Ideals in a Boolean Algebra with Transfinite Chain Condition.

By

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Let L be a generalized $\aleph$-Boolean algebra, and Ω be an ordinal number. A subset $S=(a_\alpha;\ \alpha<\Omega)$ of L is called an ascending or descending system according as $a_\alpha<a_\beta$ or $a_\alpha>a_\beta$ for all $\alpha<\beta<\Omega$. If, for every ascending and descending system S, the power of S is $<\aleph$, then we say that L satisfies the $\aleph$-chain condition. Let $\mathfrak{a}$ be an $\aleph$-ideal in L, and $L/\mathfrak{a}$ denote the set of all equivalence classes with respect to $\mathfrak{a}$. When $L/\mathfrak{a}$ satisfies the $\aleph$-chain condition, we say that L satisfies the $\aleph$-chain condition relative to $\mathfrak{a}$, and $\mathfrak{a}$ is called a basic $\aleph$-ideal of the $\aleph$-chain condition. I shall prove that L satisfies the $\aleph$-chain condition relative to $\mathfrak{a}$ if, and only if, L satisfies the following condition:

For every $T\subset L$ such that (i) $a\in T$ implies $a\notin\mathfrak{a}$, (ii) $a, b\in T$, $a\neq b$ implies $a\frown b\in\mathfrak{a}$, the power of T is $<\aleph$.

I find also that class $\mathfrak{P}_\aleph$ of all basic $\aleph$-ideals of the $\aleph$-chain condition in L is a generalized $\aleph$-Boolean algebra, and class $\Phi^*_\aleph$ of all dual $\aleph$-ideals in $\mathfrak{P}_\aleph$ is a continuous Boolean algebra.

Next I apply this result to class $\mathfrak{F}_{\aleph_1}$ of all measure functions defined in an $\aleph_1$-Boolean algebra L. Let $\mathfrak{a}_\phi$ be the class of all a such that $\phi(a)=0$. Then $\mathfrak{a}_\phi$ is a basic $\aleph_1$-ideal of the $\aleph_1$-chain condition in L. I shall prove that class $\mathfrak{Q}_{\aleph_1}$ of all $\mathfrak{a}_\phi(\phi\in\mathfrak{F}_{\aleph_1})$ is a dual $\aleph_1$-ideal in $\mathfrak{P}_{\aleph_1}$, and therefore $\mathfrak{Q}_{\aleph_1}$ is a generalized $\aleph_1$-Boolean algebra, and class $\Psi^*_{\aleph_1}$ of all dual $\aleph_1$-ideals in $\mathfrak{Q}_{\aleph_1}$ is a continuous Boolean algebra. We shall write $\psi<\phi$, when $\psi(a)$ is absolutely continuous with respect to $\phi(a)$, that is, $\psi(a)=0$ for all a such that $\phi(a)=0$. Then $\psi<\phi$ when, and only when, $\mathfrak{a}_\psi\supset\mathfrak{a}_\phi$. Hence $\mathfrak{F}_{\aleph_1}$ is dual-isomorphic to $\mathfrak{Q}_{\aleph_1}$. Therefore $\mathfrak{F}_{\aleph_1}$ is a generalized $\aleph_1$-Boolean algebra, and the class $\Psi_{\aleph_1}$ of all $\aleph_1$-ideals in $\mathfrak{F}_{\aleph_1}$ is a continuous Boolean algebra.

Lastly I shall investigate the application to the spectral theory of the complete complex Euclidean space $\mathfrak{H}$. If a family of projections $\boldsymbol{E}(a)$ is defined for all a in an $\aleph_1$-Boolean algebra L, such that

$$(\alpha)\quad \boldsymbol{E}(a)\boldsymbol{E}(b)=\boldsymbol{0} \quad \text{when} \quad a\frown b=0,$$

$$(\beta)\quad \boldsymbol{E}(a)=\boldsymbol{E}(a_1)+\boldsymbol{E}(a_2)+\cdots+\boldsymbol{E}(a_i)+\cdots \quad \text{when} \quad a=\sum_i\oplus a_i,$$

$$(\gamma)\quad \boldsymbol{E}(1)=\boldsymbol{1};$$

then we say that $\boldsymbol{E}(a)$ is a resolution of identity in the generalized sense. Let $\mathfrak{a}_f$ be the class of all a such that $\boldsymbol{E}(a)f=0$. Then $\mathfrak{a}_f$ is a basic $\aleph_1$-ideal of the $\aleph_1$-chain condition in L. I shall prove that class $\mathfrak{R}_E$ of all $\mathfrak{a}_f(f\in\mathfrak{H})$ is a dual $\aleph$-ideal in $\mathfrak{P}_{\aleph_1}$, and therefore $\mathfrak{R}_E$ is a generalized $\aleph_1$-Boolean algebra, and class θ_E^* of all dual $\aleph$-ideals in $\mathfrak{R}_E$ is a continuous Boolean algebra. Next I shall obtain a resolution of identity $\boldsymbol{F}(\mathfrak{A}^*)$ defined for all $\mathfrak{A}^*\in\theta_E^*$, which, we may say, is an extention of $\boldsymbol{E}(a)$ in the domain of definition. $\boldsymbol{F}(\mathfrak{A}^*)$ corresponds to the resolution of identity defined by F. Wecken in his remarkable paper, which appeared recently.(1)

General Properties of $\aleph$-Ideals in a Generalized $\aleph$-Boolean Algebra.(2)

1. Let us consider *generalized* $\aleph$*-Boolean algebra* L with elements $a, b, c, \ldots\ldots$, $\aleph$ being a transfinite cardinal number, that is, L satisfies the following axioms (I)–(IV):

(I) L is a *partially ordered set*; that is, in L a relation of inclusion $a\supset b$ is defined, such that

(i) $a\supset a$;

(ii) $a\supset b$, $b\supset c$ implies $a\supset c$.

We define equality $a=b$ in L as the simultaneous existence of the relations $a\supset b$, $a\subset b$. We write $a>b$, when $a\supset b$ but $a\neq b$.

(II) L is an $\aleph$*-lattice*; that is,

For every subset S of L of power $<\aleph$, there is an element $\sum(S)$ in L which is a least upper bound or join of S, i.e.

(i) $\sum(S)\supset a$ for every $a\in S$,

(ii) $c\supset a$ for every $a\in S$ implies $c\supset\sum(S)$.

For every subset S of L of power $<\aleph$, there is an element $\Pi(S)$ in L which is a greatest lower bound or meet of S, i.e.

(i) $\Pi(S)\subset a$ for every $a\in S$,

(ii) $c\subset a$ for every $a\in S$ implies $c\subset\Pi(S)$.

When $S=(a, b)$, we write $\sum(S)=a\vee b$, $\Pi(S)=a\wedge b$. If $\aleph>$ power of L, then L is called a *continuous* lattice.

(III) L is *complemented* in the generalized sense; that is,

For any three elements a, b, c such that $a\subset b\subset c$, there exists an element x such that

$$b\vee x=c, \qquad b\wedge x=a.$$

(IV) L is *distributive* in the generalized sense; that is,

For every subset S of L of power $<\aleph$ and every $b\in L$

(1) Cf. W. Wecken [1]. The numbers in square brackets refer to the list given at the end of this paper.

(2) M. H. Stone has investigated the properties of $\aleph_0$-ideals in a generalized $\aleph_0$-Boolean algebra with zero element. (Cf. M. H. Stone [1]). These investigations can be applied in our general case. In the present paper only a few properties are mentioned, which are necessary to what follows.

$$\sum(S)\wedge b=\sum(a\wedge b;\ a\in S),$$
$$\Pi(S)\vee b=\Pi(a\vee b;\ a\in S).$$

It should be observed that in the axioms given above, the existence of zero element 0 and unit element 1 is not assumed. If the existence of these elements is assumed, then the $\aleph$-lattice satisfying (III) and (IV) is equivalent to the $\aleph$-lattice satisfying the following axioms (III′) and (IV′).[1]

(III′) For any element a, there exists an element x such that

$$a\vee x=1, \quad a\wedge x=0.$$

x is called the inverse of a, and is often denoted by a'.

