Wave Geometry No. 40

Mathematical Foundations of Wave Geometry. I.

By

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Introduction and Summary.

Wave Geometry is constructed by taking $ds\psi=\gamma_i dx^i\psi\ (\gamma_{(i}\gamma_{j)}=g_{ij}I)$ as the metric of 4-dimensional manifold. This metrical form is introduced to unify general relativity and quantum mechanics in a natural way from the physical angle; but the mathematical basis for adopting such a metric representation has not yet been fully established. Hence to establish the foundations of Wave Geometry, it is very important to pursue the problem along this line.

The purpose of this paper is (1) to construct a theory of number system $\mathfrak{B}$ (or operator system) which shall be a generalization and abstraction of Dirac's γ_i or Eddington's E-number, (2) to show how this new system can imply, as a special case, Cliford's number, including Dirac's matrices γ_i and Eddington's E-numbers, and how it is related to matrix representation, and (3) to establish the methematical foundations of Wave Geometry by constructing a geometry of the manifold of this system, without using coordinates independent of number system or operator as in the geometry of operators hitherto proposed.

This investigation especially distinguished from those in Wave Geometry hitherto have previously appeared in the following respects: Elements of the ideal in this number system $\mathfrak{B}$ play the rôle of ψ (wave function) used as operands for operators; therefore all quantities in consideration belong to $\mathfrak{B}$. And, using this ideal, we introduce the conception of "norm" of any element in $\mathfrak{B}$, by which we can discuss convergency of any series in $\mathfrak{B}$.

In this paper, as the first step, we shall restrict our investigation to when the number of linearly independent elements in $\mathfrak{A}$ is finite, leaving the case of infinite linearly independent elements for future papers now in preparation.

In § 1, we construct from a general (original) operator set $\mathfrak{M}$ and a corpus $\mathfrak{K}$ a linear manifold $\mathfrak{A}$ in the field $\mathfrak{K}$ satisfying Axiom I. Then we find some relations among $\mathfrak{M}$, $\mathfrak{K}$, and $\mathfrak{A}$ (Theorem 1).

In § 2, we introduce the multiplication and addition of elements of $\mathfrak{A}$, and construct a manifold $\mathfrak{B}$ from the multiplication and addition of elements of $\mathfrak{A}$ in the field $\mathfrak{K}$; and we consider the special case in which the square of each element of $\mathfrak{A}$ becomes a definite element M of $\mathfrak{B}$ with a factor of $\mathfrak{K}$ (or, in another words, to the iteration of the element of M there corresponds an element of $\mathfrak{K}$); we call such a system "*an iteration system.*" In the iteration system a normal base system $\mathring{a}_1, \mathring{a}_2, \ldots, \mathring{a}_{n-n_0}, \mathring{a}_{n-n_0+1}, \ldots, \mathring{a}_n$ of $\mathfrak{A}$ can be introduced such that

$$\mathring{a}_{(i}\mathring{a}_{j)}=p_{ij}M\,,\quad p_{lm}=\delta_{lm}\,,\quad p_{aj}=0\quad \begin{pmatrix} l, m=1, 2, \ldots, n-n_0;\ a=n-n_0+1, \\ \ldots, n;\ j=1, 2, \ldots, n \end{pmatrix}.$$

In particular, in the above when $M=kI$, $n_0=0$ and $\mathfrak{K}$ is the complex number system, it is shown that this iteration system $\mathfrak{B}$ becomes equivalent to Eddington's E-number system.

In § 3, it is proved that in the iteration system the element M belongs to the "zentrum" of $\mathfrak{B}$, and each element of $\mathfrak{B}$ is expressed by a linear combination of $a_1, a_2, \ldots\ldots, a_{[1}\ldots a_{n]}$ with coefficients of functions of M (Theorem 6).

In § 4, assuming in the iteration system that $\beta_0=0$ follows from $M\beta_0=0 (\beta_0 \in \mathfrak{B})$, we find the zentrum of $\mathfrak{B}$ (Theorem 9). Then, by investigating the possibility of extending $\mathfrak{A}$ while preserving its iterational character by adjoining an element of $\mathfrak{B}$ (Theorem 12), we answer the question: When is $\mathfrak{A}$ the minimum base system?

In § 5, we consider the problem: Is there any relation among the elements of the base system $a_1, a_2, \ldots, a_n, \ldots, a_{[1}\ldots a_{n]}$; and if it exists, what is the expression of that relation? We show that the totality of relations can be reduced to a system of independent relations (Theorem 16–20). And in the special case when $n_0=0$, we show that for even n the relation, if it exists, is

$$g(M)=0$$

and no more; for odd n

$$\tau\Big(g(M)\tilde{g}(M)+\tilde{g}(M)\overset{I}{A}\Big)=0\,,\qquad g'(M)\tilde{g}(M)=0$$

and no more, where $\tau=1$ or 0 $(\tau \in \mathfrak{K})$ and g, g' are prime to each other.

§ 1. A central linear manifold.

Consider a set $\mathfrak{M}$ of any operators, and a corpus $\mathfrak{K}$ (this may be a ring). We assume that the multiplication (right or left) of each element of $\mathfrak{M}$ with each element of $\mathfrak{K}$, and the addition of operators thus introduced, are known. Now, by this procedure, we construct a linear manifold of operators from $\mathfrak{M}$ in the field $\mathfrak{K}$, which we call the *linear manifold* $\mathfrak{A}$, expressing it by $\mathfrak{A}\equiv[\mathfrak{K}, \mathfrak{M}]$ or $[\mathfrak{M}, \mathfrak{K}]$ according as multiplication is right or left. More precisely, if $k, k_1, k_2 \in \mathfrak{K}$ and $m, m_1, m_2 \in \mathfrak{M}$, then

$$km \in \mathfrak{A} \quad\text{and}\quad k_1m_1+k_2m_2 \in \mathfrak{A} \qquad \text{(for left multiplication).}$$

And if a is any element of $\mathfrak{A}$, it can be expressed in the form:

$$a=\sum k_i m^i\,, \quad \text{where} \quad k_i \in \mathfrak{K}\,, \quad m^i \in \mathfrak{M}\,.\text{[1]}$$

With regard to $\mathfrak{A}$, we make the following assumptions:
If $k, k_1, k_2 \in \mathfrak{K}$ and $a, a_1, a_2, \in \mathfrak{A}$, then

$$(\mathrm{I})_1 \begin{cases} a=a\,, \\ \text{if } a_1=a_2 \text{ and } a_2=a_3 \text{ then } a_1=a_3\,, \\ a_1+a_2=a_2+a_1\,, \\ a_1+a_2+a_2=a_1+(a_2+a_3)\,, \\ k_1(k_2a)=(k_1k_2)a\,, \\ k(a_1+a_2)=ka_1+ka_2\,, \\ (k_1+k_2)a=k_1a+k_2a_2\,. \end{cases}$$

$$(\mathrm{I})_2 \begin{cases} \text{If a null element 0 is contained in } \mathfrak{K}\text{, then, for the element 0,} \\ \qquad 0.\,a_1=0.\,a_2=\mathbf{O}\,; \\ \text{we call this } \mathbf{O} \text{ the null element of } \mathfrak{A}. \\ \text{For the null element } \mathbf{O}, \\ \qquad a+\mathbf{O}=a\,. \end{cases}$$

$(\mathrm{I})_2'$ If $a=a'$, then $a+a_1=a'+a_1$.

$(\mathrm{I})_3$ $ka=ak$ for any k and a.

Remark 1: In the definition above we do not always need $\mathfrak{M}\subset\mathfrak{A}$ or $1.\,m=m$, even when there exists a unit element 1 in $\mathfrak{K}$; but in latter case we know that if we put $1.\,m=\mathring{a}$, then

$$1.\,\mathring{a}=\mathring{a}$$

(for $1.\,(1.\,m)=1.\,m=\mathring{a}$ by $(\mathrm{I})_1$).

Remark 2: When a unit element 1 is contained in $\mathfrak{A}$, $a+\mathbf{O}=a$ follows from other axioms; for, from $(\mathrm{I})_1$ and the first part of $(\mathrm{I})_2$, we have

$$(1+0)a=1.\,a+0.\,a=a+\mathbf{O}$$

and $$(1+0)a=1.\,a=a\,,$$

hence $$a+\mathbf{O}=a\,.$$

Remark 3: When $(k_1+k_2)m=k_1m+k_2m(k_1, k_2\in\mathfrak{K}; m\in\mathfrak{M})$ and $k_1k_2m=k_1(k_2m)$, $(\mathrm{I})_2$ and $(\mathrm{I})_2'$ can be replaced by:

$$(\mathrm{I})_2^* \begin{cases} \text{If} \quad a=a' \text{ then} \\ \qquad a+a_1=a'+a_1 \\ \text{and } a+a_1-a_1=a\,. \end{cases}$$

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(1) In this paper, subscript does not show power of element but is a notation denoting different element; and when we want to write the powers we shall put bracket s $(a)^i$.

(Assumption $(\mathrm{I})_2^*$ can be used even when $\mathfrak{A}$ has not the null element.)
Proof. When $\mathfrak{A}$ has the null element as defined by $(\mathrm{I})_2$, it can be proved that $(\mathrm{I})_2$ and $(\mathrm{I})_2' \equiv (\mathrm{I})_2^*$.
For, since, by $(\mathrm{I})_2$, $a+0a_1=a$, we have from the assumption $(\mathrm{I})_1$

$$0.\,a_1=0\sum k_i m^i=\sum(0k_i)m^i=\sum(k_i-k_i)m^i=\sum k_i m^i-\sum k_i m^i=a_1-a_1\,,$$

hence
$$a+a_1-a_1=a\,;$$
therefore $(\mathrm{I})_2^*$ follows from $(\mathrm{I})_2$. Conversely, since, by $(\mathrm{I})_2^*$;
$$a+a_1-a_1=a\,,$$
we have, from $(\mathrm{I})_1$,
$$a+(a_1-a_1)=a \quad \text{or} \quad a+0a_1=a\,, \tag{1.1}$$
and by $(\mathrm{I})_2^*$
$$(-a)+a+0a_1=(-a)+a$$
or
$$0a+0a_1=0a\,,$$
and similarly
$$0a_1+0a=0a_1\,;$$
so that
$$0a_1=0a\,.$$

Therefore, from (1.1) and the preceding equation, we know that $(\mathrm{I})_2$ is implied in $(\mathrm{I})_2^*$.

Theorem 1. *When a unit element* 1 *is contained in* $\mathfrak{K}$, *the necessary and sufficient condition for* $\mathfrak{M}\subset\mathfrak{A}$ *is that* $1.\,m=m$, *where* $\boldsymbol{m}$ *is any element of* $\mathfrak{M}$.

Proof. Suppose $\mathfrak{M}\subset\mathfrak{A}$. Then, for each element m of $\mathfrak{M}$ there exists at least a set of element k^i of $\mathfrak{K}$ and m_i of $\mathfrak{M}$ such that
$$m=\sum k_i m^i\,; \tag{1.2}$$
multiplying both sides by 1 (the unit of $\mathfrak{K}$) we get
$$1.\,m=1\sum k_i m^i\,. \tag{1.3}$$
But, from Assumption (I),
$$1\sum k_i m^i=\sum k_i m^i\,;$$
so from (1.2) and (1.3), we have for any element m of $\mathfrak{M}$
$$1.\,m=\sum k_i m^i=m\,.$$
And conversely, if $1.\,m=m$ for any element m of $\mathfrak{M}$ and unit element 1 of then $\mathfrak{M}\subset\mathfrak{A}$, clearly. So we have proved the theorem. Q. E. D.

Let $a_1, a_2, \ldots\ldots, a_n$ be the totality of linearly independent elements of $\mathfrak{A}$; then any element of $\mathfrak{A}$ may be expressed by a linear combination of a's, the coefficients being elements of $\mathfrak{K}$. We then express $\mathfrak{A}$ by
$$\mathfrak{A}=[a_i]$$
and call it *a central linear manifold*.

§ 2. An iteration system and its normal basis.

Here we assume that the multiplication between the elements of $[a_i]$ is known such that the associative and distributive laws hold; i. e.,

$$(\mathrm{II})_0 \begin{cases} a_i a_j a_k = a_i(a_j a_k)\,, \\ a_i(a_j + a_k) = a_i a_j + a_i a_k\,, \\ (a_i + a_j)a_k = a_i a_k + a_j a_k\,. \end{cases}$$

Now consider the totality of numbers made from $\mathfrak{A}$ by multiplication and summation in the field $\mathfrak{K}$, and denote it by $\mathfrak{B}$, $\mathfrak{B} \equiv \{[a_i]\}$. But we must not say that all the elements of $\mathfrak{B}$ always belong to $\mathfrak{A}$. Further, for $\mathfrak{B}$ we make the asssumptions:

$$(\mathrm{II})_1 \begin{cases} \beta = \beta\,. \\ \text{If } \beta_1 = \beta_2\,, \quad \beta_2 = \beta_3\,, \quad \text{then } \beta_1 = \beta_3\,; \end{cases}$$

$$(\mathrm{II})_2 \begin{cases} \beta_1 + \beta_2 = \beta_2 + \beta_1\,, \\ \beta_1 + \beta_2 + \beta_3 = \beta_1 + (\beta_2 + \beta_3)\,, \\ \beta_1(\beta_2 + \beta_3) = \beta_1\beta_2 + \beta_1\beta_3\,, \\ (\beta_2 + \beta_3)\beta_1 = \beta_2\beta_1 + \beta_3\beta_1\,. \\ \text{If } \beta = \beta' \text{ then } \beta + \beta_1 = \beta' + \beta_1\,. \end{cases}$$

$(\mathrm{II})_3 \quad 0\beta_2 = 0\beta_3 = \mathbf{O}\,, \quad \mathbf{O} + \beta = \beta\,.$[1]

$(\mathrm{II})_4 \quad k(\beta_1 + \beta_2) = k\beta_1 + k\beta_2\,, \quad (k \in \mathfrak{K};\ \beta_1,\ \beta_2,\ \beta_3 \in \mathfrak{B})\,.$

From these assumptions we have

Theorem 2. (i) $\beta_1\beta_2\beta_3 = \beta_1(\beta_2\beta_3)$, (ii) $(k_1 + k_2)\beta = k_1\beta + k_2\beta$, (iii) $k_1k_2\beta = k_1(k_2\beta)$, (iv) $k\beta = \beta k$.

The theorem can easily be proved, so the proof will be omitted here; we need only notice that

(i) follows from $(\mathrm{II})_2$ and $(\mathrm{I})_1$,
(ii) follows from $(\mathrm{II})_2$, $(\mathrm{II})_4$, and $(\mathrm{I})_1$,
(iii) follows from $(\mathrm{II})_4$ and $(\mathrm{I})_1$,
(iv) follows from $(\mathrm{II})_4$ and $(\mathrm{I})_3$.

Next, in order to characterize the numbers of $\mathfrak{A}$ (or $\mathfrak{B}$), we make the following assumptions:

III To each element of $\mathfrak{A}$ there *corresponds* only one element of $\mathfrak{K}$ by an operation called iteration.[2]

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(1) Using Theorem 2 (ii), we can easily see that when a unit element 1 is contained in $\mathfrak{B}$, the latter part of Axiom $(\mathrm{II})_3$, i. e. $\beta + \mathbf{O} = \beta$, follows from other axioms. And $\mathbf{O}$ in $\mathfrak{A}$ becomes $\mathbf{O}$ in $\mathfrak{B}$.

(2) We can generalize the idea of "the iteration," if we correspond to an element a of $\mathfrak{A}$ another element $\overset{*}{a}$ of $\mathfrak{A}$ (or $\mathfrak{B}$) in a certain definite manner S. If S is needed to satisfy the condition of involution, we must take it that $(\overset{*}{a})^* = a$, and as the iteration define $\overset{*}{a}a$.

