}$, $A_i^{\cdot 5}$, A_{ij}, $A_{ij}^{\cdot\cdot 5}$ are all real, the relations above become simpler. And the corresponding fundamental equations for ψ become as follows:

$$\text{(I)}', \text{(I)}'': \quad \nabla_i\psi=\left(\frac{1}{2}\epsilon_i^{\cdot jkr}\chi_r\gamma_j\gamma_k+A_i^{\cdot 5}\gamma_5+\overset{0}{a}\gamma_i+\overset{5}{a}\gamma_i\gamma_5\right)\psi\,, \tag{N.29}$$

$$\text{(II)} \quad \nabla_i\psi=\left(\frac{1}{2}\epsilon_i^{\cdot jkr}\chi_r\gamma_j\gamma_k+A_i^{\cdot 5}\gamma_5+\overset{0}{a}\gamma_i+\overset{5}{a}\gamma_i\gamma_5-\overset{0}{f}_{ij}\gamma^j-\overset{5}{f}_{ij}\gamma^j\gamma_5\right)\psi\,, \tag{N.30}$$

$\overset{0}{f}_{ij}$ and $\overset{5}{f}_{ij}$ being antisymmetric,

$$\text{(III)} \quad \nabla_i\psi=\left(\frac{1}{2}\epsilon_i^{\cdot jkr}\chi_r\gamma_j\gamma_k+A_i^{\cdot 5}\gamma_5+\overset{0}{a}\gamma_i+\overset{5}{a}\gamma_i\gamma_5+\overset{0}{h}_{ij}\gamma^j+\overset{5}{h}_{ij}\gamma^i\gamma_5\right)\psi\,, \tag{N.31}$$

$\overset{0}{h}_{ij}$ and $\overset{5}{h}_{ij}$ being symmetric. ($\epsilon_i^{\cdot jkr}\chi_r$ is real in all cases; accordingly χ_r is a purely imaginary vector because $g<0$).

Note V.

Equation (N. 29) is already obtained in cosmology.[1] There it is seen that (N. 29) is reduced to either

$$\nabla_i\psi=\frac{1}{2}\epsilon_i^{\cdot jkr}\chi_r\gamma_j\gamma_k\psi$$

or

$$\nabla_i\psi=a\gamma_i\psi\,,$$

and the vector $\overset{1}{\lambda}{}^j=u^j/\sqrt{u^l u_l}$ satisfies the equation of geodesic line:

$$\overset{1}{\lambda}{}^l\nabla_l\overset{1}{\lambda}{}^k=0\,.$$

On the contrary, equations (N. 30) and (N. 31) give the following relations:

$$\overset{1}{\lambda}{}^l\nabla_l\overset{1}{\lambda}{}^k=-2\overset{1}{\lambda}{}^l(\overset{0}{f}_l^{\cdot k}M+\overset{5}{f}_l^{\cdot k}N)/\sqrt{M^2+N^2}\,. \tag{N.32}$$

and

$$\overset{1}{\lambda}{}^l\nabla_l\overset{1}{\lambda}{}^k=2\overset{1}{\lambda}{}^l(\overset{0}{h}_l^{\cdot k}M+\overset{5}{h}_l^{\cdot k}N)/\sqrt{M^2+N^2}-2\overset{1}{\lambda}{}^k(M\overset{0}{h}_{lm}+N\overset{5}{h}_{lm})\frac{\overset{1}{\lambda}{}^l\overset{1}{\lambda}{}^m}{\sqrt{M^2+N^2}} \tag{N.33}$$