(IV′) For all elements a, b, c in L

$$(a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c),$$
$$(a\wedge b)\vee c=(a\vee c)\wedge(b\vee c).$$

For the sake of simplicity we shall call the $\aleph$-lattice with zero and unit elements satisfying (III′) and (IV′) an *$\aleph$-Boolean algebra.*

It is also to be noted that, in axioms (I)–(IV), the *duality* holds good interchanging $\subset, \sum, \vee$ by $\supset, \Pi, \wedge$.

2. DEFINITION 2.1. In a generalized $\aleph$-Boolean algebra L, if a subset $\mathfrak{a}$ of L satisfies the following properties, then $\mathfrak{a}$ is said to be an *$\aleph$-ideal* in L:

(i) if $S\subseteqq\mathfrak{a}$ and power of $S<\aleph$, then $\sum(S)\in\mathfrak{a}$,

(ii) if $a\in\mathfrak{a}$ and $b\in L$, then $a\wedge b\in\mathfrak{a}$.

(Condition (ii) is equivalent to: "if $a\in\mathfrak{a}$, $c\in L$, $c\subset a$, then $c\in\mathfrak{a}$".)

DEFINITION 2.1.*[2] Dually, we can define a *dual $\aleph$-ideal* $\mathfrak{a}^*$ in a generalized $\aleph$-Boolean algebra L by the following properties:

(i*) if $S\subseteqq\mathfrak{a}^*$ and power of $S<\aleph$, then $\Pi(S)\in\mathfrak{a}^*$,

(ii*) if $a\in\mathfrak{a}^*$ and $b\in L$, then $a\vee b\in\mathfrak{a}^*$.

(Condition (ii*) is equivalent to: "if $a\in\mathfrak{a}^*$, $c\in L$, $c\supset a$, then $c\in\mathfrak{a}^*$".)

It is evident that class $\mathfrak{a}(a)$ of all elements $b\in L$, such that $b\subset a$, is an $\aleph$-ideal in L.

DEFINITION 2.2. We say that $\mathfrak{a}(a)$ is a *principal $\aleph$-ideal* in L.

Dually, we denote by $\mathfrak{a}^*(a)$ the class of all elements $b\in L$ such that $b\supset a$.

DEFINITION 2.2.* We say that $\mathfrak{a}^*(a)$ is a *principal dual $\aleph$-ideal in L.*

THEOREM 2.1. *Class $\mathfrak{J}$ of all $\aleph$-ideals in a generalized $\aleph$-Boolean algebra L is a lattice, closed with respect to the join of any subset of $\mathfrak{J}$ and to the meet of subsets of $\mathfrak{J}$ of power $<\aleph$. For every subset $\mathfrak{S}=(\mathfrak{a}_\alpha;\ \alpha\in I)$ of $\mathfrak{J}$, I being the set of indices, the join $\sum(\mathfrak{S})$ consists of those*

(1) Cf. J. v. Neumann [1], 6; [2], 7.

(2) * means the dual relation.

elements such that $c=\sum(a_\alpha;\ a_\alpha\in\mathfrak{a}_\alpha,\ \alpha\in I_1)$ *where* I_1 *is a subclass of* I *of power* $<\aleph$. *And for every subset* $\mathfrak{S}=(\mathfrak{a}_\alpha;\ \alpha\in I)$ *of* $\mathfrak{J}$ *of power* $<\aleph$, *the meet* $\Pi(\mathfrak{S})$ *consists of those elements such that* $c=\Pi(a_\alpha;\ a_\alpha\in\mathfrak{a}_\alpha,\ \alpha\in I)$.

$\mathfrak{J}$ *satisfies the following distributive laws:*

$$\mathfrak{a}\wedge\sum(\mathfrak{S})=\sum(\mathfrak{a}\wedge\mathfrak{a}_\alpha;\ \alpha\in I)$$

for every subset $\mathfrak{S}=(\mathfrak{a}_\alpha;\ \alpha\in I)$ *of* $\mathfrak{J}$, *and*

$$\mathfrak{a}\vee\Pi(\mathfrak{S})=\Pi(\mathfrak{a}\vee\mathfrak{a}_\alpha;\ \alpha\in I)$$

for every subset $\mathfrak{S}=(\mathfrak{a}_\alpha;\ \alpha\in I)$ *of power* $<\aleph$.

$\mathfrak{J}$ *has the unit element* $\mathfrak{e}$, *which is* L *itself.*

Especially when L *has zero element* 0, $\mathfrak{J}$ *has zero element* $\mathfrak{o}$ *which consists of* 0 *alone, and* $\mathfrak{J}$ *is closed with respect to the meet of any subset of* $\mathfrak{J}$.

PROOF. Since the following dual Theorem 2.1* is not familiar, I shall prove it. The present theorem can be proved as Theorem 2.1* with slight modifications.

THEOREM 2.1.* *Class* $\mathfrak{J}^*$ *of all dual* $\aleph$*-ideals in a generalized* $\aleph$*-Boolean algebra* L *is a lattice, closed with respect to the join of any subset of* $\mathfrak{J}^*$ *and to the meet of subsets of* $\mathfrak{J}^*$ *of power* $<\aleph$. *For every subset* $\mathfrak{S}^*=(\mathfrak{a}_\alpha^*;\ \alpha\in I)$ *of* $\mathfrak{J}^*$, *I being the set of indices, the join* $\sum(\mathfrak{S}^*)$ *consists of those elements such that* $c=\Pi(a_\alpha;\ a_\alpha\in\mathfrak{a}_\alpha^*,\ \alpha\in I_1)$ *where* I_1 *is a subclass of* I *of power* $<\aleph$. *And for every subset* $\mathfrak{S}^*=(\mathfrak{a}_\alpha^*;\ \alpha\in I)$ *of* $\mathfrak{J}^*$ *of power* $<\aleph$, *the meet* $\Pi(\mathfrak{S}^*)$ *consists of those elements such that* $c=\sum(a_\alpha;\ a_\alpha\in\mathfrak{a}_\alpha^*,\ \alpha\in I)$.

$\mathfrak{J}^*$ *satisfies the following distributive laws:*

$$\mathfrak{a}^*\wedge\sum(\mathfrak{S}^*)=\sum(\mathfrak{a}^*\wedge\mathfrak{a}_\alpha^*;\ \alpha\in I)$$

for every subset $\mathfrak{S}^*=(\mathfrak{a}_\alpha^*;\ \alpha\in I)$ *of* $\mathfrak{J}^*$, *and*

$$\mathfrak{a}^*\vee\Pi(\mathfrak{S}^*)=\Pi(\mathfrak{a}^*\vee\mathfrak{a}_\alpha^*:\ \alpha\in I)$$

for every subset $\mathfrak{S}^*=(\mathfrak{a}_\alpha^*;\ \alpha\in I)$ *of* $\mathfrak{J}^*$ *of power* $<\aleph$.

$\mathfrak{J}^*$ *has unit element* $\mathfrak{e}^*$, *which is* L *itself.*

Especially when L *has unit element* 1, $\mathfrak{J}^*$ *has zero element* $\mathfrak{o}^*$ *which consists of* 1 *alone, and* $\mathfrak{J}^*$ *is closed with respect to the meet of any subset of* $\mathfrak{J}^*$.

PROOF. (i) Let $\mathfrak{S}^*=(\mathfrak{a}_\alpha^*;\ \alpha\in I)$ be any subset of $\mathfrak{J}^*$, and $\mathfrak{c}^*$ be the class of all elements $c=\Pi(a_\alpha;\ a_\alpha\in\mathfrak{a}_\alpha^*,\ \alpha\in I_1)$ where I_1 is a subclass of I of power $<\aleph$. And let S^* be any subset of $\mathfrak{c}^*$ of power $<\aleph$; then, since all $\mathfrak{a}_\alpha^*$ are dual $\aleph$-ideals, $\Pi(S^*)$ can be written in the form $\Pi(S^*)=\Pi(b_\alpha;\ b_\alpha\in\mathfrak{a}_\alpha^*,\ \alpha\in I_2)$, where I_2 is a subclass of I of power $<\aleph$. That is, $\Pi(S^*)\in\mathfrak{c}^*$.