(i) We call the operator of squaring of an element of $\mathfrak{K}$ the *iteration* of the element.

(ii) If we take from $a_ia_j+a_ja_i$, for all suffices, all the operators, say P_λ, linearly independent of each other in the field $\mathfrak{K}$, then we have

$$(k^ia_i)^2=\sum_\lambda p^\lambda P_\lambda \qquad (p^\lambda\in\mathfrak{K}).^{(1)}$$

Thus we see that to an element k^ia_i of $\mathfrak{A}$ there *corresponds* a set of numbers p^a of $\mathfrak{K}$. When there is only one P_a, we say that to the iteration of a there *corresponds a single number p' of* $\mathfrak{K}$.

The number system $\mathfrak{B}$ which satisfies axioms **I**, **II**, **III** is called "*an iteration system.*"

Now, returning to our case, from Assumption **III** it follows that

$$a_ia_j+a_ja_i=p_{ij}M \quad \text{for} \quad i\neq j \quad \text{and} \quad a_ia_i=p_{ii}M \tag{2.1}$$

where $p_{ij}\in\mathfrak{K}$ and $M\in\mathfrak{B}$.(2)
Conversely, from (2.1), we see that each element belonging to $\mathfrak{A}$, say k^ia_i, satisfies the iteration condition, i. e.

$$k^ia_i=\sum_{i\leq j}k^ik^jp_{ij}M=kM \qquad \Big(\sum_{i\leq j}k^ik^jp_{ij}=k(\in\mathfrak{K})\Big). \tag{2.2}$$

So we see that through the iteration a number $k(\in\mathfrak{K})$ corresponds to the operator k^ia_i. So we have

Theorem 3. *In order that $\mathfrak{A}$ may satisfy Assumption* **III**, *it is necessary and sufficient that*

$$a_ia_j+a_ja_i=q_{ij}M$$

where $q_{ij}\in\mathfrak{K}$, and M is a definite non-null element of $\mathfrak{B}$.

The relation (2.1) is a fundamental condition characterizing the number of iteration system that we are going to deal with.)

Theorem 4. *When $\mathfrak{K}$ is a corpus,*(3) *in the relation*

$$\frac{1}{2}(a_ia_j+a_ja_i)=p_{ij}M \qquad (i=1, 2, \ldots\ldots, n, \text{ finite})$$

if the rank of p_{ij} is n, a_i $(i=1,\ldots\ldots,n)$ are linearly independent. But the converse is not always true.

Proof. Let the rank of p_{ij} be n, and a_i $(i=1,\ldots\ldots,n)$ be linearly dependent; then we shall have $k^ia_i=0$ (when there is no null element in $\mathfrak{A}$, make $k^ia_i+a=a$ stand for $k^ia_i=0$). Multiply the equation by a_j from the right

(1) Hereafter we shall use the convention that when the same letter appears in any term as a subscript and a superscript, it is to be understood that this letter is summed for all the values.

(2) When $\mathfrak{K}$ is a corpus, we have $\frac{1}{2}(a_ia_j+a_ja_i)=p_{ij}M$ and $(k^ia_i)^2=k^ik^jp_{ij}M$ instead of (2.1) and (2.2).

(3) Refer to footnote (1) on Page 8 for the case when $\mathfrak{K}$ is a ring.

side and from the left respectively and taking the summation of two resulting equations side by side,

$$k^i p_{ij} M = 0\,;\text{[1]}$$

therefore[2] $k^i M = 0$ (because the rank of p_{ij} is n). So that, to prove the last part of the theorem, we can take a special example.[3] If we take

$$a_1 = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & X_1 \end{array}\right), \quad a_2 = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & X_2 \end{array}\right), \ldots\ldots\ldots, a_{n-n_0} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & X_{n-n_0} \end{array}\right),$$

$$a_{n-n_0+1} = \left(\begin{array}{c|c} \begin{matrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right), \quad a_{n-n_0+2} = \left(\begin{array}{c|c} \begin{matrix} 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right), \ldots,$$

$$a_n = \left(\begin{array}{c|c} \begin{matrix} 0 & \cdots & 0 & 1 \\ 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right),$$

where X_i's $(i=1, \ldots, n-n_0)$ are square matrices satisfying the relation

$$X_{(l} X_{m)} = \delta_{lm} M \qquad (l, m = 1, \ldots, n-n_0),$$

then a_i's are all independent of each other, and the rank of p_{ij} is $n-n_0$, because in this case

$$\begin{array}{l} p_{lm} = \delta_{lm}, \\ p_{aj} = 0 \end{array} \qquad \left(\begin{array}{l} i, j = 1, \ldots, n;\ l, m = 1, \ldots, n-n_0 \\ a = n-n_0+1, \ldots, n \end{array}\right)$$

and M is of the form $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & M' \end{array}\right)$.

Theorem 5. *When $\mathfrak{K}$ is a corpus, in*

$$\frac{1}{2}(a_i a_j + a_j a_i) = p_{ij} M \qquad (i = 1, 2, \ldots, n)$$

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(1) Hereafter we denote null element in $\mathfrak{B}$ by 0 (in usual letter).

(2) When $\mathfrak{K}$ is a ring, we bave $\Delta k^i M = 0$ (where $\Delta = |p_{ij}|$) in place of $k^i M = 0$ in Theorem 4. Hence this theorem is true for $\mathfrak{B}$ in which $k=0$ or $M=0$ follows from $kM=0 (k \in \mathfrak{K})$ even when $\mathfrak{K}$ is not a corpus.

(3) These matrices X_i actually exist.

if the rank of (p_{ij}) *is* $n-n_0$, $\mathfrak{A}=[a_i]$ *contains just* $n-n_0$ *linearly independent elements which are orthogonal to each other* (i. e. $\mathring{a}_l\mathring{a}_m=\delta_{lm}M$) *and their iterations are not null.*

Proof. If the rank of the (p_{ij}) is $n-n_0$, we can find a linear transformation $\mathring{a}_j=h^i_j a_i$ of the base system $a_i\,(i=1,\dots,n)$ ($|h^i_j|\neq 0$, $h^i_j\in\mathfrak{K}$) such that

$$\mathring{p}_{lm}=\delta_{lm},\quad \mathring{p}_{ai}=0\quad \begin{pmatrix} l=1,\dots,n-n_0;\ a=n-n_0+1,\dots,n; \\ i=1,\dots,n \end{pmatrix}$$

and $\frac{1}{2}(\mathring{a}_i\mathring{a}_j+\mathring{a}_j\mathring{a}_i)=\mathring{p}_{ij}M$; in fact, $\mathring{p}_{ij}=h^h_i h^k_j p_{hk}$ and $\mathfrak{A}=[a_i]=[\mathring{a}_i]$.

So, from Theorem 4, we know that $\overset{I}{\mathfrak{A}}=[\mathring{a}_l]$ $(l=1,2,\dots,n-n_0)$ contains just $n-n_0$ linearly independent elements which are orthogonal to each other and whose iterations are non-null. Therefore $\mathfrak{A}=[a_i]$ contains at least $n-n_0$ linearly independent elements which are orthogonal and whose iterations are non-null. Hereafter we call the elements whose iteration becomes null, i. e., $a^2=0$, *a nil potent element.*

If $\mathfrak{A}=[a_i]$ contains $n-n_0+p$ $(p\neq 0)$ non-nil potent elements say $\mathring{a}_1,\dots,\mathring{a}_{n-n_0+p}$ which are linearly independent and anticommutative to each other, i. e. $\mathring{a}_{(\lambda}\mathring{a}_{\mu)}=\delta_{\lambda\mu}M$ $(\lambda,\mu=1,2,\dots,n-n_0+p)$,[1] then the rank of (p_{ij}) is greater than $n-n_0$; but this contradicts the assumption that the rank of $p(_{ij})$ is $n-n_0$. Therefore $\mathfrak{A}=[a_i]$ has just $n-n_0$ non-null bases. Q. E. D.

If in $\mathfrak{A}=(a_i]$, a's are linearly independent (in $\mathfrak{K}$) and the rank of p_{ij} in $a_{(i}a_{j)}=p_{ii}M$ is $n-n_0$, then, by the same process as that used in proving Theorem 5, we can choose the base $\mathring{a}_i$ such that

$$\begin{matrix}\mathring{a}_{(l}\mathring{a}_{m)}=\delta_{lm}M \\ \mathring{a}_{(a}\mathring{a}_{j)}=0\end{matrix}\qquad \begin{pmatrix} j=1,2,\dots,n;\ l,m=1,\dots,n-n_0; \\ a=n-n_0+1,\dots,n \end{pmatrix} \tag{2.3}$$

And since a's are linearly independent and accordingly form a base system of $\mathfrak{A}$, $\mathfrak{A}$ can be written as follows:

$$\mathfrak{A}=[\mathring{a}_i]=\overset{I}{\mathfrak{A}}+\overset{N}{\mathfrak{A}} \tag{2.4}$$

where $\overset{I}{\mathfrak{A}}\equiv]\mathring{a}_l]$ and $\overset{N}{\mathfrak{A}}\equiv[\mathring{a}_a]$ $(l=1,\dots,n-n_0;\ a=n-n_0+1,\dots,n)$. Here $\overset{I}{\mathfrak{A}}$ has the following property: each element is not anticommutative with all of $\overset{I}{\mathfrak{A}}$; but, on the other hand, $\overset{N}{\mathfrak{A}}$ has the property: each element of $\overset{N}{\mathfrak{A}}$ is anticommutative with each element of $\mathfrak{A}$ (of course $\overset{N}{\mathfrak{A}}$). Further, we can easily show that $\overset{I}{\mathfrak{A}}$ and $\overset{N}{\mathfrak{A}}$ are maximum linear manifolds in $\mathfrak{A}$ having the above-mentioned property and satisfying (2.4); and for such separation of

(1) $\mathring{a}_{(\lambda}\mathring{a}_{\mu)}\equiv\frac{1}{2}(\mathring{a}_\lambda\mathring{a}_\mu+\mathring{a}_\mu\mathring{a}_\lambda)$

$\mathfrak{A}$, $\overset{N}{\mathfrak{A}}$ is invariant but $\overset{I}{\mathfrak{A}}$ is not uniquely determined unless $\overset{N}{\mathfrak{A}}=0.$[1] From these properties of $\mathring{a}_i$ as seen in (2.3), we call this base system $\mathring{a}_i$ the *normal base system* of $\mathfrak{A}$. Hereafter we denote $\{\overset{I}{\mathfrak{A}}\}$ and $\{\overset{N}{\mathfrak{A}}\}$ by $\overset{I}{\mathfrak{B}}$ and $\overset{N}{\mathfrak{B}}$ respectively, i. e.,

$$\overset{I}{\mathfrak{B}}\equiv\{\overset{I}{\mathfrak{A}}\}\equiv\{[\mathring{a}_l]\} \quad \text{and} \quad \overset{N}{\mathfrak{B}}\equiv\{\overset{N}{\mathfrak{A}}\}\equiv\{[\mathring{a}_a]\} \quad \begin{pmatrix} l=1,\dots,n-n_0; \\ a=n-n_0+1,\dots,n \end{pmatrix}.$$

§ 3. Properties of M and canonical expression for elements of $\mathfrak{B}$ by α's.

Taking the normal base system $\mathring{a}_i$, we shall investigate some important properties of M.

When $\mathfrak{K}$ is a corpus; since

$$\begin{matrix} \mathring{a}_l\mathring{a}_l=M \\ \mathring{a}_a\mathring{a}_l+\mathring{a}_l\mathring{a}_a=0 \end{matrix} \qquad \begin{pmatrix} l=1,2,\dots,n-n_0 \\ a=n-n_0+1,\dots,n \end{pmatrix} \tag{3.1}$$

we have, multiplying by $\mathring{a}_l$ from the right-hand side,

$$\mathring{a}_lM=M\mathring{a}_l \quad \text{and} \quad \mathring{a}_aM+\mathring{a}_l\mathring{a}_a\mathring{a}_l=0\,. \tag{3.2}$$

From (3.1) the last equations become

$$\mathring{a}_aM=M\mathring{a}_a \qquad (a=n-n_0+1,\dots,n)\,.$$

From this and (3.2),

$$\mathring{a}_iM=M\mathring{a}_i \qquad (i=1,2,\dots,n)\,.$$

So we see that M is commutative with each element of $\mathfrak{A}$, i. e.

$$\alpha M=M\alpha \qquad (\alpha\in\mathfrak{A})\,.$$

Further, from Assumption II, we can easily see that M is also commutative with each element of $\mathfrak{B}$, i. e.

$$\beta M=M\beta \qquad (\beta\in\mathfrak{B})\,.$$

Also, accordingly, since $M\in\mathfrak{B}$, from **(II)**, we can deduce that any polynominal of M with coefficients from $\mathfrak{K}$ is commutative with any element of $\mathfrak{B}$, i. e.

$$\beta F(M)=F(M)\beta\,.$$

So we have

Theorem 6. *When $\mathfrak{K}$ is a corpus, any polynominal of M with coefficients from $\mathfrak{K}$ belongs to "the zentrum" of $\mathfrak{B}$.*

When $\mathfrak{K}$ is a ring, since $(k^ia_i)^2=kM$, and from the relation

$$(k^ia_i)^2(k^ia_i)=k^ia_i(k^ia_i)^2\,,$$

we have $kMk^ia_i=k^ia_ikM$ or $Mkk^ia_i=kk^ia_iM$.

---

(1) When $\mathfrak{A}=[\mathring{a}_e]+[\mathring{a}_a]=\overset{I}{\mathfrak{A}}+\overset{N}{\mathfrak{A}}$ is an above-mentioned separation,

$$\mathfrak{A}=[\mathring{a}_l']+[\mathring{a}_a]=\overset{I}{\mathfrak{A}}{}'+\overset{N}{\mathfrak{A}} \qquad (\mathring{a}_l'=\mathring{a}_l+g_l^a\mathring{a}_a\,;\ g_l^a\in\mathfrak{K})$$

is also an above-mentioned separation and $\overset{I}{\mathfrak{A}}\neq\overset{I}{\mathfrak{A}}{}'$.

This shows that $kk^i\alpha_i$ and M are commutative, and therefore $kk^i\alpha_i$ is commutative with any polynominal of M with coefficients from $\mathfrak{K}$.

Remark. When $\mathfrak{K}$ is a ring, the theorem is true if we take $\varDelta M$ for M, where $\varDelta$ is the determinant of the matrix of the highest rank in (p_{ij}). For, when $\mathfrak{K}$ is a ring, in (3.1) we can make some modifications, as follows:

$$\mathring{\alpha}_l\mathring{\alpha}_l=\varDelta M \quad \text{and} \quad \mathring{\alpha}_a\mathring{\alpha}_l+\mathring{\alpha}_l\mathring{\alpha}_a=0 .$$

So that we get (by the same procedure as in obtaining (3.3) from (3.1))

$$\mathring{\alpha}_i\varDelta M=\varDelta M\mathring{\alpha}_i$$

and

$$\beta F(\varDelta M)=F(\varDelta M)\beta \qquad (\beta\in\mathfrak{B}) . \qquad \text{Q. E. D.}$$

Accordingly, all the theorems concerning M, when $\mathfrak{K}$ is a corpus, are transfered to those when $\mathfrak{K}$ is a ring by putting $\varDelta M$ in place of M. So that we can assume, hereafter, that $\mathfrak{K}$ is a corpus, so far as we are concerned with M.

As an application of Theorem 6, we shall find "*the canonical form*" of the expression for the element of $\mathfrak{B}$ in terms of α's.