For, using the relations (N. 5) and (N. 8), we have, in general,

$$\begin{aligned}\nabla_l\left(\frac{u_j}{\sqrt{M^2+N^2}}\right)&=\frac{1}{\sqrt{M^2+N^2}}\nabla_l u_j-\frac{u_j}{\sqrt{M^2+N^2}}(M\nabla_l M+N\nabla_l N)\\ &=4a_{lj}^{\cdot\cdot q}\overset{1}{\lambda}_q+2(a_{lj}m+a_{lj}^{\cdot\cdot 5}n)\\ &\quad+2i[2ib_{lj}\overset{*}{h}\overset{2}{\lambda}{}^h+b_l^{\cdot 5}\overset{2}{\lambda}_j+2b_l^{\cdot h}(-n\overset{[1\,2]}{\lambda_j\lambda_h}+im\overset{[3\,4]}{\lambda_l\lambda_h}+2b_l^{\cdot h5}(in\overset{[3\,4]}{\lambda_j\lambda_h}+m\overset{[1\,2]}{\lambda_j\lambda_h})]\\ &\quad-2\overset{1}{\lambda}_j[-2b_l^{\cdot pq}\overset{3}{\lambda}_p\overset{4}{\lambda}_q+(a_{lh}m+a_{lh}^{\cdot\cdot 5}n)\overset{1}{\lambda}{}^h+i(b_{lh}^{\cdot\cdot 5}m-b_{lh}n)\overset{2}{\lambda}{}^h]\,,\end{aligned}$$

where $m=\dfrac{M}{\sqrt{M^2+N^2}}$ and $n=\dfrac{N}{\sqrt{M^2+N^2}}$. Hence

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(1) T. Sibata: This Journal, 8 (1938), 206.

$$\begin{aligned}\overset{1}{\lambda}{}^l \nabla_l \overset{1}{\lambda}_j = & 4a_{ljq}\overset{1}{\lambda}{}^l\overset{1}{\lambda}{}^q + 2i^l(a_{lj}m + a_{lj}^{\cdot\cdot 5}n) - 4b^{*}_{ljh}\overset{1}{\lambda}{}^l\overset{2}{\lambda}{}^h \\ & + 2i\overset{2}{\lambda}_j\{b_l^{\cdot 5}\overset{1}{\lambda}{}^l + (nb_{lh} - mb_{lh}^{\cdot\cdot 5})\overset{1}{\lambda}{}^l\overset{1}{\lambda}{}^h\} \\ & - 2(mb_{lh} + nb_{lh}^{\cdot\cdot 5})(\overset{1}{\lambda}{}^j\overset{4}{\lambda}{}^h\overset{3}{\lambda}_j - \overset{1}{\lambda}{}^l\overset{3}{\lambda}{}^h\overset{4}{\lambda}_j) \\ & + 2\overset{1}{\lambda}_j\{2b_{lpq}\overset{1}{\lambda}{}^l\overset{3}{\lambda}{}^p\overset{4}{\lambda}{}^q - (ma_{lh} + na_{lh}^{\cdot\cdot 5})\overset{1}{\lambda}{}^l\overset{1}{\lambda}{}^h\};\end{aligned}$$

so that for equation (N. 30) and (N. 31) we have (N. 32) and (N. 33) respectively.

By putting $\overset{1}{\lambda}{}^k = \dfrac{dx^k}{ds}$, (N. 32) and (N. 33) are expressed in the form:

$$\frac{d^2x^k}{ds^2} + \left\{{k \atop lm}\right\}\frac{dx^l}{ds}\frac{dx^m}{ds} = F_l^{\cdot k}\frac{dx^l}{ds}, \qquad \text{(N. 32)}'$$

where
$$F_{lk} = -2(\overset{0}{f}_{lk}M + \overset{5}{f}_{lk}N)/\sqrt{M^2+N^2},$$

and
$$\frac{d^2x^k}{ds^2} + \left\{{k \atop lm}\right\}\frac{dx^l}{ds}\frac{dx^m}{ds} = H_l^{\cdot k}\frac{dx^l}{ds} - \frac{dx^k}{ds}H_{lm}\frac{dx^l}{ds}\frac{dx^m}{ds}, \qquad \text{(N. 33)}'$$

where
$$H_{kl} = 2(\overset{0}{h}_{lk}M + \overset{5}{h}_{lk}N)/\sqrt{M^2+N^2}.$$

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