Let d be any element of L and $c\in\mathfrak{c}^*$. Then

$$\begin{aligned} d\vee c&=d\vee\Pi(a_\alpha;\ a_\alpha\in\mathfrak{a}_\alpha^*,\ \alpha\in I_1)\\ &=\Pi(d\vee a_\alpha;\ a_\alpha\in\mathfrak{a}_\alpha^*,\ \alpha\in I_1)^{(1)}\in\mathfrak{c}^* \end{aligned}$$

(1) By Axiom (IV) of Sec. 1.

since $d \vee a_\alpha \in \mathfrak{a}_\alpha^*$. Hence $\mathfrak{c}^*$ is a dual $\aleph$-ideal. Since $\mathfrak{c}^*$ is the smallest dual $\aleph$-ideal which contains all $\mathfrak{a}^*$ in $\mathfrak{S}^*$, we have

$$\mathfrak{c}^* = \sum(\mathfrak{S}^*).$$

(ii) Next let $\mathfrak{S}^* = (\mathfrak{a}_\alpha^*;\ \alpha \in I)$ be any subset of $\mathfrak{J}^*$ of power $< \aleph$. The intersection of all these $\mathfrak{a}_\alpha^*$ in $\mathfrak{S}^*$ consists of those elements c such that $c = \sum(a_\alpha;\ a_\alpha \in \mathfrak{a}_\alpha^*,\ \alpha \in I)$. For any such element is common to all $\mathfrak{a}_\alpha^*$ of $\mathfrak{S}^*$, and if c is common to all $\mathfrak{a}_\alpha^*$ of $\mathfrak{S}^*$, then c is written in the form above, where $a_\alpha = c$ for all $\alpha \in I$. And this intersection is the required meet $\Pi(\mathfrak{S}^*)$. Especially when L has unit element 1, the intersection of any subset of $\mathfrak{J}^*$ is non-void, and is the meet $\Pi(\mathfrak{S}^*)$.

(iii) Let $\mathfrak{S}^* = (\mathfrak{a}_\alpha^*;\ \alpha \in I)$ be any subset of $\mathfrak{J}^*$. Since

$$\mathfrak{a}^* \wedge \sum(\mathfrak{S}^*) \supset \mathfrak{a}^* \wedge \mathfrak{a}_\alpha^* \quad \text{for all} \quad \alpha \in I,$$

we have
$$\mathfrak{a}^* \wedge \sum(\mathfrak{S}^*) \supset \sum(\mathfrak{a}^* \wedge \mathfrak{a}_\alpha^*;\ \alpha \in I).$$

Next, by (i) and (ii), any element c in $\mathfrak{a}^* \wedge \sum(\mathfrak{S}^*)$ is expressible in the form $c = a \vee \Pi(b_\alpha;\ b_\alpha \in \mathfrak{a}_\alpha^*,\ \alpha \in I_1)$, where $a \in \mathfrak{a}^*$ and I_1 is a subclass of I of power $< \aleph$. But, by Axiom (IV) of Sec. 1,

$$c = \Pi(a \vee b_\alpha;\ b_\alpha \in \mathfrak{a}_\alpha^*,\ \alpha \in I_1).$$

Since $a \vee b_\alpha \in \mathfrak{a}^* \wedge \mathfrak{a}_\alpha^*$, we have $c \in \sum(\mathfrak{a}^* \wedge \mathfrak{a}_\alpha^*;\ \alpha \in I_1)$. Therefore

$$\mathfrak{a}^* \wedge \sum(\mathfrak{S}^*) \subset \sum(\mathfrak{a}^* \wedge \mathfrak{a}_\alpha^*;\ \alpha \in I).$$

Consequently
$$\mathfrak{a}^* \wedge \sum(\mathfrak{S}^*) = \sum(\mathfrak{a}^* \wedge \mathfrak{a}_\alpha^*;\ \alpha \in I).$$

(iv) As in (iii), we can prove that

$$\mathfrak{a}^* \vee \Pi(\mathfrak{S}^*) = \Pi(\mathfrak{a}^* \vee \mathfrak{a}_\alpha^*;\ \alpha \in I)$$

for any subset $\mathfrak{S}^* = (\mathfrak{a}_\alpha^*;\ \alpha \in I)$ of $\mathfrak{J}^*$ of power $< \aleph$.

Theorem 2.2. *Class $\mathfrak{P}$ of all principal $\aleph$-ideals in a generalized $\aleph$-Boolean algebra L is isomorphic to L in accordance with the following relations:*

(i) $\mathfrak{a}(a) = \mathfrak{a}(b)$ *if, and only if,* $a = b$,

(ii) $\mathfrak{a}\big(\sum(S)\big) = \sum\big(\mathfrak{a}(a);\ a \in S\big)$,

(iii) $\mathfrak{a}\big(\Pi(S)\big) = \Pi\big(\mathfrak{a}(a);\ a \in S\big)$,

where S is any subset of L with power $< \aleph$. Hence $\mathfrak{P}$ is also a generalized $\aleph$-Boolean algebra.

Proof. (i) $a = b$ obviously implies $\mathfrak{a}(a) = \mathfrak{a}(b)$; and $\mathfrak{a}(a) = \mathfrak{a}(b)$ implies $a \subset b$, $b \subset a$, and hence $a = b$.

(ii) Since $\sum(S) \supset a$ for all $a \in S$, $\mathfrak{a}\big(\sum(S)\big) \supset \sum\big(\mathfrak{a}(a);\ a \in S\big)$. Next, by Theorem 2.1, $\sum(S) \in \sum\big(\mathfrak{a}(a);\ a \in S\big)$; hence $\mathfrak{a}\big(\sum(S)\big) \subset \sum\big(\mathfrak{a}(a);\ a \in S\big)$.

(iii) Similarly to (ii).

Theorem 2.2.* *Class $\mathfrak{P}^*$ of all principal dual $\aleph$-ideals in a generalized $\aleph$-Boolean algebra L is dual-isomorphic to L in accordance with the following relations:*

(i) $\mathfrak{a}^*(a)=\mathfrak{a}^*(b)$ *if, and only if,* $a=b$,

(ii) $\mathfrak{a}^*\big(\sum(S)\big)=\Pi\big(\mathfrak{a}^*(a);\ a\in S\big)$,

(iii) $\mathfrak{a}^*\big(\Pi(S)\big)=\sum\big(\mathfrak{a}^*(a);\ a\in S\big)$,

where S is any subset of L with power $<\aleph$. Hence $\mathfrak{P}^*$ is also a generalized $\aleph$-Boolean algebra.

PROOF. Cf. proof of Theorem 2.2.

THEOREM 2.3. *In a generalized $\aleph$-Boolean algebra L with zero element* 0, *let $\mathfrak{a}$ be an $\aleph$-ideal, and let $\mathfrak{a}'$ be the class of all elements b such that*

$$b\wedge a=0 \quad \textit{for all} \quad a\in\mathfrak{a}.$$

Then $\mathfrak{a}'$ is an $\aleph$-ideal in L, and

$$\mathfrak{a}\wedge\mathfrak{a}'=\mathfrak{o}.$$

PROOF. This theorem can be proved as the following dual case.

THEOREM 2.3.* *In a generalized $\aleph$-Boolean algebra L with unit element* 1, *let $\mathfrak{a}^*$ be a dual $\aleph$-ideal, and let $\mathfrak{a}^{*\backprime}$ be the class of all elements b such that*

$$b\vee a=1 \quad \textit{for all} \quad a\in\mathfrak{a}^*.$$

Then $\mathfrak{a}^{*\backprime}$ is a dual $\aleph$-ideal in L and

$$\mathfrak{a}^*\wedge\mathfrak{a}^{*\backprime}=\mathfrak{o}^*.$$

PROOF. (i) Let S^* be any subset of $\mathfrak{a}^{*\backprime}$ of power $<\aleph$. Then, by Axiom (IV) in Sec. 1,

$$\Pi(S^*)\vee a=\Pi(b\vee a;\ b\in S^*)=1 \quad \text{for all} \quad a\in\mathfrak{a}^*.$$

Hence $\Pi(S^*)\in\mathfrak{a}^{*\backprime}$.

(ii) If $b\in\mathfrak{a}^{*\backprime}$ and $c\supset b$, then $c\in\mathfrak{a}^{*\backprime}$, since $c\vee a\supset b\vee a=1$ for all $a\in\mathfrak{a}^*$.

From (i) and (ii), $\mathfrak{a}^{*\backprime}$ is a dual $\aleph$-ideal in L.