For convenience we use the symbolical notation Ⓘ such that

$$\beta+\beta\beta_1=\beta(\text{Ⓘ}+\beta_1)$$
$$\beta+\beta_1\beta=(\text{Ⓘ}+\beta_1)\beta ,$$

even when there exists no unit element of $\mathfrak{B}$ and, accordingly, in such a case Ⓘ has no real meaning in itself, but has meaning when it operates to each element of $\mathfrak{B}$. Using this symbol, we have

Theorem 7. *Each element of $\mathfrak{B}$ can be expressed linearly by the base elements* $M, \alpha_1, \alpha^2, \ldots\ldots, \alpha_{[1}\alpha_{2]}, \ldots\ldots, \alpha_{[1}, \ldots\ldots \alpha_{n]}$,(1) *the coefficients being polynomials of* M, i. e.,

$$\beta=f(M)+f^i\alpha_i+f^{i_1i_2}\alpha_{[i_1}\alpha_{i_2]}+\cdots\cdots+f^{i_1\ldots\ldots i_n}\alpha_{[i_1}\ldots\ldots\alpha_{i_n]} \qquad (3.5)$$

where $f(M)$ is a polynomial of M, other f's are polynomials of M and Ⓘ *with the coefficients from $\mathfrak{K}$.*

Remark. Though $f^{i_1\ldots\ldots i_r}$ may contain the meaningless unit Ⓘ, each term of (3.5) has real meaning, since they operate to, or are operated by, α's, the elements of $\mathfrak{B}$.

Proof. Any element of $\mathfrak{B}$ can be expressed in a polynomial of normal base elements $\mathring{\alpha}$'s, i. e.

$$\beta=\sum_{a i_1\ldots\ldots a i_p} p^{a_{i_1}\ldots\ldots a_{i_p}}(\mathring{\alpha}_{i_1})^{a_{i_1}}\ldots\ldots(\mathring{\alpha}_{i_p})^{a_{i_p}} \qquad (p\text{'s}\in\mathfrak{K}) \qquad (3.6)$$

But, from (3.1),

$$p^{a_{i_1}\ldots\ldots a_{i_p}}(\mathring{\alpha}_{i_1})^{a_{i_1}}\ldots\ldots(\mathring{\alpha}_{i_p})^{a_{i_p}}=q^{a_1'\ldots\ldots a_n'}(\mathring{\alpha}_1)^{a_1'}\ldots\ldots(\mathring{\alpha}_n)^{a_n'} \quad \left(\begin{array}{l} q\text{'s}\in\mathfrak{K};\ \sum a_{i_r}=\sum a_i'; \\ p^{a_{i_1}\ldots\ldots a_{i_p}}=\pm q^{a_1'\ldots\ldots a_n'}; \\ \text{not summed by } a_{i_r} \end{array}\right)$$

(1) This notation is the same in Der Ricci-Kalkül (J. A. Schouten), p. 26.

provided that $(\mathring{a}_i)^0=\text{①}$. Further, from **III** and (3.3),

$$q^{a'_1\dots a'_n}(\mathring{a}_1)^{a'_1}\dots(\mathring{a}_n)^{a'_n}$$

can be written, using the symbol $(M)^0\equiv\text{①}$,

$$\tau(M)^a q^{a'_1\dots a'_n}(\mathring{a}_1)^{c_1}\dots(\mathring{a}_n)^{c_n}$$

where $c_i=1$ or 0; $a=\sum_{l=}^{n-n_0} p'_l$ when we write $a'_l=2p'_l+c_l$; if all indices $a'_\alpha<2$ $(\alpha=n-n_0+1,\dots,n)$ then $\tau=\text{①}$, and, in the other case, $\tau=0$. So, (3.6) becomes

$$\beta=g(M)+g^{i_1}\mathring{a}_{i_1}+g^{i_1i_2}\mathring{a}_{i_1}\mathring{a}_{i_2}+\dots+g^{i_1\dots i_n}\mathring{a}_{i_1}\dots\mathring{a}_{i_n} \tag{3.7}$$

where $g^{i_1\dots i_r}$'s are polynomials of M and ① with the coefficients from $\mathfrak{K}$, provided that $i_\lambda<i_\mu$ for $\lambda<\mu$.

On the other hand, since $a_{(j}a_{k)}=0$ $(j\neq k)$, for $i_\lambda\neq i_\mu$ $(\lambda<\mu)$, we have

$$\begin{aligned}\mathring{a}_{i_1}\dots\mathring{a}_{i_p}&=\mathring{a}_{i_1}\dots\mathring{a}_{[i_a}\mathring{a}_{|i_{a+1}}\dots\mathring{a}_{i_{b-1}|}\mathring{a}_{i_b]}\dots\mathring{a}_{i_p}\\&\quad+\mathring{a}_{i_1}\dots\mathring{a}_{(i_a}\mathring{a}_{|i_{a+1}}\dots\mathring{a}_{i_{b-1}|}\mathring{a}_{i_b)}\dots\mathring{a}_{i_p}\\&=\mathring{a}_{i_1}\dots\mathring{a}_{[i_a}\mathring{a}_{|i_{a+1}}\dots\mathring{a}_{i_{b-1}|}\mathring{a}_{i_b]}\dots\mathring{a}_{i_p}\\&\quad+(-1)^{b-a-1}\mathring{a}_{i_1}\dots\mathring{a}_{i_{a-1}}\mathring{a}_{i_{a+1}}\dots\mathring{a}_{i_{b-1}}\mathring{a}_{i_a}\mathring{a}_{i_b}\dots\mathring{a}_{i_p}\\&=\mathring{a}_{i_1}\dots\mathring{a}_{[i_a}\mathring{a}_{|i_{a+1}}\dots\mathring{a}_{i_{b-1}|}\mathring{a}_{i_b]}\dots\mathring{a}_{i_p}.^{(1)}\end{aligned}$$

Hence, for $i_\lambda\neq i_\mu$ $(\lambda<\mu)$

$$\mathring{a}_{i_1}\dots\mathring{a}_{i_p}=\mathring{a}_{[i_1}\dots\mathring{a}_{i_p]}. \tag{3.8}$$

Therefore (3.7) becomes

$$\beta=f(M)+g^i\mathring{a}_i+g^{i_1i_2}\mathring{a}_{[i_1}\mathring{a}_{i_2]}+\dots+g^{i_1\dots i_n}\mathring{a}_{[i_1}\dots\mathring{a}_{i_n]}.$$

If we transform back the normal base system to the original base system a's by $\mathring{a}_j=\bar{h}^i_j a_j$, we have

$$\beta=f(M)+f^i a_i+f^{i_1i_2}a_{[i_1}a_{i_2]}+\dots+f^{i_1\dots i_n}a_{[i_1}\dots a_{i_n]}$$

where $f(M)$ is a polynomial of M and other f^{i_r} are polynomial of M and ① with coefficients from $\mathfrak{K}$.

Hereafter this expression of β is called "*a canonical form*" of β.

§ 4. The "Zentrum" of $\mathfrak{B}$ and the extension of $\mathfrak{A}$ by an element of $\mathfrak{B}$.

We shall prove the following theorem for the purpose of finding the zentrum of $\mathfrak{B}$.

(1) We apply some notations used in tensor calculus. See Der Ricci-Kalkül (J. A. Schouten), p. 25, 26.

Theorem 7. *An element P_C of $\mathfrak{B}$ which is commutative with all the elements of $\mathfrak{B}$ is, except for a certain additional term β_0 satisfying $M^{n-n_0}\beta_0=0$, expressed in the form*

(i) (*when* $n-n_0=even$)

$$P_C=h_2+\tilde{h}_1\overset{I}{A}$$

(ii) (*when* $n-n_0=odd$)

$$P_C=h_2+\tilde{h}_2\overset{I}{A}$$

where $\overset{I}{A}\equiv a_{[1}\ldots\ldots a_{n-n_0]}$, h's $\in\overset{N}{\mathfrak{B}}$, h_1 *is a polynomial of odd degree in* a_a *and* h_2, $\tilde{h}_2$ *are of even degree in* a_a *satisfying the relations*

$$M^{n-n_0}a_a\tilde{h}_1\overset{I}{A}=0\,,\qquad M^{n-n_0}a_a\tilde{h}_2\overset{I}{A}=0\qquad (a=n-n_0+1,\ldots\ldots,n)\,.$$

Proof. Since P_C is commutative with $\mathring{a}_i$, we have

$$\left.\begin{array}{l}\mathring{a}_lP_C=P_C\mathring{a}_l\\ \mathring{a}_aP_C=P_C\mathring{a}_a\end{array}\right\}\quad\left(\begin{array}{l}l=1,2,\ldots\ldots,n-n_0\,;\\ a=n-n_0+1,\ldots\ldots,n\end{array}\right). \tag{4.1}$$

Multiplying the first equation by $\mathring{a}_l$ from the left-hand side, we have

$$MP_C-\mathring{a}_lP_C\mathring{a}_l=0\,; \tag{4.2}$$

and, using Theorem 6,

$$P_C=p(M)+p^{i_1}\mathring{a}_{i_1}+\cdots\cdot p^{i_1\cdots\cdot i_n}(M)\mathring{a}_{[i_1}\ldots\ldots\mathring{a}_{i_n]}\,.$$

But each term of $\mathring{a}_lP_C\mathring{a}_l$ is the same as the corresponding term of MP_C, except for the signs, the ambiguity of the sign being determined by the following rule (by (3.1)): In MP_C and $\mathring{a}_lP_C\mathring{a}_l$

(a) terms of MP_C of even degree with respect to $\mathring{a}_i$ $(i=1,\ldots\ldots,n)$ containing $\mathring{a}_l$,

(b) terms of MP_C of odd degree with respect to $\mathring{a}_i$ $(i=1,\ldots\ldots,n)$ not containing $\mathring{a}_l$ (4.3)

have different signs, and the other terms have the same sign.

Now, expressing the sum of terms (a) by MP_{Cl}, and the sum of the terms of (b) by MP'_{Cl}, we have, from (4.2),)

$$MP_{Cl}+MP'_{Cl}=0\,. \tag{4.4}$$

If we define $MP_C^{(l)}\equiv MP_C-M(P_{Cl}+P'_{Cl})$, then, from the equation above,

$$MP_C^{(l)}=MP_C\,. \tag{4.4$'$}$$

By the same process, $\qquad M^2P_{Cl_2}^{(l_1)}+M^2P_{Cl_2}^{(l_1)'}=0\,.$

Also, if we put

$$M^2P_C^{(l_1)(l_2)}\equiv M^2P_C-M^2P_C-M^2(P_{Cl_2}^{(l_1)}+P_{Cl_2}^{(l_1)'})\,,\qquad (l_1\neq l_2)$$

then, from the equation above,

$$M^2 P_C^{(l_1)(l_2)} = M^2 P_C .$$

Proceeding like this, we finally arrive at

$$M^{n-n_0} P_C^{(1)(2)\dots\dots(n-n_0)} = M^{n-n_0} P_C .$$

But from the meaning of indices of $P_C^{(l_1)(l_2)\dots\dots}$ we see that $P_C^{(1)(2)\dots\dots(n-n_0)}$ is composed of two parts, the one being of terms of odd degree with respect to $\mathring{a}_i\,(i=1, 2, \dots, n)$ containing all $\mathring{a}_1, \dots\dots, \mathring{a}_{n-n_0}$ at the same time, and the other being of those of even degree with respect to $\mathring{a}_i\,(i=1, \dots\dots, n)$ containing non of $\mathring{a}_1, \mathring{a}_2, \dots\dots, \mathring{a}_{n-n_0}$.

So we have

$$\left.\begin{array}{lll} \text{(i)} & \text{(when } n-n_0 \text{ is even)} & M^{n-n_0} P_C = M^{n-n_0}(h_2 + \tilde{h}_1 \mathring{a}_{[1} \dots\dots \mathring{a}_{n-n_0]}) , \\ \text{(ii)} & \text{(when } n-n_0 \text{ is odd)} & M^{n-n_0} P_C = M^{n-n_0}(h_2 = \tilde{h}_2 \mathring{a}_{[1} \dots\dots \mathring{a}_{n-n_0]}) \end{array}\right\} \quad (4.5)$$

where h's $\{\overset{N}{\mathfrak{A}}\}$ and $\tilde{h}_1$ is a polynomial of odd degree in $\mathring{a}_a$, and h_2 and $\tilde{h}_2$ are also polynomials of even degree in $\mathring{a}_a$.

Substituting the second equation of (4.1) into (4.5), and using (3.1), we have

$$\left.\begin{array}{l} \text{(i) (when } n-n_0 \text{ is even)} \\ \qquad M^{n-n_0}\left\{\mathring{a}_a h_2 - h_2 \mathring{a}_a + (\mathring{a}_a \tilde{h}_1 - \tilde{h}_1 \mathring{a}_a)\mathring{a}_{[1} \dots\dots \mathring{a}_{n-n_0]}\right\} = 0 \\ \text{(ii) (when } n-n_0 \text{ is odd)} \\ \qquad M^{n-n_0}\left\{\mathring{a}_a h_2 - h_2 \mathring{a}_a + (\mathring{a}_a \tilde{h}_2 + \tilde{h}_2 \mathring{a}_a)\mathring{a}_{[1} \dots\dots \mathring{a}_{n-n_0]}\right\} = 0 \end{array}\right\} \quad (4.5)'$$

But in general,

$$\mathring{a}_{b_1} \dots\dots \mathring{a}_{b_{r'}} \mathring{a}_{a_1} \dots\dots \mathring{a}_{a_r} = (-1)^{rr'} \mathring{a}_{a_1} \dots\dots \mathring{a}_{a_r} \mathring{a}_{b_1} \dots\dots \mathring{a}_{b_{r'}}$$
$$(a_s, b_s = n-n_0+1, \dots\dots, n) ,$$

therefore we have

$$\left.\begin{array}{l} \mathring{a}_{b_1} \dots\dots \mathring{a}_{b_{r'}} \mathring{a}_{a_1} \dots\dots \mathring{a}_{a_r} + \mathring{a}_{a_1} \dots\dots \mathring{a}_{a_r} \mathring{a}_{b_1} \dots\dots \mathring{a}_{b_{r'}} = 0 \text{ when } rr' = \text{odd}, \\ \mathring{a}_{b_1} \dots\dots \mathring{a}_{b_{r'}} \mathring{a}_{a_1} \dots\dots \mathring{a}_{a_r} + \mathring{a}_{a_1} \dots\dots \mathring{a}_{a_r} \mathring{a}_{b_1} \dots\dots \mathring{a}_{b_{r'}} = 2\mathring{a}_{b_1} \dots\dots \mathring{a}_{b_{r'}} \mathring{a}_{a_1} \dots\dots \mathring{a}_{a_r} \\ \hfill \text{when } rr' = \text{even}. \end{array}\right\} \quad (4.6)$$

But since the degree of $\tilde{h}_1$ is odd with respect to $\mathring{a}_a$'s, the degree of $h_2, \tilde{h}_2$ are even, and $\tilde{h}_1, h_2, \tilde{h}_2 \in \overset{N}{\mathfrak{B}}$, we have, from (4.6)

$$\mathring{a}_a h_2 - h_2 \mathring{a}_a = 0 , \quad \mathring{a}_a \tilde{h}_1 - \tilde{h}_1 \mathring{a}_a = 2\mathring{a}_a \tilde{h}_1 \quad \text{and} \quad \mathring{a}_a \tilde{h}_2 + \tilde{h}_2 \mathring{a}_a = 2\mathring{a}_a \tilde{h}_2 .$$

So that (4.5)′ becomes

$$\left.\begin{array}{ll} M^{n-n_0} \mathring{a}_a \tilde{h}_1 \mathring{a}_{[1} \dots\dots \mathring{a}_{n-n_0]} = 0 & \text{for } n-n_0 = \text{even} \\ M^{n-n_0} \mathring{a}_a \tilde{h}_2 \mathring{a}_{[1} \dots\dots \mathring{a}_{n-n_0]} = 0 & \text{for } n-n_0 = \text{odd}. \end{array}\right\} \quad (4.7)$$

Therefore, from (4.5) and (4.7),

$$\left.\begin{aligned} &P_C=h_2+\tilde{h}_1\mathring{a}_{[1}\ldots\ldots\mathring{a}_{n-n_0]}+\beta_0\,,\\ &\qquad M^{n-n_0}\mathring{a}_a\tilde{h}_1\mathring{a}_{[1}\ldots\ldots\mathring{a}_{n-n_0]}=0\quad\text{for}\quad n-n_0=\text{even},\\ &P_C=h_2+\tilde{h}_2\mathring{a}_{[1}\ldots\ldots\mathring{a}_{n-n_0]}+\beta_0\,,\\ &\qquad M^{n-n_0}\mathring{a}_a\tilde{h}_2\mathring{a}_{[1}\ldots\ldots\mathring{a}_{n-n_0]}=0\quad\text{for}\quad n-n_0=\text{odd}\\ &\qquad\qquad\qquad\qquad(a=n-n_0+1,\ldots\ldots,n)\end{aligned}\right\}\tag{4.8}$$

where β_0 is a certain element satisfying $M^{n-n_0}\beta_0=0$. Thus we know that the element of $\mathfrak{B}$ which is commutative with each element of $\mathfrak{A}$ must be of the form (4.8).(1) On the other hand, any element of $\mathfrak{B}$ which is commutative with all elements of $\mathfrak{A}$ must be also commutative with all elements of $\mathfrak{B}$, and conversely. So that any element which is commutative with all elements of $\mathfrak{B}$ is given by (4.8). Thus we have proved the theorem.