(iii) By Theorem 2.1*, $\mathfrak{a}^*\wedge\mathfrak{a}^{*\backprime}$ consists of those elements $a\vee b$ such that $a\in\mathfrak{a}^*$, $b\in\mathfrak{a}^{*\backprime}$. Since $a\vee b=1$ for all $a\in\mathfrak{a}^*$, $b\in\mathfrak{a}^{*\backprime}$, we have

$$\mathfrak{a}^*\wedge\mathfrak{a}^{*\backprime}=\mathfrak{o}^*.$$

DEFINITION 2.3. The $\aleph$-ideal $\mathfrak{a}'$ associated with an $\aleph$-ideal $\mathfrak{a}$ in the manner indicated in Theorem 2.3 is called the *orthocomplement* of $\mathfrak{a}$.

DEFINITION 2.3.* The dual $\aleph$-ideal $\mathfrak{a}^{*\backprime}$ associated with a dual $\aleph$-ideal $\mathfrak{a}^*$ in the manner indicated in Theorem 2.3* is called the *dual orthocomplement* of $\mathfrak{a}^*$.

THEOREM 2.4. *In a generalized $\aleph$-Boolean algebra L with zero element* 0, *if every $\aleph$-ideal contained in a principal $\aleph$-ideal is also principal, then class $\mathfrak{J}$ of all $\aleph$-ideals in L is a continuous Boolean algebra.*

PROOF. Cf. the following dual case.

THEOREM 2.4.* *In a generalized $\aleph$-Boolean algebra L with unit element* 1, *if every dual $\aleph$-ideal contained in a principal dual $\aleph$-ideal is also principal, then class $\mathfrak{J}^*$ of all dual $\aleph$-ideals in L is a continuous Boolean algebra.*

PROOF. From Theorem 2.1*, $\mathfrak{J}^*$ is a distributive continuous lattice, that is, $\mathfrak{J}^*$ is a continuous lattice which satisfies Axiom (IV′) in Sec. 1. Hence it is sufficient to show that $\mathfrak{J}^*$ is complemented, that is, that $\mathfrak{J}^*$ satisfies Axiom (III′) in Sec. 1.

Let $\mathfrak{a}^*$ be any element in $\mathfrak{J}^*$, and $\mathfrak{a}^{*\backprime}$ be the dual orthocomplement of $\mathfrak{a}^*$. Then, by Theorem 2.3*,

$$\mathfrak{a}^* \wedge \mathfrak{a}^{*\backprime} = \mathfrak{o}^* .$$

Next we shall prove that

$$\mathfrak{a}^* \vee \mathfrak{a}^{*\backprime} = \mathfrak{e}^* .$$

For this purpose, from Theorem 2.1*, it is sufficient to show that any element $c \in L$ is expressed in the form

$$c = a \wedge b \quad \text{where} \quad a \in \mathfrak{a}^*,\ b \in \mathfrak{a}^{*\backprime}. \tag{1}$$

If $c \in \mathfrak{a}^{*\backprime}$, then

$$c = 1 \wedge c \quad \text{where} \quad 1 \in \mathfrak{a}^*,\ c \in \mathfrak{a}^{*\backprime}.$$

If $c \notin \mathfrak{a}^{*\backprime}$, then there exists an element $a \in \mathfrak{a}^*$ such that $c \vee a \neq 1$. Hence

$$\mathfrak{a}^*(c) \wedge \mathfrak{a}^* \neq \mathfrak{o}^* .$$

Since $\mathfrak{a}^*(c) \wedge \mathfrak{a}^* \subset \mathfrak{a}^*(c)$, by assumption there exists an element $d \neq 1$ such that

$$\mathfrak{a}^*(c) \wedge \mathfrak{a}^* = \mathfrak{a}^*(d). \tag{2}$$

Let d_1 be such that

$$d \vee d_1 = 1, \qquad d \wedge d_1 = c .$$

Then $d_1 \in \mathfrak{a}^{*\backprime}$. For if not, then, as above, there exists an element $f \neq 1$ such that

$$\mathfrak{a}^*(d_1) \wedge \mathfrak{a}^* = \mathfrak{a}^*(f). \tag{3}$$

Then $d_1 \subset f$, and $d \vee f = 1$. Hence

$$\mathfrak{a}^*(d) \wedge \mathfrak{a}^*(f) = \mathfrak{o}^* . \tag{4}$$

Since $f \supset d_1 \supset c$, $\mathfrak{a}^*(f) \subset \mathfrak{a}^*(c)$. And by (3) $\mathfrak{a}^*(f) \subset \mathfrak{a}^*$. Hence by (2)

$$\mathfrak{a}^*(f) \subset \mathfrak{a}^*(c) \wedge \mathfrak{a}^* = \mathfrak{a}^*(d) .$$

But this contradicts (4). Consequently

$$c = d \wedge d_1 \quad \text{where} \quad d \in \mathfrak{a}^*,\ d_1 \in \mathfrak{a}^{*\backprime}.$$

Consequently (1) holds good for any $c \in L$. And the theorem is completely proved.

3. DEFINITION 3.1. If $\mathfrak{a}$ is an $\aleph$-ideal in a generalized $\aleph$-Boolean algebra L, then a and b are *equivalent modulo* $\mathfrak{a}$ $\big(a \sim b \pmod{\mathfrak{a}}\big)$ when there exist $u, v \in \mathfrak{a}$ with $a \vee u = a \vee v$.[1]

LEMMA 3.1. *The relation* $a \sim b \pmod{\mathfrak{a}}$ *is reflexive, symmetric, and transitive.*

(1) We can define the equivalence modulo $\mathfrak{a}^*$ $(a \sim b \pmod{\mathfrak{a}^*})$ when there exist $u, v \in \mathfrak{a}^*$ with $a \wedge u = b \wedge v$, and the set $L/\mathfrak{a}^*$ of all equivalence classes. But I omit investigations into these dual properties. Cf. p. 20, footnote (3).

PROOF. Obviously $a \sim a$ (mod $\mathfrak{a}$); and $a \sim b$ (mod $\mathfrak{a}$) implies $b \sim a$ (mod $\mathfrak{a}$). To prove the transitivity, let $a \sim b$ (mod $\mathfrak{a}$), $b \sim c$ (mod $\mathfrak{a}$) i.e., $a \vee u_1 = b \vee v_1$, $b \vee u_2 = c \vee v_2$, $u_1, v_1, u_2, v_2 \in \mathfrak{a}$. Then

$$a \vee u_1 \vee u_2 = b \vee v_1 \vee u_2 = c \vee v_1 \vee v_2,$$

and $u_1 \vee u_2$, $v_1 \vee v_2 \in \mathfrak{a}$. Therefore $a \sim c$ (mod $\mathfrak{a}$).

DEFINITION 3.2. Let A_a be the set of all $x \in L$ with $x \sim a$ (mod $\mathfrak{a}$). The system $(A_a;\ a \in L)$ is a mutually exclusive and exhaustive partition of L into equivalence classes. We denote the equivalence classes by $A, B, C, \ldots,$ and the set of all equivance classes by $L/\mathfrak{a}$.

DEFINITION 3.3. $A \subset B$ means the existence of $a \in A$, $b \in B$ with $a \subset b$.

LEMMA 3.2. *When $A \subset B$, for every $a \in A$, $b \in B$ there exists u such that $a \subset b \vee u$, $u \in \mathfrak{a}$ and $b \vee u \in B$.*

PROOF. Since $A \subset B$, there exist $a_1 \in A$, $b_1 \in B$ such that $a_1 \subset b_1$. Since $a \sim a_1$ (mod $\mathfrak{a}$), $b \sim b_1$ (mod $\mathfrak{a}$), there exist u_1, v_1, u_2, v_2 in $\mathfrak{a}$ such that

$$a \vee u_1 = a_1 \vee v_1, \qquad b \vee u_2 = b_1 \vee v_2.$$

Put $u = v_1 \vee u_2$; then, since $u \in \mathfrak{a}$, $b \vee u \in B$; and

$$a \subset a_1 \vee v_1 \subset b_1 \vee v_1 \subset b \vee v_1 \vee u_2 = b \vee u.$$

THEOREM 3.1. *When L is a generalized $\aleph$-Boolean Algebra, $L/\mathfrak{a}$ is a generalized $\aleph$-Boolean algebra with zero element.*[1]

PROOF. (i) To show that $L/\mathfrak{a}$ is a partially ordered set, we need only to prove that $A \supset B$, $B \supset C$ imply $A \supset C$. By Lemma 3.2, since $A \supset B$, $B \supset C$, there exist $a \in A$, $b \in B$, $c \in C$ such that $a \supset b$, $b \supset c$. Hence $a \supset c$. Consequently, by Definition 3.3, $A \supset C$. It is evident that class $\mathfrak{a}$ itself is zero element of $L/\mathfrak{a}$. (When L has unit element 1, then A_1 is unit element of $L/\mathfrak{a}$).

(ii) Next we shall show that $L/\mathfrak{a}$ is an $\aleph$-lattice. Consider a subset $\mathfrak{S}$ of $L/\mathfrak{a}$ with power $< \aleph$. For each $A \in \mathfrak{S}$ select an element $a \in A$, and denote the set of these elements by S. Then, the power of $S < \aleph$, and $\mathfrak{S} = (A_a;\ a \in S)$. Thus $\sum(S)$ and $\Pi(S)$ exist, and for every $a \in S$, $\Pi(S) \subset a \subset \sum(S)$, $A_{\Pi(S)} \subset A_a \subset A_{\Sigma(S)}$.

Suppose now that $C = A_{a_0} \supset A_a$ for every $a \in S$. Then, by Lemma 3.2, there exists for each $a \in S$ an element $u_a \in \mathfrak{a}$ with $a_0 \vee u_a \supset a$. Thus

$$a_0 \vee \sum(u_a;\ a \in S) \supset \sum(S).$$

Since $\mathfrak{a}$ is an $\aleph$-ideal, it follows that $\sum(u_a;\ a \in S) \in \mathfrak{a}$. Hence

$$C = A_{a_0} \supset A_{\Sigma(S)}.$$

Consequently $$\sum(\mathfrak{S}) = \sum(A_a;\ a \in S) = A_{\Sigma(S)}. \qquad (1)$$

(1) J. v. Neumann has proved this theorem when L has zero and unit elements. Cf. J. v. Neumann [2], 10–11. Here I consider the general case, and the proof is somewhat simplified.

Next suppose that $C=A_{a_0}\subset A_a$ for every $a\in S$. Then, by Lemma 3.2, there exists for each $a\in S$ an element $v_a\in\mathfrak{a}$ with $a_0\subset a\vee v_a$. Thus

$$a_0\subset a\vee\sum(v_b;\ b\in S)\quad\text{for every}\quad a\in S,$$

and consequently $$a_0\subset\Pi\big(a\vee\sum(v_b;\ b\in S);\ a\in S\big);$$

that is, $$a_0\subset\Pi(S)\vee\sum(v_b;\ b\in S)$$

by Axiom (IV) in Sec. 1. As before, $\sum(v_b;\ b\in S)\in\mathfrak{a}$, whence $A_{a_0}\subset A_{\Pi(S)}$. Consequently $$\Pi(\mathfrak{S})=\Pi(A_a;\ a\in S)=A_{\Pi(S)}. \tag{2}$$

Thus $L/\mathfrak{a}$ is an $\aleph$-lattice.

(iii) $L/\mathfrak{a}$ is complemented in the generalized sense. For let A, B, C be such that $A\subset B\subset C$; then, by Lemma 3.2, there exist $a\in A$, $b\in B$, $c\in C$ such that $a\subset b\subset c$. Then, by Axiom (III) of Sec. 1, there exists an element x in L such that

$$b\vee x=c,\qquad b\wedge x=a.$$

Hence, by (1) and (2),

$$A_b\vee A_x=A_c,\qquad A_b\wedge A_x=A_a.$$

Thus there exists an element $X=A_x$ in $L/\mathfrak{a}$ such that

$$B\vee X=C,\qquad B\wedge X=A.$$

(iv) $L/\mathfrak{a}$ is distributive in the generalized sense. To prove this, consider a subset $\mathfrak{S}$ of $L/\mathfrak{a}$ with power $<\aleph$. Then, as in (i), we have a subset S of L, the power of $S<\aleph$, and $\mathfrak{S}=(A_a;\ a\in S)$. And let $B=A_b$ be any element in $L/\mathfrak{a}$. Then, by Axiom (IV) in Sec. 1, and (1) and (2), we have

$$\begin{aligned}\sum(\mathfrak{S})\wedge B&=A_{\Sigma(S)}\wedge A_b=A_{\Sigma(S)\wedge b}=A_{\Sigma(a\wedge b;\ a\in S)}\\&=\sum(A_{a\wedge b};\ a\in S)=\sum(A_a\wedge B;\ a\in S).\end{aligned}$$

Similarly we have $$\Pi(\mathfrak{S})\vee B=\Pi(A_a\vee B;\ a\in S).$$

Theorem 3.2. *If $\mathfrak{b}$ is an $\aleph$-ideal in a generalized $\aleph$-Boolean algebra L, then the image $\mathfrak{B}$ of its elements under the homomorphism $L\to L/\mathfrak{a}$ constitutes an $\aleph$-ideal in $L/\mathfrak{a}$. And if $\mathfrak{C}$ is an $\aleph$-ideal in $L/\mathfrak{a}$, then the class $\mathfrak{c}$ of all elements of L with images in $\mathfrak{C}$ is an $\aleph$-ideal in L.*

Proof. (i) Let $\mathfrak{S}$ be any subset of $\mathfrak{B}$ with power $<\aleph$. Let Ω be the smallest ordinal corresponding to this power, and replace $\mathfrak{S}$ by the system $(A^\alpha;\ \alpha<\Omega)$. Then there exist a_α in $\mathfrak{b}$ such that $A_{a_\alpha}=A^\alpha$ for all $\alpha<\Omega$. By (1) in the proof of Theorem 3.1,

$$\sum(\mathfrak{S})=\sum(A_{a_\alpha};\ \alpha<\Omega)=A_{\Sigma(a_\alpha;\ \alpha<\Omega)}.$$

Since $\sum(a_\alpha;\ \alpha<\Omega)\in\mathfrak{b}$, $\sum(\mathfrak{S})\in\mathfrak{B}$. Next, let $A\in\mathfrak{B}$, $B\in L/\mathfrak{a}$. Then there exist $a\in\mathfrak{b}$, $b\in L$ such that $A=A_a$, $B=A_b$. By (2) in the proof of Theorem 3.1, $A\wedge B=A_{a\wedge b}$. But, since $a\wedge b\in\mathfrak{b}$, $A\wedge B\in\mathfrak{B}$. Consequently $\mathfrak{B}$ is an $\aleph$-ideal in $L/\mathfrak{a}$.

(ii) Let S be any subset of $\mathfrak{c}$ with power $<\aleph$; and, as in (i), replace S by the system $(a_\alpha;\ \alpha<\Omega)$. Then $A_{a_\alpha}\in\mathfrak{C}$ for all $\alpha<\Omega$. Hence

$\sum(A_{a_\alpha}; \alpha<\Omega)\in\mathfrak{C}$. Since $\sum(A_{a_\alpha}; \alpha<\Omega)=A_{\Sigma(a_\alpha;\, \alpha<\Omega)}$, we have $\sum(a_\alpha; \alpha<\Omega)\in\mathfrak{c}$. Next, let $a\in\mathfrak{c}$, $b\in L$. Then, since $A_a\in\mathfrak{C}$, $A_b\in L/\mathfrak{a}$, $A_a\wedge A_b\in\mathfrak{C}$. Hence, since $A_a\wedge A_b=A_{a\wedge b}$, we have $a\wedge b\in\mathfrak{c}$. Thus $\mathfrak{c}$ is an $\aleph$-ideal in L, as we wished to prove.