Theorem 9. *When $\beta_0=0$ folllows from $M\beta_0=0$, the element P_C of $\mathfrak{B}$ which is commutative with all the elements of $\mathfrak{B}$ is given by*

(i) (*when $n-n_0=even$*) $P_C=h_2+\tilde{h}_1\overset{I}{A}$

(ii) (*when $n-n_0=odd$*) $P_C=h_2+\tilde{h}_1\overset{I}{A}$

where h's $\in\overset{N}{\mathfrak{B}}$, $\tilde{h}_1$ is a polynomial of odd degree in $\mathring{a}_a$ and h_2, $\tilde{h}_2$ are of even degree in $\mathring{a}_a$ satisfying the relations

$$\mathring{a}_a\tilde{h}_1=0\,,\qquad \mathring{a}_a\tilde{h}_2=0\qquad(a=n-n_0+1,\ldots\ldots,n)\,.$$

Proof. In this case (4.5) and (4.7) become (multiplying (4.7) by $\overset{I}{A}$)

$$\left.\begin{aligned} &P_C=h_2+\tilde{h}_1\overset{I}{A}\qquad \mathring{a}_a\tilde{h}_1=0\quad\text{for}\quad n-n_0=\text{even},\\ &P_C=h_2+\tilde{h}_2\overset{I}{A}\qquad \mathring{a}_a\tilde{h}_2=0\quad\text{for}\quad n-n_0=\text{odd}\end{aligned}\right\}\tag{4.9}$$

where $\overset{I}{A}\equiv\mathring{a}_{[1}\ldots\ldots\mathring{a}_{n-n_0]}$. And we can easily see that P_C given as above is commutative with all the elements of $\mathfrak{B}$. Q. E. D.

From the foregoing theorem we have

Theorem 10. *When $\beta_0=0$ follows from $M\beta_0=0$, "the zentrum" of $\mathfrak{B}$ is the totality of such P's as are given by*

$$\left.\begin{aligned} &P_C=h_2\tilde{h}_1\overset{I}{A}\quad\textit{for}\quad n-n_0=\textit{even}\\ &P_C=h_2\tilde{h}_2\overset{I}{A}\quad\textit{for}\quad n-n_0=\textit{odd}\end{aligned}\right\}\tag{4.10}$$

where h_1 is any polynomial of odd degree in $\mathring{a}_a$, and the other h's are any polynomial of even degree in $\mathring{a}_a$, only restricted by the relations:

$$\mathring{a}_a\tilde{h}_1=0\quad\text{and}\quad\mathring{a}_a\tilde{h}=0\qquad(a=n-n_0+1,\ldots\ldots,n)\,.$$

(1) But we have not yet succeeded in proving that this result is sufficient, because of the presence of the additional term β_0.

Remark. We can easily see that under any transformation of normal basis in $\overset{I}{\mathfrak{A}}$, $h\overset{I}{A}$ in which $a_a h=0(a_a\in\overset{N}{\mathfrak{A}})$ is invariant i.e. $h\overset{I}{A}=h\overset{I}{A}'$. And in general $h\overset{I}{A}=p_0 h a_{[1}\ldots\ldots a_{n-n_0]}$ $(a_l\in\overset{N}{\mathfrak{A}};\ p_0\in\mathfrak{K})$.

An element of $\mathfrak{B}$ can be expressed by a linear combination of the bases $M, a_1,\ldots\ldots, a_{[1}\ldots\ldots a_{n]}$ with coefficients of polynomials of M and Ⓘ in the field $\mathfrak{K}$. Here we have two cases to consider; the one in which $M, a_1,\ldots\ldots, a_{[1}\ldots\ldots a_{n]}$ are all linearly independent (coefficients being polynomials of M and Ⓘ), the other in which $M, a_1,\ldots\ldots, a_{[1}\ldots\ldots a_{n]}$ are linearly dependent.

Theorem 11. *When $\beta_0=0$ follows from $M\beta_0=0$ if, $M, a_1, a_2,\ldots\ldots, a_{[1}\ldots\ldots a_{n]}$ are all linearly independent, an element P_C of "the zentrum" of $\mathfrak{B}$ is given by*

$$P=h_2(M)+p(M)\mathring{a} \qquad \textit{for} \quad n=\textit{odd}$$
$$P=h_2(M)+p(M)\mathring{a}_{[1}\ldots\ldots\mathring{a}_{n]} \quad \textit{for} \quad n=\textit{even}$$

where $p(M)$ is any polynomial of M and Ⓘ.

Proof. By Theorem 9,

$$\left.\begin{aligned} &\mathring{a}_a\tilde{h}_1=0 \quad \text{in} \quad P_C=h_2+\tilde{h}_1\overset{I}{A} \quad \text{for} \quad n-n_0=\text{even};\\ &\mathring{a}_a\tilde{h}_2=0 \quad \text{in} \quad P_C=h_2+\tilde{h}_2\overset{I}{A} \quad \text{for} \quad n-n_0=\text{odd}\end{aligned}\right\} \tag{4.11}$$

First, we shall show that $\tilde{h}_1, \tilde{h}_5$ have the expression of the form

$$\tilde{h}_\epsilon^{n-n_0+1,\ldots\ldots,n}(M)\mathring{a}_{[n-n_0+1}\ldots\ldots\mathring{a}_{n]}\,.$$

For, if not, one of the terms of the lowest degree in $\tilde{h}_\epsilon(\epsilon=1 \text{ or } 2)$ must have the form:

$$\tilde{h}_\epsilon^{a_1\ldots\ldots a_r}(M)\mathring{a}_{[a_1}\ldots\ldots\mathring{a}_{a_r]} \quad (r<n_0 \text{ and not summed by } a_1,\ldots\ldots,a_r) \tag{4.12}$$

where $a_1,\ldots\ldots,a_r$ is a certain set taken from $n-n_0+1,\ldots\ldots,n$. Multiply (4.11) by $\mathring{a}_{a_{r+1}}\ldots\ldots\mathring{a}_{a_{r_{n_0-1}}}$, product all the $\mathring{a}$'s not contained in (4.12) except for any one, say $\mathring{a}_{a_{n_0}}$; then we have

$$\mathring{a}_{a_{r+1}}\ldots\ldots\mathring{a}_{a_{n_0-1}}\mathring{a}_a\tilde{h}=0 \qquad (a=n-n_0+1,\ldots\ldots,n)\,.$$

If we put, here $a=a_{n_0}$, we have

$$\tilde{h}_\epsilon^{a_1\ldots\ldots a_r}(M)\mathring{a}_{n-n_0+1}\ldots\ldots\mathring{a}_n=0 \qquad (\text{not summed by } a_1,\ldots\ldots,a_r)$$

or

$$\tilde{h}_\epsilon^{a_1\ldots\ldots a_r}(M)\mathring{a}_{[n-n_0+1}\ldots\ldots\mathring{a}_{n]}=0\,;$$

which contradicts the assumption of independence of $M, \mathring{a}_1, \mathring{a}_2,\ldots\ldots,\mathring{a}_{[1}\ldots\ldots\mathring{a}_{n]}$. So it must be true that

$$\tilde{h}_\epsilon=\tilde{h}_\epsilon^{n-n_0+1,\ldots\ldots n}(M)\mathring{a}_{n-n_0+1}\ldots\ldots\mathring{a}_n\,.$$

Next, if we substitute the expression of $\tilde{h}_\epsilon$ above into P_C of Theorem 10, the equation $\mathring{a}_a\tilde{h}_\epsilon=0$ is satisfied identically. But, by Theorem 10, when

$n-n_0$=even, $\tilde{h}_1$ must be of odd degree in $\mathring{a}_a$; and when $n-n_0$=odd, $\tilde{h}_2$ must be of even degree. On the other hand, from (4.10), $\tilde{h}_e$ is of n_0-th degree in $\mathring{a}_a$. So we see that n must be odd in either case $n-n_0$=even or odd; and otherwise, when n is even, necessarily $\tilde{h}_e\equiv 0$; so the theorem is proved.

Theorem 12. *When $\beta_0=0$ follows from $M\beta_0=0$, an element P_A of $\mathfrak{B}$ which is anticommutative with all the elements of $\mathfrak{B}$ is given by*

$$\left.\begin{aligned} P_A&=h_1+\tilde{h}_2\overset{I}{A} \quad \textit{for} \quad n-n_0=\textit{even} \\ P_A&=h_1+\tilde{h}_1\overset{I}{A} \quad \textit{for} \quad n-n_0=\textit{odd}, \end{aligned}\right\}$$

where $\tilde{h}_2$ is a polynomial of even degree in $\mathring{a}_a$ $(a=n-n_0+1,\ldots\ldots; n)$ and the other h's are those of odd degree satisfying the relations $\mathring{a}_a\tilde{h}_2=0$ and $\mathring{a}_a\tilde{h}_1=0$.

Proof. In exactly the same way as in the proof of Theorem 8, if we take $\bar{P}_{Al}$ and $\bar{P}'_{Al}$ instead of P_{Cl} and P'_{Cl} in (4.3) as follows: P_{Al} is the totality of terms of odd degree with respect to $\mathring{a}_i$ $(i=1, 2,\ldots\ldots, n)$ containing $\mathring{a}_l$, and $\bar{P}'_{Al}$ is the totality of terms of even degree with respect to $\mathring{a}_i$ not containing $\mathring{a}_l$; we can prove that the element P_A which is anticommutative with all elements of $\mathfrak{B}$ is given by the expression stated in the theorem.

Next we proceed to investigate the extension of $\mathfrak{A}$ by adjoining an element P of $\mathfrak{A}$ preserving Axiom **III** satisfied by $\mathfrak{A}$. If we assume that we have $\mathfrak{A}'_m$ by adjoining m linearly independent elements (together $\mathfrak{A}$) to $\mathfrak{A}$, expressing $\mathfrak{A}'_m\equiv[\beta_\lambda]$ $(\lambda=1, 2,\ldots\ldots, n, n+1,\ldots\ldots, n+m)$, by the usual method of successive normalization for vectors we have the normal basis of $\mathfrak{A}'_m$ in the following form:

$$\mathring{a}_1, \ldots\ldots, \mathring{a}_n, \mathring{a}_{n+1}, \ldots\ldots, \mathring{a}_{n+m};$$

$$\mathring{a}_{(\lambda}\mathring{a}_{\mu)}=\delta_{\lambda\mu}M \quad \text{or} \quad 0.$$

From this we may conclude that, if we can adjoin m linearly independent (together $\mathfrak{A}$) elements to $\mathfrak{A}$, $r\,(r\leqq m)$ elements can be adjoined to $\mathfrak{A}$ (e.g. $\mathring{a}_{n+1},\ldots\ldots, \mathring{a}_{n+r}$). Further, we may conclude that, if $\mathfrak{A}'_r$ is an extension of $\mathfrak{A}$ by adjoining r elements, then we are able to get $\mathfrak{A}'_r$ by adjoining $r-r\,(r_1<r)$ suitable elements to $\mathfrak{A}'_{r_1}$, which is obtained by adjoining r_1 suitable elements (e.g. $\mathring{a}_{n+1},\ldots\ldots, \mathring{a}_{n+r_1}$), r_1 being any integer $(<r)$. From these conclusions we see that the problem of extension of $\mathfrak{A}$ is reduced to one of extension by one element. Let $\mathfrak{A}'$ be an extension of $\mathfrak{A}$ by P, and a' an element of $\mathfrak{A}'$; then, necessarily

$$a'=a+kP,$$

where a and k are any elements of $\mathfrak{A}$ and $\mathfrak{K}$, respectively; and, from Axiom **III**,

$$(\alpha+kP)^2=\alpha' M' \qquad (\alpha'\in\mathfrak{K},\ M'\in\mathfrak{B}).$$

But since M' is independent of $\alpha(\in\mathfrak{A})$ and k, and $\alpha^2=\alpha M$ when $k=0$, then consequently

$$qM=M', \qquad (q\in\mathfrak{K})$$

so that

$$(\alpha+kP)^2=\alpha'' M. \tag{4.13}$$

Accordingly, if we put $\alpha=1$, $k=1$, above,

$$P^2=pM \qquad p\in\mathfrak{K}. \tag{4.14}$$

If we put $\alpha=\mathring{a}_i$, $k=1$,

$$(\mathring{a}_i+P)^2=\alpha_i'' M \qquad (\alpha_i''\in\mathfrak{K}), \tag{4.15}$$

and, using (4.14) and (4.15),

$$(\mathring{a}_i P+P\mathring{a}_i)=(\alpha_i''-p-\delta_i)M\equiv a_i M, \tag{4.16}$$

where $\delta_i=1$ for $i=1,2,\dots,n-n_0$, and $\delta_i=0$ for $n-n_0+1,\dots,n$.

In the same way as we obtained (4.4) from (4.2) in Theorem 8, we have, from the equation above

$$M\bar{P}_{l_1}+MP'_{l_1}=a_{l_1}M\mathring{a}_{l_1} \qquad (l_1=1,\dots,n-n_0) \tag{4.17}$$

where $\bar{P}_{l_1}$ is the sum of all terms in P of odd degree with respect to $\mathring{a}_i$ $(i=1,\dots,n)$ containing $\mathring{a}_{l_1}$, and $\bar{P}'_{l_1}$ is the sum of all terms in P of even degree with respect to $\mathring{a}_i$ not containing $\mathring{a}_{l_1}$.

Hence, if we put $\quad M\bar{P}^{(l_1)}=MP-\dfrac{1}{2}M(\bar{P}_{l_1}+\bar{P}'_{l_1}),$

then, from (4.16), $\quad M\bar{P}^{(l_1)}=MP-\dfrac{1}{2}Ma_{l_1}\mathring{a}_{l_1} \qquad$ (not summed by l_1).

But also, from (4.15), $\quad \mathring{a}_{l_2}MP+MP\mathring{a}_{l_2}=a_{l_2}M^2.$

Substituting (4.17) into the above we have

$$\mathring{a}_{l_2}(MP^{(l_1)})+(M\bar{P}^{(l_1)})\mathring{a}_{l_2}=a_{l_2}M^2 \qquad (l_1\neq l_2).$$

By the same process as that by which we obtained (4.16) from (4.15), we have, from the foregoing equation,

$$M^2\bar{P}^{(l_1)}_{l_2}+M^2\bar{P}^{(l_1)\prime}_{l_2}=a_{l_2}M^2\mathring{a}_{l_2}. \tag{4.18}$$

Also, if we put $\quad M^2\bar{P}^{(l_1)(l_2)}\equiv M^2P^{(l_1)}-\dfrac{1}{2}M^2(P^{(l_1)}_{l_2}+P^{(l_1)\prime}_{(l_2)}),$

we have from (4.18)

$$M^2\bar{P}^{(l_1)(l_2)}=M^2\bar{P}^{(l_1)}-\frac{1}{2}M^2a_{l_2}a_{l_2} \qquad \text{(not summed by } l_2).$$

So, from (4.17),

$$M^2\bar{P}^{(l_1)(l_2)}=M^2P-\frac{1}{2}M^2(a_{l_1}\mathring{a}_{l_1}+a_{l_2}\mathring{a}_{l_2}) \qquad (l_1\neq l_2)\,.$$

Continuing like this, we finally arrive at

$$M^{n-n_0}\bar{P}^{(l_1)\cdots\cdots(l_{n-n_0})}=M^{n-n_0}P-\frac{1}{2}M^{n-n_0}\sum_1^{n-n_0}a_l\mathring{a}_l \quad (l_i\neq l_j \text{ for } i\neq j)\,.$$

Hence $$M^{n-n_0}\bar{P}^{(1)\cdots\cdots(n-n_0)}=M^{n-n_0}P-\frac{1}{2}M^{n-n_0}\sum_1^{n-n_0}a_l\mathring{a}_l\,.$$

As seen from the meaning of indices, $\bar{P}^{(1)\cdots\cdots(n-n_0)}$ is composed of two parts, one being of even degree with respect to $\mathring{a}_i$ $(i=1,\ldots\ldots,n)$ containing all $\mathring{a}_1,\ldots\ldots,\mathring{a}_{n-n_0}$ at the same time, and the other being of odd degree with respect to $\mathring{a}_i$ $(i=1,\ldots\ldots,n)$ not containing any of $\mathring{a}_1,\ldots\ldots,\mathring{a}_{n-n_0}$. Therefore $\bar{P}^{(1)\cdots\cdots(n-n_0)}$ can be written in the form

$$\left.\begin{aligned}&\bar{P}^{(1)\cdots\cdots(n-n_0)}=h_1+\tilde{h}_2\overset{I}{A} \quad \text{for} \quad n-n_0=\text{even},\\ &\bar{P}^{(1),\cdots\cdots(n-n_0)}=h_1+\tilde{h}_1\overset{I}{A} \quad \text{for} \quad n-n_0=\text{odd},\end{aligned}\right\}$$

where h_1, $\tilde{h}_1$ and $\tilde{h}_2\in\overset{N}{\mathfrak{B}}$ and h_1, $\tilde{h}_1$ are polynomials of $\mathring{a}_a$ $(a=n-n_0+1,\ldots\ldots,n)$ of odd degree, and $\tilde{h}_2$ is of even degree; hence P is written in the form

$$\left.\begin{aligned}&P=h_1+\tilde{h}_2\overset{I}{A}+\frac{1}{2}\sum_1^{n-n_0}a_l\mathring{a}_l+\beta_0 \quad \text{for} \quad n-n_0=\text{even},\\ &P=h_1+\tilde{h}_1\overset{I}{A}+\frac{1}{2}\sum_1^{n-n_0}a_l\mathring{a}_l+\beta_0 \quad \text{for} \quad n-n_0=\text{odd},\end{aligned}\right\}$$

where β_0 is any element in $\mathfrak{B}$ satisfying $M^{n-n_0}\beta_0=0$.
When $\beta_0=0$ follows from $M\beta_0=0$, P is written

$$\left.\begin{aligned}&P=h_1+\tilde{h}_2\overset{I}{A}+\frac{1}{2}\sum_1^{n-n_0}a_la_l \quad \text{for} \quad n-n_0=\text{even},\\ &P=h_1+\tilde{h}_1\overset{I}{A}+\frac{1}{2}\sum_1^{n-n_0}a_la_l \quad \text{for} \quad n-n_0=\text{odd},\end{aligned}\right\} \tag{4.19}$$

where h_1, $\tilde{h}_1$ and $\tilde{h}_2\in\overset{N}{\mathfrak{B}}$ and h_1, $\tilde{h}_1$ are polynomials $\mathring{a}_a$ $(a=n-n_0+1,\ldots\ldots,n)$ of odd degree, and $\tilde{h}_2$ is of even degree.

P in (4.19) was obtained by using the condition of (4.16) for $i=1, 2,\ldots\ldots,n-n_0$. We shall next find, therefore, the condition under which P must satisfy the remaining equations of (4.16) for $i=1, 2,\ldots\ldots,n-n_0$. Substituting (4.19) in (4.16), and using $\mathring{a}_a\mathring{a}_a=0$, we have

$$\left.\begin{aligned}&\mathring{a}_a\tilde{h}_2\overset{I}{A}=a_aM \quad \text{for} \quad n-n_0=\text{even},\\ &\mathring{a}_a\tilde{h}_1\overset{I}{A}=a_aM \quad \text{for} \quad n-n_0=\text{odd} \quad (a=n-n_0+1,\ldots\ldots,n);\end{aligned}\right\} \tag{4.20}$$

Multiplying these equations by $\mathring{a}_a$, we have, for both,

$$a_a \mathring{a}_a M = 0 \quad \text{i. e.} \quad a_a = 0 \qquad (a = n - n_0 + 1, \ldots\ldots, n). \tag{4.21}$$

Hence, from (4.20),

$$\left.\begin{aligned} &\mathring{a}_a \tilde{h}_2 = 0 \quad \text{for} \quad n - n_0 = \text{even}, \\ &\mathring{a}_a \tilde{h}_1 = 0 \quad \text{for} \quad n - n_0 = \text{odd} \quad (a = n - n_0 + 1, \ldots\ldots, n). \end{aligned}\right\} \tag{4.22}$$

Conversely, if (4.22) holds good in (4.19), (4.16) is satisfied by P in (4.19); therefore (4.19) with (4.22) is the general form of P satisfying (4.16).

(i) When $n - n_0 =$ even. Further substituting (4.19) in (4.14), and using (4.22), we have

$$pM = \frac{1}{4} \sum (a_l)^2 M + (-1)^{\frac{n-n_0}{2}} \tilde{h}_2^0(M) \tilde{h}_2 M^{n-n_0}, \tag{4.23}$$

where $\tilde{h}_2^0$ is the term of 0-th degree in $\mathring{a}_a$ contained in $\tilde{h}_2$.
Multiplying the first equation above by $\mathring{a}_a$ $(a = n - n_0 + 1, \ldots\ldots, n)$, and using (4.22), we have, for $n - n_0 =$ even,

$$\mathring{a}_a \Big(p - \frac{1}{4} \sum (a_l)^2\Big) = 0 \qquad (a = n - n_0 + 1, \ldots\ldots, n),$$

from which

$$\left.\begin{aligned} &ap = \frac{1}{4} \sum (a_l)^2 \\ \text{or} \qquad &\mathring{a}_a = 0. \end{aligned}\right\} \tag{4.24}$$

But the latter case cannot occur because of independence of basis $\mathring{a}_a$ unless it does not exist originally, i. e. $n_0 = 0$.
When $n_0 = 0$, we have

$$pM = \frac{1}{4} \sum (a_l)^2 M + (-1)^{\frac{n}{2}} \Big(\tilde{h}_2(M)\Big)^2 M^n \qquad (a_l \in \mathfrak{R}); \tag{4.25}$$

and this equation is not an identity unless $\tilde{h}_2 \equiv 0$, for the lowest term of $(\tilde{h}_2)^2 M^n$ is even degree in M; and when $p = \frac{1}{4} \sum (a_l)^2$, we have

$$\tilde{h}_2^0 \tilde{h}_2 = 0. \tag{4.26}$$

Conversely, if (4.22) and (4.26) hold good in P in (4.19), (4.13) is satisfied by P in (4.19); therefore P in (4.19) with (4.22) and (4.25) or (4.26) is the general form of P satisfying (4.13).

(ii) When $n - n_0 =$ odd. Substituting (4.19) in (4.14), and using (4.22), we have

$$pM = \frac{1}{4} \sum (a_l)^2 M. \tag{4.26$'$}$$

Conversely, if (4.22) holds good of P in (4.19), (4.13) is satisfied by this

P; therefore (4.19) with (4.22) is the general form of P satisfying (4.13). So we have

Theorem 13. *When* $\beta_0=0$ *follows from* $M\beta_0=0(\beta_0\in\mathfrak{B})$, *in order that* $\mathfrak{A}$ *may be extended by adjoining an element* P of $\mathfrak{B}$ preserving Axiom **III**, *it is necessary and sufficient that*

When $n-n_0=even$,

either $$P=h_1+\tilde{h}_2\overset{I}{A}+\frac{1}{2}\sum_{l=1}^{n-n_0}a_l\mathring{a}_l$$

where $h_1, \tilde{h}_2\in\overset{N}{\mathfrak{B}}$ *and* h_1 *is a polynomial of odd degree in* $\mathring{a}_a$, *and* $\tilde{h}_2$ *is of even degree, satisfying*

$$\mathring{a}_a\tilde{h}_2=0 \quad and \quad \tilde{h}_2^0\tilde{h}_2=0\,,$$

or $$\left.\begin{aligned} n_0=0 \quad and \quad & P=\tilde{h}_2(M)\overset{I}{A}+\frac{1}{2}\sum_{l=1}^{n-n_0}a_l\mathring{a}_l \\ & \tilde{h}_2^2(M)M^{n-n_0}=bM \quad (b\in\mathfrak{K})\,; \end{aligned}\right\} \tag{4.27}$$

when $n-n_0=odd$

$$P=h_1+\tilde{h}_1\overset{I}{A}+\frac{1}{2}\sum_{l_1=1}^{n-n_0}a_la_l\,, \tag{4.28}$$

where h_1*'s are any polynomials of odd degree in* $\mathring{a}_a$, *satisfying* $\mathring{a}_a\tilde{h}_1=0$.

When $n-n_0=$even, if P in Theorem 12 is anticommutative with each element of $\mathfrak{A}$, and $P^2\neq 0$ (i. e. $pM\neq 0$), then $a_i=0$ in (4.16) ($i=1, 2, \ldots\ldots, n$); hence $n_0=0$. From (4.24), and from (4.23), (4.27) we have

$$P=h_1(M)+\tilde{h}_2(M)\overset{I}{A}\,, \quad n_0=0 \quad \text{and} \quad pM=(-1)^{\frac{n}{2}}(\tilde{h}_2)^2M^n\,,$$

where $$P^2=pM\,.$$

When $n-n_0=$odd, if P in Theorem 12 is anticommutative to $\mathfrak{A}$, then, from (4.16), $a_i=0$, so that, from (4.23),

$$p=0\,;$$

hence $P^2=0$; and from (4.26),

$$P=h_1+\tilde{h}_1\overset{I}{A}\,, \qquad \mathring{a}_a\tilde{h}_1=0\,.$$

So we have

Theorem 14. *When* $n-n_0=even$, *in order that* $\mathfrak{A}$ *may be extended by adjoining a non-nill potent element* P *of* $\mathfrak{B}$ *which is orthogonal to* $\mathfrak{A}$, *it is necessary and sufficient that* $n_0=0$ *and there exists a relation of the form*

$$pM=(=1)^{\frac{n}{2}}[h(M)]^2M^n\,, \qquad (not\ identity)$$

where h *is any function of* M, *and* p *is any non-null number of* $\mathfrak{K}$. *When* $n-n_0$ *is odd, in order that* $\mathfrak{A}$ *may be extended by adjoining an element* P *of* $\mathfrak{B}$ *which is anticommutative with* $\mathfrak{A}$, P *must be a nill potent element in* $\mathfrak{B}$, *and it is given by*

$$P=h_1+\tilde{h}_1\overset{I}{A} \quad and \quad \mathring{a}_a\tilde{h}_1=0\,,$$

where h's are any polynomials of $\mathring{a}_a$ $(\mathring{a}_a = n - n_0 + 1, \ldots, n)$ *of odd degree.*

Further, with respect to an extension of $\mathfrak{A}$ by elements in $\mathfrak{B}$, we have the following

Theorem 15. *When* $\mathfrak{A}'$ *is an extension of* $\mathfrak{A}$, *where*

$$\mathfrak{A}' = \overset{I}{\mathfrak{A}}{}' + \overset{N}{\mathfrak{A}}{}' = [\mathring{a}'_{l'}] + [\mathring{a}'_{a'}] \quad (l' = 1, 2, \ldots, n - n_0 + r;\ a' = n - n_0 + r + 1, \ldots, n')$$
$$\mathfrak{A} = \overset{I}{\mathfrak{A}} + \overset{N}{\mathfrak{A}} = [\mathring{a}_l] + [\mathring{a}_a] \quad (l = 1, 2, \ldots, n - n_0;\ a' = n - n_0 + 1, \ldots, n),$$

the necessary and sufficient condition for $\overset{I}{\mathfrak{A}} \subset \overset{I}{\mathfrak{A}}{}'$ (or $r \neq 0$) *is that*

$$n = even, \quad n_0 = 0, \quad pM = (-1)^{\frac{n}{2}} \big(h(M)\big)^2 M^n, \quad r = 1,$$

where p is any non-null number in $\mathfrak{K}$, *and h is any polynomial of M; and then the adjoining element P is given by*

$$P = h(M)\overset{I}{A} + a^i a_i$$

Proof. As an independent basis in $\mathfrak{A}'$, we can take (from (4.16))

$$\mathring{a}_1, \ldots, \mathring{a}_n, h_{\epsilon_1, \lambda} + \tilde{h}_{\epsilon_2, \lambda} \overset{I}{A} + a^l_\lambda a_l \qquad (\lambda = 1, 2, \ldots, r)$$

or
$$\mathring{a}_1, \ldots, \mathring{a}_n, h_{\epsilon_1, \lambda} + \tilde{h}_{\epsilon_2, \lambda} \overset{I}{A}.$$

When $n - n_0 = $ odd $(n_0 \geqq 0)$, from (4.26), we have $p_\lambda = 0$ (where $(P_\lambda)^2 = p_\lambda M$); hence $(P_\lambda)^2 = 0$, therefore necessarily

$$\overset{I}{\mathfrak{A}} = \overset{I}{\mathfrak{A}}{}', \tag{5.29}$$

therefore $n - n_0$ must be even for $\overset{I}{\mathfrak{A}} \subset \overset{I}{\mathfrak{A}}{}'$.

When $n - n_0 = $ even, for $n_0 \neq 0$, from (4.24) $p_\lambda = 0$; hence $(P_\lambda)^2 = 0$, and therefore $\overset{I}{\mathfrak{A}} = \overset{I}{\mathfrak{A}}{}'$; so, in order that $\overset{I}{\mathfrak{A}} \subset \overset{I}{\mathfrak{A}}{}'$ necessarily $n_0 = 0$; hence $P_\lambda = h(M)\overset{I}{A}$. But since necessarily $(P_\lambda)^2 = pM$, we have

$$pM (= (h_\lambda \overset{I}{A})^2 = (=1)^{\frac{n}{2}} \big(h_\lambda(M)\big)^2 M^n, \tag{5.30}$$

which is a relation of M. This proves that it is impossible to take the adjoining element more than one at most (i.e. $r \not> 1$). Conversely, if $n = $ even, $n_0 = 0$, $r = 1$, and (5.30) is satisfied, we have

$$[\mathring{a}_1, \ldots, \mathring{a}_n, h_1(M)\overset{I}{A}] \equiv \overset{I}{\mathfrak{A}}{}' \supset \overset{I}{\mathfrak{A}}$$

In this case ($n - n_0 = $ even), if we can get $\mathfrak{A}'_r$ by adjoining non-nil (linearly independent with $\mathfrak{A}$) elements, we are able to get $\mathfrak{A}_r$ by adjoining non-nil $r-1$ elements to $\mathfrak{A}_1$, which is obtained by adjoining one non-nil element.