THEOREM 3.3, *Let $\mathfrak{a}, \mathfrak{b}$ be $\aleph$-ideals in a generalized $\aleph$-Boolean algebra L such that $\mathfrak{a}\subset\mathfrak{b}$. And let $\mathfrak{B}$ be the image of $\mathfrak{b}$ under the homomorphism $L\to L/\mathfrak{a}$. Then $L/\mathfrak{b}$ is isomorphic to $(L/\mathfrak{a})/\mathfrak{B}$ with respect to the join and the meet of subsets of power $<\aleph$.*

PROOF. We denote by A_a the set of all $x\in L$ with $x\sim a \pmod{\mathfrak{a}}$; and by $\bar{A}_a$ the set of all $x\in L$ with $x\sim a \pmod{\mathfrak{b}}$, and by $\bar{\bar{A}}_A$ the set of all $B\in L/\mathfrak{a}$ with $B\sim A \pmod{\mathfrak{B}}$. Then $L/\mathfrak{a}=(A_a;\ a\in L)$, $L/\mathfrak{b}=(\bar{A}_a;\ a\in L)$, and $(L/\mathfrak{a})/\mathfrak{B}=(\bar{\bar{A}}_A;\ A\in L/\mathfrak{a})$.

(i) If $A_u\in\mathfrak{B}$, then $u\in\mathfrak{b}$.

For, since $\mathfrak{B}$ is the image of $\mathfrak{b}$, there exists an element $u_0\in\mathfrak{b}$, such that $A_u=A_{u_0}$. Then $u\sim u_0 \pmod{\mathfrak{a}}$. Hence $u\vee w=u_0\vee w_0$ where $w, w_0\in\mathfrak{a}$. Since $\mathfrak{a}\subset\mathfrak{b}$, $u\subset u_0\vee w_0\in\mathfrak{b}$. Therefore $u\in\mathfrak{b}$.

(ii) If $a\sim b \pmod{\mathfrak{b}}$, then $A_a\sim A_b \pmod{\mathfrak{B}}$.

For, since $a\vee u=b\vee v$ where $u, v\in\mathfrak{b}$, we have $A_{a\vee u}=A_{b\vee v}$, that is, $A_a\vee A_u=A_b\vee A_v$(1) and $A_u, A_u\in\mathfrak{B}$. Hence $A_a\sim A_b \pmod{\mathfrak{B}}$.

(iii) If $A_a\sim A_b \pmod{\mathfrak{B}}$, then $a\sim b \pmod{\mathfrak{b}}$.

For there exist $A_u, A_v\in\mathfrak{B}$ such that $A_a\vee A_u=A_b\vee A_v$. That is, $A_{a\vee u}=A_{b\vee v}$(1) and $a\vee u\sim b\vee v \pmod{\mathfrak{a}}$. That is $a\vee u\vee u_1=b\vee v\vee v_1$ where $u_1, v_1\in\mathfrak{a}$. Since, by (i), $u, v\in\mathfrak{b}$, and $\mathfrak{a}\subset\mathfrak{b}$, we have $u\vee u_1\in\mathfrak{b}$, $v\vee v_1\in\mathfrak{b}$. Therefore $a\sim b \pmod{\mathfrak{b}}$.

(iv) By (ii) and (iii) there exists one-to-one correspondence between the elements in $L/\mathfrak{b}$ and $(L/\mathfrak{a})/\mathfrak{B}$ such that $\bar{A}_a\leftrightarrow\bar{\bar{A}}_{A_a}$. Consider a subset $(\bar{A}_a;\ a\in S)$ of $L/\mathfrak{b}$ with power $<\aleph$. Then $\sum(\bar{A}_a;\ a\in S)=\bar{A}_{\Sigma(a;\, a\in S)}$ corresponds to $\bar{\bar{A}}_{A_{\Sigma(a;\, a\in S)}}=\bar{\bar{A}}_{\Sigma(A_a;\, a\in S)}=\sum(\bar{\bar{A}}_{A_a};\ a\in S)$.(1) Similarly for $\Pi(\bar{A}_a;\ a\in S)$. Hence $L/\mathfrak{b}$ is isomorphic to $(L/\mathfrak{a})/\mathfrak{B}$ with respect to the join and the meet of subsets of power $<\aleph$.

THEOREM 3.4. *If $\mathfrak{J}$ and $\bar{\mathfrak{J}}$ are the classes of all $\aleph$-ideals in a generalized $\aleph$-Boolean algebra L and in $L/\mathfrak{a}$ respectively, then the homomorphism $L\to L/\mathfrak{a}$ induces a homomorphism $\mathfrak{J}\to\bar{\mathfrak{J}}$ with respect to the join of any subset and to the meet of subsets of power $<\aleph$. If $\mathfrak{b}$ and $\mathfrak{c}$ are $\aleph$-ideals in L with respective images $\mathfrak{B}$ and $\mathfrak{C}$ in $L/\mathfrak{a}$, then $\mathfrak{b}\vee\mathfrak{a}=\mathfrak{c}\vee\mathfrak{a}$ and $\mathfrak{B}=\mathfrak{C}$ are equivalent.*

PROOF. (i) Let $\mathfrak{S}=(\mathfrak{a}_\alpha;\ \alpha\in I)$ be any subset of $\mathfrak{J}$, and let $\bar{\mathfrak{S}}=(\mathfrak{A}_\alpha;\ \alpha\in I)$(2) be the image of $\mathfrak{S}$ in $\bar{\mathfrak{J}}$. By Theorem 2.1, $\sum(\mathfrak{S})$ consists of those elements

(1) By (1) in the proof of Theorem 3.1.

(2) In this expression some $\mathfrak{A}_\alpha$ coincide, in spite of different indices, $\mathfrak{A}_\alpha$ being the image of $\mathfrak{a}_\alpha$.

such that $c=\sum(a_\alpha;\ a_\alpha \in \mathfrak{a}_\alpha,\ \alpha \in I_1)$, where I_1 is a subclass of I of power $<\aleph$. Hence the image of $\sum(\mathfrak{S})$ consists of those elements such that $A_c=\sum(A_\alpha;\ A_\alpha \in \mathfrak{A}_\alpha,\ \alpha \in I_1)$. Hence image of $\sum(\mathfrak{S}) \subset \sum(\bar{\mathfrak{S}})$. Conversely, $\sum(\bar{\mathfrak{S}})$ consists of those elements such that $C=\sum(A_\alpha;\ A_\alpha \in \mathfrak{A}_\alpha,\ \alpha \in I_1)$, where I_1 is a subclass of I of power $<\aleph$. There exist $a_\alpha \in \mathfrak{a}_\alpha$ such that $A_\alpha = A_{a_\alpha}$ for all $\alpha \in I_1$. Put $c=\sum(a_\alpha;\ \alpha \in I_1)$; then $c \in \sum(\mathfrak{S})$ and $C=A_c$. That is, $C \in$ image of $\sum(\mathfrak{S})$. Hence $\sum(\bar{\mathfrak{S}}) \subset$ image of $\sum(\mathfrak{S})$. Consequently $\sum(\bar{\mathfrak{S}})=$ image of $\sum(\mathfrak{S})$.

(ii) In a similar manner we can prove that if $\mathfrak{S}=(\mathfrak{a}_\alpha;\ \alpha \in I)$ be any subset of $\mathfrak{J}$ of power $<\aleph$, and $\bar{\mathfrak{S}}=(\mathfrak{A}_\alpha;\ \alpha \in I)$ be the image of $\mathfrak{S}$, then $\Pi(\bar{\mathfrak{S}})=$ image of $\Pi(\mathfrak{S})$.

(iii) Since in the homomorphism $L \to L/\mathfrak{a}$, the image of $\mathfrak{a}$ is the ideal $\mathfrak{O}$ consisting of zero element in $L/\mathfrak{a}$ alone. Hence $\mathfrak{b} \vee \mathfrak{a}$ and $\mathfrak{c} \vee \mathfrak{a}$ have respective imarges $\mathfrak{B} \vee \mathfrak{O} = \mathfrak{B}$ and $\mathfrak{C} \vee \mathfrak{O} = \mathfrak{C}$. It follows that $\mathfrak{b} \vee \mathfrak{a} = \mathfrak{c} \vee \mathfrak{a}$ implies $\mathfrak{B}=\mathfrak{C}$. Furthermore, if b is any element in L with image B in $\mathfrak{B}$, then there exists an element b_0 in $\mathfrak{b}$ which also has B as its image. The fact that b and b_0 have the same images implies that $b \sim b_0 \pmod{\mathfrak{a}}$, that is, $b \vee u = b_0 \vee v$ where $u, v \in \mathfrak{a}$. Hence $b \in \mathfrak{b} \vee \mathfrak{a}$. Thus, $\mathfrak{b} \vee \mathfrak{a}$ is the class of all elements in L with images in $\mathfrak{B}$. Similarly $\mathfrak{c} \vee \mathfrak{a}$ is the class of all elements im L with images in $\mathfrak{C}$. In consequence, $\mathfrak{B}=\mathfrak{C}$ implies $\mathfrak{b} \vee \mathfrak{a} = \mathfrak{c} \vee \mathfrak{a}$.