But since, when $n - n_0 - $odd in $\mathfrak{A}$, the extension of $\mathfrak{A}$ is impossible by one non-nil element, extension of $\mathfrak{A}'_1$ by $r-1$ non-nil elements is impossible,

for the corresponding $n'-n_0'$ $(=n+1-n_0)$ of $\mathfrak{A}_1'$ is odd. Therefore, necessarily $r \not> 1$.
Thus we have proved the theorem. Q. E. D.

And from this theorem we have

Corollary. *When* $\mathfrak{A}'$ *is an extension of* $\mathfrak{A}$ *when* $n-n_0=odd$, *then necessarily* $\overset{I}{\mathfrak{A}}=\overset{I}{\mathfrak{A}'}$, $\overset{N}{\mathfrak{A}} \subset \overset{N}{\mathfrak{A}'}$.

§ 5. The basis of $\mathfrak{B}$.

By Theorem 7 we know that, from the assumption that for the iteration of α there corresponds a single number k of $\mathfrak{K}$, $\mathfrak{B}$ forms a linear manifold (the coefficient being polynomials of M and Ⓘ with field $\mathfrak{K}$) with a basis constituting the independent elements taken from

$$M, \alpha_1, \alpha_2, \ldots, \alpha_{[1} \ldots \alpha_{n]}. \tag{5.1}$$

Now we come to the questions: In what condition are $M, \alpha_1, \ldots, \alpha_{[1} \ldots \alpha_{n]}$ linearly independent or not, and in what form is the relation when (5.1) are dependent?

For the latter question we have

Theorem 16. *Relations among* (5.1), *if they exist, are expressed by linear combinations of the forms*

$$f_\lambda \alpha_{[j_1} \ldots \alpha_{j_r]} + g_\lambda \alpha_{[j_{r+1}} \ldots \alpha_{j_{n-n_0}]} \begin{pmatrix} \lambda=1,2,\ldots;\ j_l=1,\ldots,n-n_0;\ r=1,\ldots,n-n_0 \\ \text{and } j_l\text{'s are all different};\ \alpha_a \in \overset{N}{\mathfrak{A}} \end{pmatrix}$$

satisfying the relation $M^{n-n_0}(f_\lambda \alpha_{[j_1} \ldots \alpha_{j_r]} + g_\lambda \alpha_{[j_{r+1}} \ldots \alpha_{j_{n-n_0}]}=0$ *where* f_λ, $g_\lambda \in \overset{N}{\mathfrak{B}}$, *and when* $n-n_0-even$, f_λ *and* g_λ *are both at the same time polynomials, even or odd degree in* $\mathring{\alpha}_a$, *and when* $n-n_0=odd$, *the one is even and the other is odd, satisfying the relations*

$$\alpha_a M^{n-n_0} f_\lambda = 0, \qquad \alpha_a M^{n-n_0} g_\lambda = 0 \qquad (\alpha_a \in \overset{N}{\mathfrak{A}}).$$

Proof. If there is a linear relation among $M, \alpha_1, \ldots, \alpha_{[1} \ldots \alpha_{n]}$, it may be of the form:

$$\begin{aligned} P \equiv\ & f(M) + f^i(M)\alpha_i + f^{i_1 i_2}\alpha_{[i_1}\alpha_{i_2]} \\ & + \cdots + f^{i_1 \ldots i_r}\alpha_{[i_1} \ldots \alpha_{i_r]} + \cdots + f^{i_1 \ldots i_n}\alpha_{[i_1} \ldots \alpha_{i_n]} = 0 \end{aligned}$$

i. e. $$P \equiv h(M, \mathring{\alpha}_a) + h^l \mathring{\alpha}_l + \cdots + h^{l_1 \ldots l_{n-n_0}}(M, \mathring{\alpha}_a)\mathring{\alpha}_{[l_1} \ldots \mathring{\alpha}_{l_{n-n_0}]} = 0$$

$$(f^{i_1 \ldots i_r} \neq 0;\ h\text{'s} \in \mathfrak{B});$$

then we have

$$\left.\begin{aligned} & 0 = \mathring{\alpha}_l(\mathring{\alpha}_l P + P\mathring{\alpha}_l) \equiv 2\mathring{\alpha}_l\mathring{\alpha}_l P_1^{(l)} = 2MP_1^{(l)} \\ \text{and}\quad & 0 = \mathring{\alpha}_l(\mathring{\alpha}_l P - P\mathring{\alpha}_l) \equiv 2\mathring{\alpha}_l\mathring{\alpha}_l P_2^{(l)} = 2MP_2^{(l)} \quad (l=1,\ldots,n-n_0). \end{aligned}\right\} \tag{5.2}$$

Here $P_1^{(l)}$ must be the sum of terms, in P, of odd degree with respect to $\mathring{a}_i$ $(i=1,\ldots\ldots,n)$ containing $\mathring{a}_l$, and those of even degree not containing $\mathring{a}_l$, and $P_2^{(l)}$ the sum of remaining terms in P, i. e. $P_1^{(l)}+P_2^{(l)}=P$.

Next, taking $MP_1^{(l)}$ or $MP_2^{(l)}$ as P in (5.2), we have

$$0=\mathring{a}_{l_2}(\mathring{a}_{l_2}MP_{\epsilon_1}^{l_1}\pm MP_{\epsilon_1}^{(l_1)})\equiv 2M^2P_{\epsilon_1\epsilon_2}^{(l_1)(l_2)} \qquad (l_1\neq l_2)\,,$$

where $\epsilon_2=1$ or 2 according as the sign of the second form in the above is $+$ or $-$, and $P_{\epsilon_1\epsilon_2}^{(l_1)(l_2)}\equiv(P_{\epsilon_1}^{(l_1)})_{\epsilon_2}^{(l_1)}$.

Carrying out the same procedure as above, we have, generally,

$$M^jP_{\epsilon_1\ldots\ldots\epsilon_j}^{(l_1)\ldots\ldots(l_j)}=0 \quad (1\leqq j\leqq n-n_0;\ \epsilon_i=1 \text{ or } 2;\ l_j=1,2,\ldots\ldots,n-n_0;\ l_1,l_2,\ldots\ldots,\neq).$$

Specially, if we take, above, $j=n-n_0$, $\epsilon_1=\epsilon_2=\cdots\cdots=\epsilon_r$, $\epsilon_{r+1}=\cdots\cdots=\epsilon_{n-n_0}$, we have

$$M^{n-n_0}P_{1\ldots\ldots\ldots 1\ 2\ldots\ldots\ldots\ldots 2}^{(l_1)\ldots\ldots(l_r)(l_{r+1})\ldots\ldots(l_{n-n_0})}=0 \quad \text{and} \quad M^{n-n_0}P_{2\ldots\ldots\ldots 2\ 1\ldots\ldots\ldots\ldots 1}^{(l_1)\ldots\ldots(l_r)(l_{r+1})\ldots\ldots(l_{n-n_0})}=0\,, \tag{5.3}$$

where l_j's are all different. And, from the meaning of suffices ϵ's, the foregoing can be written in the following forms respectively:

$$\left.\begin{aligned}
&(\text{when } n-n_0=\text{even})\\
&\qquad M^{n-n_0}(\overset{1}{h}{}^{l_1\ldots\ldots l_r}\mathring{a}_{l_1}\ldots\ldots\mathring{a}_{l_r}+\overset{\widetilde{2}}{h}{}^{l_{r+1}\ldots\ldots l_{n-n_0}}\mathring{a}_{l_{r+1}}\ldots\ldots\mathring{a}_{l_{n-n_0}})=0\\
&\text{and}\qquad M^{n-n_0}(\overset{2}{h}{}^{l_1\ldots\ldots l_r}\mathring{a}_{l_1}\ldots\ldots\mathring{a}_{l_r}+\overset{\widetilde{1}}{h}{}^{l_{r+1}\ldots\ldots l_{n-n_0}}\mathring{a}_{l_{r+1}}\ldots\ldots\mathring{a}_{l_{n-n_0}})=0\,,\\
&(\text{when } n-n_0=\text{odd})\\
&\qquad M^{n-n_0}(\overset{1}{h}{}^{l_1\ldots\ldots l_r}\mathring{a}_{l_1}\ldots\ldots\mathring{a}_{l_r}+\overset{\widetilde{1}}{h}{}^{l_{r+1}\ldots\ldots l_{n-n_0}}\mathring{a}_{l_{r+1}}\ldots\ldots\mathring{a}_{l_{n-n_0}})=0\\
&\text{and}\qquad M^{n-n_0}(\overset{2}{h}{}^{l_1\ldots\ldots l_r}\mathring{a}_{l_1}\ldots\ldots\mathring{a}_{l_r}+\overset{\widetilde{2}}{h}{}^{l_{r+1}\ldots\ldots l_{n-n_0}}\mathring{a}_{l_{r+1}}\ldots\ldots\mathring{a}_{l_{n-n_0}})=0\\
&\qquad\qquad\qquad\qquad\left(\begin{matrix}l_1l_2\ldots\ldots\neq 0\,;\\ \text{not summed by } l\text{'s}\end{matrix}\right),
\end{aligned}\right\} \tag{5.4}$$

where $\overset{1}{h}{}^{l_1\ldots\ldots l_r}$ and $\overset{\widetilde{1}}{h}{}^{l_{r+1}\ldots\ldots l_{n-n_0}}$ are the sum of all terms of odd degree in $\mathring{a}_a$ contained in $h^{l_1\ldots\ldots l_r}$ and $h^{l_{r+1}\ldots\ldots l_{n-n_0}}$ respectively, and $\overset{2}{h}{}^{l_1\ldots\ldots l_r}$ and $\overset{\widetilde{2}}{h}{}^{l_{r+1}\ldots\ldots l_{n-n_0}}$ are the sum of all terms of even degree in $\mathring{a}_a$ contained in $h^{l_1\ldots\ldots l_r}$ and $h^{l_{r+1}\ldots\ldots l_{n-n_0}}$ respectively. Since superfix $l_1,l_2\ldots\ldots$ can be any one of $1,2,\ldots\ldots,n-n_0$ P can be expressed by a linear combination of the forms (returning to the original base system)

$$f_\lambda a_{[l_1}\ldots\ldots a_{l_r]}+g_\lambda a_{[l_{r+1}}\ldots\ldots a_{l_{n-n_0}]}$$

satisfying $M^{n-n_0}(f_\lambda a_{[l_1}\ldots\ldots a_{l_r]}+g_\lambda a_{[l_{r+1}}\ldots\ldots a_{l_{n-n_0}]})=0$.

So we have proved the first part of the theorem.

In order to show the second part of the theorem, we write simply

$h_{1,\lambda}$, $h_{2,\lambda}$, $\tilde{h}_{1,\lambda}$ and $\tilde{h}_{2,\lambda}$ respectively, instead of $\overset{1}{h}{}^{l_1\cdots\cdots l_r}$, $\overset{2}{h}{}^{l_1\cdots\cdots l_r}$, $\overset{\tilde{1}}{h}{}^{l_{r+1}\cdots\cdots l_{n-n_0}}$ and $\overset{\tilde{2}}{h}{}^{l_{r+1}\cdots\cdots l_{n-n_0}}$, expressing the suffices by a single suffix notation λ. By this notation, (5.4) can be written in a single equation

$$M^{n-n_0}(h_{\epsilon_1,\lambda}\mathring{a}_{l_1}\cdots\cdots\mathring{a}_{l_r}+\tilde{h}_{\epsilon_2,\lambda}\mathring{a}_{l_{r+1}}\cdots\cdots\mathring{a}_{l_{n-n_0}})=0 \quad (\epsilon=1,2;\ \lambda=1,2,\ldots\ldots) \tag{5.5}$$

satisfying $n-n_0+\epsilon_1+\epsilon_2=\text{odd}$. Multiplying (5.5) by $\mathring{a}_a\,(a=n-n_0+1,\ldots\ldots,n)$ from the left-hand and right-hand sides, we have, respectively,

$$\mathring{a}_a M^{n-n_0}(h_{\epsilon_1,\lambda}\mathring{a}_{l_1}\cdots\cdots\mathring{a}_{l_r}+\tilde{h}_{\epsilon_2,\lambda}\mathring{a}_{l_{r+1}}\cdots\cdots\mathring{a}_{l_{n-n_0}})=0$$

and
$$\mathring{a}_a M^{n-n_0}(h_{\epsilon_1,\lambda}\mathring{a}_{l_1}\cdots\cdots\mathring{a}_{l_r}-\tilde{h}_{\epsilon_2,\lambda}\mathring{a}_{l_{r+1}}\cdots\cdots\mathring{a}_{l_{n-n_0}})=0:$$

from which follows

$$\mathring{a}_a M^{n-n_0}h_{\epsilon_1,\lambda}\mathring{a}_{l_1}\cdots\cdots\mathring{a}_{l_r}=0 \qquad \mathring{a}_a M^{n-n_0}\tilde{h}_{\epsilon_2,\lambda}\mathring{a}_{l_{r+1}}\cdots\cdots\mathring{a}_{l_{n-n_0}}=0; \tag{5.6}$$

and therefore, returning to the original base system a_i, we have

$$a_a M^{n-n_0}f_\lambda a_{[j_2}\cdots\cdots a_{j_r]}=0,\quad a_a M^{n-n_0}g_\lambda a_{[j_{r+1}}\cdots\cdots a_{j_{n-n_0}]}=0 \quad (a_a\in\overset{N}{\mathfrak{A}}).$$

So we have proved the theorem.

Next, with regard to the properties of $h_{\epsilon,\lambda}$ and $\tilde{h}_{\epsilon,\lambda}$, we shall show

Theorem 17. *When $n_0\geqq 1$ and $M, a_1,\ldots\ldots,a_{[1}\cdots\cdots a_{n]}$ are not independent, there exists a relation $l(M)\mathring{a}_1\cdots\cdots\mathring{a}_n=0$ for even $n-n_0$, and $l(M)\mathring{a}_1\cdots\cdots\mathring{a}_n=l'(M)\mathring{a}_{n-n_0+1}\cdots\cdots\mathring{a}_n$ for odd $n-n_0$, where l and l' are polynomials of M.*

Proof. From Theorem 15, the relations must be satisfied when $M, a_1,\ldots\ldots,$ $a_{[1}\cdots\cdots a_{n]}$ are not independent

$$M^{n-n_0}(h_{\epsilon_1,\lambda}\mathring{a}_{l_1}\cdots\cdots\mathring{a}_{l_r}+\tilde{h}_{\epsilon_2,\lambda}\mathring{a}_{l_{r+1}}\cdots\cdots\mathring{a}_{l_{n-n_0}})=0 \quad (\epsilon_1+\epsilon_1+n-n_0=\text{odd}). \tag{5.7}$$

(i) When $n-n_0=\text{even}$.

Since $h_{\epsilon_1,\lambda}$ and $h_{\epsilon_2,\lambda}$ are respectively odd and even in $\mathring{a}_a\,(a=n-n_0+1,\ldots\ldots,n)$ (or even and odd degree), the $2n_0$ equations in (5.6) must not be identical.[1]

If we use the notation $h_\lambda\equiv\mathring{a}_a h_{\epsilon_1,\lambda}+\mathring{a}_{\epsilon_2,\lambda}$ and express any one term of the lowest degree in h_λ with respect to $\mathring{a}_a$ by $l_\lambda^{a_1\cdots\cdots a_s}\mathring{a}_{a_1}\cdots\cdots\mathring{a}_{a_s}$ (not summed by a's) when $s_0=n_0$, we have, from (5.7), multiplying by $\mathring{a}_{l_{r+1}}\cdots\cdots\mathring{a}_{l_{n-n_0}}$ or $\mathring{a}_{l_1}\cdots\cdots\mathring{a}_{l_r}$,

$$M^{n-n_0}l_\lambda^{a_1\cdots\cdots a_{n_0}}(M)\mathring{a}_{n-n_0+1}\cdots\cdots\mathring{a}_n\mathring{a}_{a_1}\cdots\cdots\mathring{a}_{a_{n_0}}=0 \quad (\text{not summed by } a\text{'s})$$

i. e.
$$l(M)\mathring{a}_1\cdots\cdots\mathring{a}_n=0;$$

---

(1) Since we know that of $h_{\epsilon_1,\lambda}$ and $h_{\epsilon_2,\lambda}$ one is odd and the other even degree in $\mathring{a}_a$ when $n-n_0=\text{even}$, it cannot be true that $h_{\epsilon_1,\lambda}$ and $h_{\epsilon_2,\lambda}$ are at the same time of the following form

$$\left.\begin{array}{l} p_{\epsilon_1,\lambda}\mathring{a}_{n-n_0+1}\cdots\cdots\mathring{a}_n \\ p_{\epsilon_1,\lambda}\mathring{a}_{n-n_0+1}\cdots\cdots\mathring{a}_n \end{array}\right\}\ (p_{\epsilon_1,\lambda}\in\mathfrak{K}).$$

and when $s<n_0$, multiplying (5.7) by $\mathring{\alpha}_{a_{s_0+1}}\dots\dots\mathring{\alpha}_{a_{n_0}}$, the product of all the elements α's, which are not contained in $\mathring{\alpha}_{a_1},\dots\dots,\mathring{\alpha}_{a_{s_0}}$, we see that all the terms in the resulting equation except for those induced from

$M^{n-n_0}l_\lambda^{a_1\dots\dots a_{s_0}}\mathring{\alpha}_{a_1}\dots\dots\mathring{\alpha}_{a_{s_0}}\mathring{\alpha}_{l_1}\dots\dots\mathring{\alpha}_{l_r}$ (not summed by a's) vanish identically,

so that we have
$$l(M)\mathring{\alpha}_1\dots\dots\mathring{\alpha}_n=0\,. \tag{5.8}$$

Remark: Further, from (5.7), iterating it and using (5.6), we have

$$\left.\begin{aligned} &M^{2(n-n_0)}h_{1,\lambda}\tilde{\mathring{h}}_{2,\lambda}(M)\mathring{\alpha}_1\dots\dots\mathring{\alpha}_{n-n_0}=0\\ \text{and}\quad &M^{2(n-n_0)}\tilde{h}_{1,\lambda}\mathring{h}_{2,\lambda}(M)\mathring{\alpha}_1\dots\dots\mathring{\alpha}_{n-n_0}=0\,, \end{aligned}\right\} \tag{5.9}$$

where $h_{2,\lambda}(M)$ and $\tilde{\mathring{h}}_{2,\lambda}(M)$ are the term of 0-th degree of $\mathring{\alpha}_a$ in $h_{2,\lambda}$ and $\tilde{h}_{2,\lambda}$.

(ii) When $n-n_0=$odd, we have, similarly, the equations corresponding to (5.8) and (5.9):

$$\left.\begin{aligned} &l(M)\mathring{\alpha}_1\dots\dots\mathring{\alpha}_n=l'(M)\mathring{\alpha}_{n-n_0+1}\dots\dots\mathring{\alpha}_n\\ \text{and}\quad &M^{2(n-n_0)}(\tilde{h}_{2,\lambda}\mathring{h}_{2,\lambda}+h_{2,\lambda}\tilde{\mathring{h}}_{2,\lambda})\mathring{\alpha}_1\dots\dots\mathring{\alpha}_{n-n_0}=0\,. \end{aligned}\right\} \tag{5.10}$$

So we have proved the theorem.

Special case.

When we can conclude that from $M\beta=0$ there follows $\beta=0$, from equation (5.5) in Theorem 15:

$$M^{n-n_0}(h_{\epsilon_1,\lambda}\mathring{\alpha}_{l_1}\dots\dots\mathring{\alpha}_{l_r}+\tilde{h}_{\epsilon_2,\lambda}\mathring{\alpha}_{l_{r+1}}\dots\dots\mathring{\alpha}_{l_{n-n_0}})=0\,,$$

we have
$$h_{\epsilon_1,\lambda}\mathring{\alpha}_{l_1}\dots\dots\mathring{\alpha}_{l_r}+\tilde{h}_{\epsilon_2,\lambda}\mathring{\alpha}_{l_{r+1}}\dots\dots\mathring{\alpha}_{l_{n-n_0}}=0\,,$$

becoming
$$(-1)^{\epsilon_1+\epsilon_3}h_{\epsilon_1,\lambda}M^r+h_{\epsilon_2,\lambda}\overset{I}{A}=0\,,$$

where $n-n_0+\epsilon_1+\epsilon_2$ is odd and $\epsilon_1,\epsilon_2=1,2$.

Similarly, (5.6), (5.8), and (5.9) become respectively

$$\mathring{\alpha}_a h_{\epsilon_1,\lambda}=0 \quad\text{and}\quad \mathring{\alpha}_a\tilde{h}_{\epsilon_2,\lambda}=0\,,$$

$$l(M)\mathring{\alpha}_{n-n_0+1}\dots\dots\mathring{\alpha}_n=0\,,$$

and $h_{1,\lambda}\tilde{\mathring{h}}_{2,\lambda}(M)=0$ and
$$\tilde{h}_{1,\lambda}\mathring{h}_{2,\lambda}(M)=0\,. \tag{5.11}$$

Moreover, (5.10) becomes

$$l(M)\mathring{\alpha}_1\dots\dots\mathring{\alpha}_n=l'(M)\mathring{\alpha}_{n-n_0+1}\dots\dots\mathring{\alpha}_n \quad\text{and}\quad \tilde{h}_{2,\lambda}\mathring{h}_{2,\lambda}+h_{2,\lambda}\tilde{\mathring{h}}_{2,\lambda}=0\,. \tag{5.12}$$

So we have

Theorem 18. *When we can assume that from $M\beta=0$ follows $\beta=0$, the relations among the basis of $\mathfrak{B}$ $M, \alpha_1,\dots\dots\alpha_{[1}\dots\dots\alpha_{n]}$, if they exist, are as follows:*

$$h_{\epsilon_1,\lambda}(M,\alpha_a)+\tilde{h}_{\epsilon_2,\lambda}(M,\alpha_a)\overset{I}{A}=0 \quad (\alpha_a\in\overset{N}{\mathfrak{A}};\ a=n-n_0+1,\dots\dots,n)\,, \tag{5.13}$$

where $n-n_0+\epsilon_1+\epsilon_2=\text{odd}$, ϵ_b *is 1 or 2 according as* h_{ϵ_b} *is a polynomial of odd or even degree with respect to* $\mathring{a}_a$, *and* $h_{\epsilon,\lambda}$, $\tilde{h}_{\epsilon,\lambda}\in\overset{N}{\mathfrak{B}}$ *satisfy the relations*

$$a_a h_{\epsilon,\lambda}=0 \quad \text{and} \quad a_a\tilde{h}_{\epsilon,\lambda}=0 \quad (\lambda=1,2,\ldots.).$$

Precisely, we have

Theorem 18′. *When* $n-n_0=\text{even}$, *and the relations exist we have, from* (5.13),

$$l(M)a_{n-n_0+1}\ldots.a_n=0, \quad h_{1,\lambda}\overset{\circ}{\tilde{h}}_{2,\lambda}(M)=0 \quad \text{and} \quad \tilde{h}_{1,\lambda}\mathring{h}_{2,\lambda}(M)=0;$$

also, when $n-n_0=\text{odd}$,

$$l(M)\mathring{a}_{n-n_0+1}\ldots.\mathring{a}_n=l'(M)\mathring{a}_1\ldots.\mathring{a}_n \quad \text{and} \quad \tilde{h}_{2,\lambda}\mathring{h}_{2,\lambda}+h_{2,\lambda}\overset{\circ}{\tilde{h}}_{2,\lambda}=0$$

where $\mathring{h}_{2,\lambda}(M)$ *is the term of* 0-*th degree of* a_a *in* $h_{2,\lambda}$.

The difference between the general case and the special case is that in the former a power of M^{n-n_0} always remains as a factor of equations, but in the latter it can be removed. Although this difference is not essential in treating expressions, in the former case the resulting expressions become exceedingly complex compared with those in the latter. For this reason we shall hereafter consider the problem in the special case, and shall state those in the general case as necessity requires.

Next we shall find all the independent relations within (5.13):

$$h_{\epsilon_1,\lambda}+\tilde{h}_{\epsilon_2,\lambda}\overset{I}{A}=0$$

and arrange term in good order.

For this purpose we introduce the following notations:
Express each of

$$M,\ \mathring{a}_{n-n_0+1},\ \ldots.,\ \mathring{a}_{[n-n_0+1}\ldots.\mathring{a}_{n]}$$

by A_m simply, and set them in order according to the following rule:
Writing

$$A_r\equiv a_{a_1}\ldots.a_{a_s} \qquad (a_1<a_2<\ldots.<a_s),$$
$$A_{r'}\equiv a_{b_1}\ldots.a_{b_t} \qquad (b_1<b_2<\ldots.<b_t),$$

for the integer r and r',

if $a_s\gtrless b_t$, then $r\gtrless r'$,
if $a_s=b_t$ and $a_{s-1}\gtrless b_{t-1}$, then $r\gtrless r'$
if $a_s=b_t$, $a_{s-1}=b_{t-1}$ and $a_{s-2}\gtrless b_{t-2}$, then $r\gtrless r'$
. . . . ,
. . . .

and assuming that $A_0=M$, we know that A_r can be defined when, and only when, $A_1,\ldots.,A_{r-1}$ are defined. Thus $A_1,\ldots.,A_N$ $(N=2^{n_0})$ can just cover $a_{a_1},\ldots.,a_{[a_1}\ldots.a_{a_n]}$ in the unique way.
Using the notation A_r given above we have the following theorem with regard to a reduction of the relations of (5.13)

Theorem 19. *When $n-n_0=$even, the totality of the equations* (5.13):

$$h_{\epsilon_1,\lambda}+\tilde{h}_{\epsilon_2,\lambda}\overset{I}{A}=0$$

can be reduced to the system of relations of the form:

$$\left.\begin{array}{l} g_0+\left(\tilde{g}_0+f_0(M)A_{p_0}\right)\overset{I}{A}=0 \\ g_1+\left(\tilde{g}_1+f_1(M)A_{p_1}\right)\overset{I}{A}=0 \\ \cdots\cdots\cdots\cdots \\ g_r+\left(\tilde{g}_r+f_r(M)A_{p_r}\right)\overset{I}{A}=0 \end{array}\right\} \quad (0\leqq p_0<p_1<\ldots\ldots<p_r\leqq 2^{n_0})$$

where g_p, $\tilde{g}_p$ are polynomials of M of odd and even degree in a_a $(a_a\in\overset{N}{\mathfrak{A}})$ respectively and are linear combinations of $A_{p_0+1},\ldots\ldots,A_{2^{n_0}}$, and f's satisfy the relation of the form $f_{p_{s_1}}=a_{s_1s_2}f_{p_{s_2}}$ for p_{s_1} and p_{s_2} related by $A_{p_{s_2}}=A_{p_{s_1}}A_{r_1}$ (for suitable r').

Proof. The totality of all relations (5.13) and those obtained from (5.13) by multiplying with $\overset{I}{A}$ (from whichever side), can be written in the form

$$k_\lambda+\tilde{k}_\lambda\overset{I}{A}=0 \qquad (\lambda=1,2,\ldots\ldots,s), \tag{5.14}$$

where k_λ and $\tilde{k}$ are linear combinations of A_j; and then the totality of the relations in (5.14) containing A_{q_0}, q_0 being the smallest suffix of A_j contained in (5.14), must have the form:

$$l_i+\left(\tilde{l}_i+m_i(M)A_{q_0}\right)\overset{I}{A}=0 \qquad (i=1,2,\ldots\ldots,c_1) \tag{5.15}$$

where l_i, $\tilde{l}_i$ are linear combinations of $A_{q_0+1}\ldots\ldots A_{2^{n_0}}$.
When $c_1=1$, (5.15) takes the form

$$l+\left(\tilde{l}+m(M)A_{q_0}\right)\overset{I}{A}=0.$$

And when $c_1\geqq 2$, we take any two from (5.15), say

$$l_1+\left(\tilde{l}_1+m_1(M)A_{q_0}\right)\overset{I}{A}=0 \tag{5.16$_1$}$$

and

$$l_2+\left(\tilde{l}_2+m_2(M)A_{q_0}\right)\overset{I}{A}=0. \tag{5.16$_2$}$$

Since m_1, m_2 are polynomials, they may be expressed in the form

$$m_1=m_1^0+m_1^1M+\cdots\cdots+m_1^{r_1}(M)^{r_1}$$

and $m_2=m_2^0+m_2^\prime M+\cdots\cdots+m_2^{r_2}(M)^{r_2}$ (assuming that $0\leqq r_1\leqq r_2$; $m_a^i\in\mathfrak{R}$)

By eliminating the term $m_2^{r_2}(M)^{r_2}A_{q_0}\overset{I}{A}$ in $(5.16)_2$ from $(5.16)_1$ and $(5.16)_2$,

---

(1) Since $A_r=\mathring{a}_{a_1}\ldots\ldots\mathring{a}_{a_\lambda}$, for given r,s $(r,s\leqq 2^{n_0})$ there holds the relation

$A_rA_s=A_t$ if A_r,A_s have no common $\mathring{a}_a$'s,

$A_rA_s=0$ if A_r,A_s have common $\mathring{a}_a$'s.

$$(M)^{r_2-r_1}m_2^{r_2}(l_1+\tilde{l}_1\overset{I}{A})-m_1^{r_1}(l_2+\tilde{l}_2\overset{I}{A})+\left((M)^{r_2-r_1}m_2^{r_2}m_1-m_1^{r_1}m_2\right)A_{q_0}\overset{I}{A}=0$$

or, shortly, $$l_1'+(\tilde{l}_1'+m_1'A_{q_0})\overset{I}{A}=0\,, \tag{5.17}$$

where the degree of the coefficient m_1' of A_{q_0} is less than r_2, and the system of relations $(5.16)_1$ and (5.17) is equivalent to that of $(5.16)_1$ and $(5.16)_2$. Similarly, by eliminating the term of the greatest power of M in the coefficient of $A_{q_0}\overset{I}{A}$ from (5.17) and $(5.16)_1$—the equation in which the coefficient of $A_{q_0}\overset{I}{A}$ is of lower degree in M than that in $(5.15)_3$—we have

$$l_1''+(\tilde{l}_1''+m_i''\overset{I}{A}_{q_0})=0\,, \tag{5.18}$$

where the degree of m_1'' in M is less than the greatest degree of m_1' and m_2'. Continuing this process, by which we obtained (5.18) from $(5.15)_1$ and $(5.15)_2$, at last we have the following two equations, equivalent to $(5.16)_1$ and $(5.16)_2$, as seen by the procedure used above:

$$l_1+\left(\bar{\tilde{l}}_1+\overline{m}_1(M)\right)\overset{I}{A}=0$$
$$\bar{l}_2+\bar{\tilde{l}}_2\overset{I}{A}=0$$

where l's are linear combinations of $A_{q_0+1},\ldots,A_{2^{n_0}}$, and $\overline{m}_1$ is the greatest common factor of m_1 and m_2, as easily seen from this procedure. This procedure is nothing but an elimination process of A_{q_0} from $(5.16)_1$ and $(5.16)_2$, preserving the system of relations to be equivalent as a whole. We call such a procedure "*the elimination for* A_{q_0}."