THEOREM 3.5. *Let $\mathfrak{Y}$ be the class of all $\aleph$-ideals in a generalized $\aleph$-Boolean algebra L which contains $\mathfrak{a}$, and $\mathfrak{J}$ be the class of all $\aleph$-ideals in $L/\mathfrak{a}$. The homomorphism $L \to L/\mathfrak{a}$ induces an isomorphism $\mathfrak{Y} \leftrightarrow \mathfrak{J}$ with respect to the join of any subset and to the meet of subsets of power $<\aleph$.*

PROOF. By Theorem 3.2, to any element $\mathfrak{b}$ in $\mathfrak{Y}$ there corresponds only one element $\mathfrak{B}$ in $\mathfrak{J}$. Next, let $\mathfrak{b}$ and $\mathfrak{c}$ be two elements in $\mathfrak{Y}$ which have the same image $\mathfrak{B}$ in $\mathfrak{J}$; then, by Theorem 3.4, $\mathfrak{b} \vee \mathfrak{a} = \mathfrak{c} \vee \mathfrak{a}$. Since $\mathfrak{b} \supset \mathfrak{a}$, $\mathfrak{c} \supset \mathfrak{a}$, we have $\mathfrak{b}=\mathfrak{c}$. That is, there exists one-to-one correspondence between $\mathfrak{Y}$ and $\mathfrak{J}$. And Theorem 3.4 shows that this correspondence is an isomorphism with respect to the join of any subset and to the meet of subsets of power $<\aleph$

Generalized $\aleph$-Boolean Algebra with $\aleph$-Chain Condition.

4. DEFINITION 4.1 (4.1*). Let L be a generalized $\aleph$-Boolean algebra and Ω be an ordinal number. A subset $S=(a_\alpha;\ \alpha<\Omega)$ of L is called an *ascending* (*descending*) *system* when $a_\alpha < a_\beta$ ($a_\alpha > a_\beta$) for all $\alpha<\beta<\Omega$.

DEFINITION 4.2 (4.2*). In L, if for every ascending (descending) system S, the power of S is $<\aleph$, then we say that L satisfies the $\aleph$-*ascending* (*descending*) *chain condition.*

DEFINITION 4.3. If L satisfies both $\aleph$-ascending and -descending chain conditions, then we say that L satisfies the $\aleph$-*chain condition.*

THEOREM 4.1. *If a generalized $\aleph$-Boolean algebra L with zero element 0 satisfies the $\aleph$-ascending chain condition, then L satisfies, also, the $\aleph$-descending chain condition.*

PROOF. Let $S=(a_\alpha;\ \alpha<\Omega)$ be any descending system. Let $b_\alpha\ (\alpha<\Omega)$ be such that

$$a_\alpha \vee b_\alpha = a_0, \qquad a_\alpha \wedge b_\alpha = 0.$$

Then $S'=(b_\alpha;\ \alpha<\Omega)$ is an ascending system.(1) Since the power of S' is $<\aleph$, the power of S is $<\aleph$.

THEOREM 4.1*. *If a generalized $\aleph$-Boolean algebra L with unit element 1 satisfies the $\aleph$-descending chain condition, then L satisfies, also, the $\aleph$-ascending chain condition.*

PROOF. Let $S=(a_\alpha;\ \alpha<\Omega)$ be any ascending system. Let $b_\alpha\ (\alpha<\Omega)$ be such that

$$a_\alpha \vee b_\alpha = 1, \qquad a_\alpha \wedge b_\alpha = a_0.$$

Then $S'=(b_\alpha;\ \alpha<\Omega)$ is a descending system, and the theorem holds good.

THEOREM 4.2. *If a generalized $\aleph$-Boolean algebra L satisfies the $\aleph$-chain condition, then L is a continuous Boolean algebra, and every $\aleph$-ideal and dual $\aleph$-ideal in L is principal.*

PROOF. In order to prove this theorem, we must show that $\sum(S)$ and $\Pi(S)$ exist for any subset S of L. If there exists a subset S such that $\sum(S)$ does not exist, then there exists a set $\bar{S}$ of minimum power having this property. Let Ω be the smallest ordinal corresponding to this power, and replace $\bar{S}$ by the system $(a_\alpha;\ \alpha<\Omega)$. By the definition of $\bar{S}$, $b_\alpha=\sum(a_\gamma;\ \gamma<\alpha)$ exists for each $\alpha<\Omega$. But, by assumption, $\sum(b_\alpha;\ \alpha<\Omega)$ does not exist. Now, for each $\alpha<\Omega$ there exists a smallest $\alpha'<\Omega$ such that $b_{\alpha'}=b_\alpha$; let us denote this α' by α^*. Let K be the set of all $\alpha^*<\Omega$. Then the sets $(b_\alpha;\ \alpha<\Omega)$, $(b_{\alpha^*};\ \alpha^*\in K)$ are equal, and $\sum(b_{\alpha^*};\ \alpha^*\in K)$ does not exist. Therefore the power of K is $\geqq\aleph$. But $(b_{\alpha^*};\ \alpha^*\in K)$ is an ascending system, and, by assumption, the power of K is $<\aleph$, which is absurd.

Similarly we can prove the existence of $\Pi(S)$ for every subset S of L.

Next, let $\mathfrak{b}$ be any $\aleph$-ideal in L. Let Ω_0 be the smallest ordinal corresponding to the power of $\mathfrak{b}$, and denote all the elements of $\mathfrak{b}$ by $(a_\alpha;\ \alpha<\Omega_0)$. Since L is continuous, $b=\sum(a_\alpha;\ \alpha<\Omega_0)$ and $b_\alpha=\sum(a_\gamma;\ \gamma<\alpha)\ (\alpha<\Omega_0)$ exist. As before, for each $\alpha<\Omega_0$ there exists a smallest $\alpha'<\Omega_0$ such that $b_{\alpha'}=b_\alpha$; let us denote this α' by α^*. Let K_0 be the set of all $\alpha^*<\Omega$. Then

(1) For, when $\alpha<\beta<\Omega$, since $b_\alpha\wedge a_\beta\subset b_\alpha\wedge a_\alpha=0$,

$$b_\alpha=b_\alpha\wedge a_0=b_\alpha\wedge(a_\beta\vee b_\beta)=(b_\alpha\wedge a_\beta)\vee(b_\alpha\wedge b_\beta)=b_\alpha\wedge b_\beta,$$

that is, $b_\alpha\subset b_\beta$. If $b_\alpha=b_\beta$, then

$$a_\alpha=a_0\wedge a_\alpha=(a_\beta\vee b_\alpha)\wedge a_\alpha=(a_\beta\wedge a_\alpha)\vee(b_\alpha\wedge a_\alpha)=a_\beta,$$

which is absurd. Consequently $b_\alpha<b_\beta$.

$$b=\sum(a_\alpha;\ \alpha<\Omega_0)=\sum(b_\alpha;\ \alpha<\Omega_0)=\sum(b_{\alpha^*};\ \alpha\in K_0). \qquad (1)$$

Since $(b_{\alpha^*};\ \alpha^*\in K_0)$ is an ascending system, the power of K_0 is $<\aleph$. Hence, $\mathfrak{b}$ being an $\aleph$-ideal, by (1) $b\in\mathfrak{b}$. Consequently $\mathfrak{b}=\mathfrak{a}(b)$, as we wished to prove.

Similarly we can prove that every dual $\aleph$-ideal is principal.