Thus, in (5.14), applying elimination for A_{q_0} successively, finally we have the system of relations equivalent to (5.14):

$$\left.\begin{array}{ll} u_{q_0}+(\tilde{u}_q+m'A_{q_0})\overset{I}{A}=0 & (5.19)_1 \\ k_\lambda'+\tilde{k}_\lambda'\overset{I}{A}=0 & (5.19)_2 \end{array}\right\}\ (\lambda=1,\ldots,s-1) \tag{5.19}$$

where k_λ', $\tilde{k}_\lambda'$, u_{q_0} and $\tilde{u}_{q_0}$ are linear combinations of $A_{q_0+1},\ldots,A_{2^{n_0}}$ and m' is the greatest common factor of m_i in (5.15).

The totality of all relations (5.19) and those obtained from $(5.19)_1$ by multiplying with $A_1, A_2,\ldots, A_{2^{n_0}}$ (from whichever side) can be written in the form

$$\left.\begin{array}{ll} u_{q_0}+(\tilde{u}_{q_0}+m'A_{q_0})\overset{I}{A}=0 & (5.20)_1 \\ k_\lambda''+\tilde{k}_\lambda''\overset{I}{A}=0 & (5.20)_2 \end{array}\right\}\ (\lambda=1,\ldots,s'-1) \tag{5.20}$$

which is equivalent to (5.19) (i. e. (5.14)) and in which $\tilde{k}_\lambda''$; k are linear combinations of $A_{q_0+1},\ldots,A_{2^{n_0}}$.

Next, let the smallest suffix of A_j contained in $(5.20)_2$ be q_1', if applying the elimination for $A_{q_1}\overset{I}{A}$, to the system of relations $(5.20)_2$ we have the following system of relations, which is equivalent to $(5.20)_2$,

$$\left.\begin{array}{l} u_{q_1}+\left(\tilde{u}_{q_1}+m''(M)A_{q_1}\right)\overset{I}{A}=0 \\ k_\lambda'''+\tilde{k}_\lambda'''\overset{I}{A}=0 \end{array}\right\} \quad (\lambda=1,2,\ldots.)$$

where u_{q_1}, $\tilde{u}_q$ and k'''s are linear combinations of $A_{q_1+1},\ldots, A_{2^{n_0}}$ and m'' is the greatest common factor of equations including $A_{q_1}\overset{I}{A}$ in $(5.20)_2$.[1]

Continuing this process, we finally reach the system of relations equivalent to (5.14), as follows:

$$\left.\begin{array}{l} u_{q_0}+\left(\tilde{u}_{q_0}+m^{(0)}(M)A_{q_0}\right)\overset{I}{A}=0 \\ u_{q_0}+\left(\tilde{u}_{q_1}+m'\ (M)A_{q_1}\right)\overset{I}{A}=0 \\ \cdots\cdots\cdots \\ u_{q_r}+\left(\tilde{u}_{q_r}+m^{(r)}(M)A_{q_r}\right)\overset{I}{A}=0 \end{array}\right\} (0\leqq q_0<q_1\ldots.<q_r\leqq 2^{n_0};\ q\text{'s are integer}) \tag{5.21}$$

where u_q and $\tilde{u}_q$ are linear combinations of $A_{q+1},\ldots\ldots, A_{2^{n_0}}$ (with the coefficients of polynomials of M) and $m^{(i)}$ is the greatest common factor of the term in the relations in which the smallest suffix of A_j is q_i. And from the process by which (5.14) and (5.20) are deduced from (5.13) and (5.19) respectively, we can easily see that, when $A_{q_{s_2}}=A_{q_{s_1}}A_i$, the relation obtained by multiplying with A_i the equation $u_{q_{s_1}}+(\tilde{u}_{q_{s_1}}+m^{(s_1)}\overset{I}{A}_{q_{s_1}})A=0$, i. e. $A_i\left\{u_{q_{s_1}}+(\tilde{u}_{q_{s_1}}+m^{(s_1)}A_{q_{s_1}})\overset{I}{A}\right\}=0$, can be expressed by a linear combination of relations in (5.21)

$$u_{q_{s_2}}+(\tilde{u}_{q_{s_2}}+m^{(s_2)}A_{q_{s_2}})\overset{I}{A}=0$$

and
$$u_{q_s}+(\tilde{u}_{q_s}+m^{(s)}A_{q_s})\overset{I}{A}=0 \qquad (q_s=q_{s_2+1},\ldots., q_r).$$

So, from the property of $m^{(i)}$ mentioned above $m^{(s_2)}$ is a factor of $m^{(s_1)}$, i. e. $m^{(s_1)}=a_{s_1s_2}m^{(s_2)}$ when q_{s_1} and q_{s_2} are related in $A_{q_{s_2}}=A_{q_{s_1}}A_i$ (for suitable i).

If $a_{s_1s_2}=\textcircled{\scriptsize I}$, the relation

$$u_{q_{s_2}}+(\tilde{u}_{q_{s_2}}+m^{(s_2)}A_{q_{s_2}})\overset{I}{A}=0$$

is expressed by

$$A_i\left\{A_{q_{s_1}}+(\tilde{u}_{q_{s_1}}+m^{(s_1)}A_{q_{s_1}})\overset{I}{A}\right\}=0$$

and
$$u_{q_i}+(\tilde{u}_{q_s}+m^{(s)}A_{q_s})\overset{I}{A}=0 \qquad (q_s=q_{s_2+1},\ldots., q_r).$$

Therefore, when $a_{s_1s_2}=\textcircled{\scriptsize I}$, ommiting the relation $u_{q_{s_1}}+(\tilde{u}_{q_{s'}}+m^{(s_1)}A_{q_{s_1}})\overset{I}{A}=0$

(1) If the number of relations in $(5.20)_2$ containing $A_{q_1}\overset{I}{A}$ is only one, then we apply the elimination for $A_{q_2}\overset{I}{A}$ to the system of relations containing $A_{q_2}\overset{I}{A}$ with the next smallest suffix, say q_2.

from (5.21), and applying this process to all q's, we have the following system of relations, which is equivalent to (5.21) (i. e. (5.14))

$$\left.\begin{array}{l} g_0+(\tilde{g}_0+f_0A_{p_0})\overset{I}{A}=0 \\ g_1+(\tilde{g}_1+f_1A_{p_1})\overset{I}{A}=0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ g_{r'}+(\tilde{g}_{r'}+f_{r'})A=0A_{p_{r_1}} \end{array}\right\} (0\leqq p_0<p_1<\ldots\ldots<p_r\leqq 2^{n_0}). \quad (5.22)$$

Theorem 20. *When $n-n_0=odd$, all the relations among the elements of $\mathfrak{B}$ are reduced to the set of equations of the form:*

$$\left.\begin{array}{l} g_0+n_0f_0A_{p_0}+\left(\tilde{u}_0+f_0(M)A_{p_0}\right)\overset{I}{A}=0 \\ g_1+n_1f_1A_{p_1}+\left(\tilde{g}_1+f_1(M)A_{p_1}\right)\overset{I}{A}=0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ g_r+n_rf_rA_{p_r}+(\tilde{g}_{p_r}+f_{p_r}A_{p_r})\overset{I}{A}=0 \\ h_0+(\tilde{h}_0+\bar{f}_0\bar{A}_{t_0})\overset{I}{A}=0 \\ h_1+(\tilde{h}_1+\bar{f}_1A_{t_1})\overset{I}{A}=0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ h_{t_{r'}}+(\tilde{h}_{r'}+\bar{f}_{r'}A_{t_{r'}})=0 \end{array}\right\} \begin{pmatrix} 0\leqq p_0<p_1<\ldots\ldots<p_r\leqq 2^{n_0}\,; \\ 0\leqq t_0<t_1<\ldots\ldots<t_r\leqq 2^{n_0} \end{pmatrix} \quad (5.23)$$

where g_i, $\tilde{g}_i$, h_i, $\tilde{h}_i$ are linear combinations of $A_{p_i+1},\ldots\ldots,A_{2^{n_0}}$, the coefficients being polynomials of M, if $p_i=t_j$, then $\bar{f}_{t_j}=f_{p_i}n'_{p_i}$ and n_{p_i}, n'_{p_i} are polynomials prime to each other, and $a_{s_1s_2}f_{p_{s_1}}=f_{p_{s_2}}$ when $A_{p_{s_1}}=A_{p_{s_2}}A_i$, A_i being taken suitably.

Proof. Using the same process as in Theorem 18, and taking into account that the relations including A_{q_0}, q_0 being the smallest suffix of A_i in (5.14), are all written in the form $w_i+(\tilde{\omega}_i+v_iA_{p_0})\overset{I}{A}=0$ $(i=1,2,\ldots\ldots,c)$, we have the following relations corresponding to (5.19)

$$\left.\begin{array}{ll} u_0+n'_0(M)m_0A_{q_0}+(\tilde{u}_0+m_0A_{q_0})\overset{I}{A}=0 & (5.24)_1 \\ k'_\lambda+\tilde{k}'_\lambda\overset{I}{A}=0 & (5.24)_2 \end{array}\right\} \quad (5.24)$$

where u_0, k'_λ are linear combinations of A_{q_0}, $A_{q_0+1},\ldots\ldots,A_{2^{n_0}}$, and $\tilde{u}_0$ and $\tilde{k}'_\lambda$ are linear combinations of $A_{q_0+1},\ldots\ldots,A_{2^{n_0}}$, and n'_0 may be null.

When $n'_0(M)\neq 0$: multiplying by $\overset{I}{A}$ (from whichever side) all the equations of $(5.24)_2$, and by elimination for $A_{q_0}\overset{I}{A}$ from the resulting equations, we have

$$\left.\begin{array}{ll} u_1'+(\tilde{u}_1'+n_1'm_0A_{q_0})\overset{I}{A}=0 & (5.25)_1 \\ k_\lambda''+\tilde{k}_\lambda''\overset{I}{A}=0 & (5.25)_2 \end{array}\right\} \qquad (5.25)$$

where u_1', $\tilde{u}_1'$, k_λ'' and $\tilde{k}_\lambda''$ are linear combinations of $A_{q_0+1},\ldots\ldots, A_{2^{n_0}}$, and $n_1'm_0$ is a factor of all the coefficients of terms which contains $A_{q_0}\overset{I}{A}$ in (5.24).

Further, by elimination for $A_{q_0}\overset{I}{A}$ from $(5.25)_1$ and the equation obtained by multiplying $(5.24)_1$ with $\overset{I}{A}$, we have the following relations equivalent to (5.14)

$$\left.\begin{array}{ll} u_0+n_0'(M)m_0(M)A_{q_0}+(\tilde{u}_0+m_0A_{q_0})\overset{I}{A}=0 & (5.26)_1 \\ v_0'+(\tilde{v}_0'+\tilde{n}_0'm_0\bar{A}_{q_0})\overset{I}{A}=0 & (5.26)_2 \\ k_\lambda''+\tilde{k}_\lambda''\overset{I}{A}\equiv 0 & (5.26)_3 \end{array}\right\}, \ (\lambda=1,2,\ldots.) \qquad (5.26)$$

where u_0, v_0', $\tilde{u}_0$, $\tilde{v}_0'$, k_λ'' and $\tilde{k}_\lambda''$ are linear combinations of $A_{q_0+1},\ldots\ldots, A_{2^{n_0}}$. If n_0' and $\tilde{n}_0'$ are not prime, let the greatest common factor n_0' and $\tilde{n}_0'$ be $a(M)$ then $n_0'=a\mathring{n}$, $\tilde{n}_0'=a\tilde{\mathring{n}}$. Substituting this into $(5.26)_1$ and multiplying the resulting equation by $\overset{I}{A}$ and $\mathring{n}$, we have

$$\tilde{\mathring{n}}\left\{(-1)^{\frac{n-n_0+1}{2}}(\tilde{u}_0+m_0A_{q_0})M^{n-n_0}+u_0\overset{I}{A}\right\}+\tilde{\mathring{n}}a\mathring{n}A_{q_0}\overset{I}{A}=0\,;$$

and using $(5.26)_2$ we have from the equation above (by elimination for $\tilde{n}_0'\mathring{n}A_{p_0}\overset{I}{A}$ and multiplication by $\overset{I}{A}$ the resulting equation.

$$(\tilde{\mathring{n}}m_0A_{q_0}+h)\overset{I}{A}+\tilde{h}=0$$

where h, $\tilde{h}$ are linear combinations of $A_{p_0+1},\ldots\ldots, A_{2^{n_0}}$. And by elimination for A_{q_0} from $(5.26)_2$ and the above we have the following equations equivalent to

$$\left.\begin{array}{l} u_0+n_0(M)m_0(M)A_{q_0}+(\tilde{u}_0+m_0A_{q_0})\overset{I}{A}=0 \\ v_0+(\tilde{v}_0+\tilde{n}_0m_0A_{q_0}\overset{I}{A}=0 \\ k_\lambda'''+\tilde{k}_\lambda'''\overset{I}{A}=0 \end{array}\right\} (\lambda=1,2,\ldots.) \qquad (5.27)$$

where u_0, v_0, $\tilde{u}_0$, $\tilde{v}_0$ k_λ''' and $\tilde{k}_\lambda'''$ are linear combinations of $A_{q_0+1},\ldots\ldots, A_{2^{n_0}}$; and n_0, $\tilde{n}_0$ are polinomials prime to each other except for a power of M (but $\tilde{n}_0$ it may be that).

When $n_0'=0$: Multiplying by $\overset{I}{A}$ all the equations of $(5.24)_2$, and eliminating $A_{q_0}\overset{I}{A}$ from the resulting equations and (5.24), we have

$$\left.\begin{array}{l} v_0+(\tilde{v}_0+m_0A_{q_0})\overset{I}{A}=0 \\ k_\lambda''+\tilde{k}_\lambda''\overset{I}{A}=0 \end{array}\right\}, \qquad (5.28)$$

where v_0, $\tilde{v}_0$, k''_λ and $\tilde{k}''_\lambda$ and linear combinations of $A_{q_0+1}, \ldots, A_{2^{n_0}}$.

Continuing the process of elimination by which we obtained from (5.14), (5.27) or (5.28), we finally get (5.23).

Corollary. *When $n-n_0=$ odd, if there is a relation $g(M)+\tilde{g}(M)\overset{I}{A}=0$, there always exists the relation $f(M)=0$.*

Proof. By Theorem 19, the relation $g(M)+\tilde{g}(M)\overset{I}{A}$ is reduced to

$$g'(M)\tilde{g}_0(M)+\tilde{g}(M)\overset{I}{A}=0 \qquad (p_0=0 \text{ in Theorem } 19).$$

Eliminating $\overset{I}{A}$ from $\qquad g'(M)\tilde{g}_0(M)+\tilde{g}_0(M)\overset{I}{A}=0$

and $\qquad \overset{I}{A}\left\{g'\tilde{g}_0+\tilde{g}(M)\overset{I}{A}\right\}=0 \quad$ i. e. $\quad (-1)^{\frac{n-n_0+1}{2}} M^{n-n_0}\tilde{g}+g'\tilde{g}_0\overset{I}{A}=0$

we have

$$\left\{(g')^2-(-1)^{\frac{n-n_0+1}{2}} M^{n-n_0}\right\}(\tilde{g}_0)^2=0\,. \tag{5.28}$$

But, since $n-n_0$ is odd, this equation can not be an identity. So we have the equation of M, say $f(M)=0$, which is a factor of the right-hand side of (5.28). So the corollary has been proved.

Special case, $n_0=0$:

When $n_0=0$, from Theorems (18) and (19) and the corollary, we see that, if there exist relations in $\mathfrak{B}$, then necessarily:

(when $n=$even) $g(M)=0$, and there is no more relation.

(when $n=$odd) either $f(M)=0$

or $\qquad \tilde{g}\overset{I}{A}=g(M)\tilde{g}(M) \quad$ and $\quad g'(M)\tilde{g}=0\,, \quad g'(M)\neq 0\,,$

and there is no more relation where $g(M)$ and $g'(M)$ have no common factor.

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