DEFINITION 4.4. If a generalized $\aleph$-Boolean algebra L with zero element satisfies the following condition:

(T_1) For every subset T of L such that

(α_1) $a\in T$ implies $a\neq 0$,

(β_1) $a, b\in T$, $a\neq b$ implies $a\wedge b=0$,

the power of T is $<\aleph$;

then we say that L satisfies the $\aleph$-*independence condition.*(1)

THEOREM 4.3. *A generalized $\aleph$-Boolean algebra L satisfies the $\aleph$-chain condition, if, and only if, L has zero element and satisfies the $\aleph$-independence condition.*

PROOF. (i) First, assume that L satisfies the $\aleph$-chain condition. Then, by Theorem 4.2, L is a continuous Boolean algebra. Let T be a subset of L which satisfies (α_1) and (β_1) of (T_1). And let Ω be the smallest ordinal corresponding to the power of T, and denote T by $(a_\alpha;\ \alpha<\Omega)$. Then, for every $\alpha<\Omega$, $b_\alpha=\sum(a_\gamma;\ \gamma<\alpha)$ exists. Now, by (β_1),

$$a_\alpha\wedge b_\alpha=a_\alpha\wedge\sum(a_\gamma;\ \gamma<\alpha)=\sum(a_\alpha\wedge a_\gamma;\ \gamma<\alpha)=0.$$

Hence $a_\alpha\not\subset b_\alpha$ and $a_\alpha\vee b_\alpha>b_\alpha$. If $\alpha<\beta<\Omega$, then

$$b_\beta\supset a_\alpha\vee b_\alpha>b_\alpha.$$

Therefore $(b_\alpha;\ \alpha<\Omega)$ is an ascending system. Since L satisfies the $\aleph$-chain condition, the power of $(b_\alpha;\ \alpha<\Omega)$ is $<\aleph$. Hence the power of $T<\aleph$.

(ii) Conversely, assume that L has zero element 0 and satisfies the $\aleph$-independence condition. If L does not satisfy the $\aleph$-chain condition, then there exists an ascending system $S=(b_\alpha;\ \alpha<\Omega)$ with power $\aleph$. Let a_α be such that

$$b_\alpha\vee a_\alpha=b_{\alpha+1},\qquad b_\alpha\wedge a_\alpha=0.$$

Then the power of $T=(a_\alpha;\ \alpha<\Omega)$ is $\aleph$ and

(α_1) $a_\alpha\neq 0$, since $b_\alpha<b_{\alpha+1}$;

(β_1) $a_\alpha\wedge a_\beta=0$ when $\alpha<\beta<\Omega$, since $a_\alpha\wedge a_\beta\subset b_\beta\wedge a_\beta=0.$

But, by condition (T_1), the power of $T<\aleph$, which is absurd.

DEFINITION 4.5. Let L be a continuous Boolean algebra. An element

(1) J. v. Neumann introduced this condition, and proved that, if an $\aleph$-Boolean algebra L satisfies the $\aleph$-independence condition, then L is a continuous Boolean algebra. (Cf. J. v. Neumann [2], 8). From Theorem 4.3, we may say that Theorem 4.2 is a generalization of J. v. Neumann's theorem.

$a \neq 0$ of L is *atomistic* if $b < a$ implies $b=0$. The set L_{at} is the set of all atomistic elements of L; L is *atomistic* if $\sum(L_{at})=1$.(1)

THEOREM 4.4. *If a generalized $\aleph_0$-Boolean algebra L satisfies the $\aleph_0$-chain condition, then L is an atomistic Boolean algebra which is lattice-isomorphic to the class of all subsets of a finite set.*

PROOF. By Theorem 4.2, L being continuous, L has zero element 0 and unit element 1. Now, we can prove that $\sum(L_{at})=1$. If $\sum(L_{at}) \neq 1$, then there exists an element $b \neq 0$ in L such that $\sum(L_{at}) \wedge b=0$. Since L satisfies the $\aleph_0$-chain condition, there exists an atomistic element a such that $a \subset b$. But $a \subset \sum(L_{at})$, contradicting $\sum(L_{at}) \wedge b=0$. Consequently L is an atomistic Boolean algebra.

If $a, b \in L_{at}$ and $a \neq b$, then $a \wedge b=0$. Since, by Theorem 4.3, L satisfies the $\aleph_0$-independence condition, L_{at} is a finite set. And L is lattice-isomorphic to the class of all subsets of L_{at}.

DEFINITION 4.6. Let $\mathfrak{a}$ be an $\aleph$-ideal in a generalized $\aleph$-Boolean algebra L; if $L/\mathfrak{a}$(2) satisfies the $\aleph$-chain condition, then we say that L satisfies the *$\aleph$-chain condition relative to* $\mathfrak{a}$, and $\mathfrak{a}$ is called a *basic $\aleph$-ideal of the $\aleph$-chain condition.*(3)

THEOREM 4.5. *If a generalized $\aleph$-Boolean algebra L satisfies the $\aleph$-chain condition relative to $\mathfrak{a}$, then $L/\mathfrak{a}$ is a continuous Boolean algebra, and every $\aleph$-ideal and every dual $\aleph$-ideal in $L/\mathfrak{a}$ is principal.*

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(1) Cf. J. v. Neumann [2], 19.

(2) By Theorem 3.1, $L/\mathfrak{a}$ is a generalized $\aleph$-Boolean algebra with zero element.

(3) Dually, we can define as follows:

DEFINITION 4.6.* Let $\mathfrak{a}^*$ be a dual $\aleph$-ideal in a generalized $\aleph$-Boolean algebra L; if $L/\mathfrak{a}^*$ satisfies the $\aleph$-chain condition, then we say that L satisfies the *$\aleph$-chain condition relative to* $\mathfrak{a}^*$, and $\mathfrak{a}^*$ is called a *basic dual $\aleph$-ideal of the $\aleph$-chain condition.*

And we have the following theorem:

THEOREM 4.6.* *A generalized $\aleph$-Boolean algebra L satisfies the $\aleph$-chain condition relative to $\mathfrak{a}^*$ if, and only if, L satisfies the following condition:*

(T_2^*) *For every subset T of L such that*

(α_2) *$a \in T$ implies $a \notin \mathfrak{a}^*$,*

(β_2) *$a, b \in T$, $a \neq b$ implies $a \vee b \in \mathfrak{a}^*$,*

the power of T is $< \aleph$.

Corresponding to Sec. 5, in an $\aleph$-Boolean algebra L ($\aleph=\aleph_0$ or $\aleph_1$), we have lattice functions which are multiplicative, instead of additive, as follows:

(i) Case $\aleph=\aleph_0$: $\phi(a)=\dfrac{1}{n}$, $n=1, 2, \ldots .$,

Case $\aleph=\aleph_1$: $0 < \phi(a) \leqq 1$;

(ii) $\phi(a)=\Pi\phi(a_i)$ when $a=\underset{i}{\Pi} \boxtimes a_i$.

($\underset{i}{\Pi} \boxtimes a_i$ means $\Pi(a_i; i=1, 2, \ldots .)$ when $a_i \vee a_j=1$ for all $i \neq j$).

THEOREM 5.1.* *Set $\mathfrak{a}_\phi^*$ of all elements a such that $\phi(a)=1$ is a dual $\aleph$-ideal in L.*

THEOREM 5.2.* *L satisfies the $\aleph$-chain condition relative to $\mathfrak{a}_\phi^*$.*

In this way we have dual theorems for those in Sec. 4 and 5; but I omit them.

PROOF. This theorem is evident from Definition 4.6 and Theorem 4.2.

DEFINITION 4.7. Let $\mathfrak{a}$ be an $\aleph$-ideal in a generalized $\aleph$-Boolean algebra L. If L satisfies the following condition:

($\mathbf{T}_2$) For every subset T of L such that

(α_2) $a \in T$ implies $a \notin \mathfrak{a}$,

(β_2) $a, b \in T$, $a \neq b$ implies $a \wedge b \in \mathfrak{a}$,

the power of T is $< \aleph$;

then we say that L satisfies the *$\aleph$-independence condition relative to* $\mathfrak{a}$.

THEOREM 4.6. *In a generalized $\aleph$-Boolean algebra L, the $\aleph$-chain condition relative to $\mathfrak{a}$ is equivalent to the $\aleph$-independence condition relative to $\mathfrak{a}$.*

PROOF. If a subset T of L satisfies (α_2) and (β_2) of ($\mathbf{T}_2$), then $\mathfrak{T} = (A_a;\ a \in T)$ satisfies (α_1) and (β_1) of ($\mathbf{T}_1$). Since $a \neq b$ implies $A_a \neq A_b$[(1)] T and $\mathfrak{T}$ have the same power. Next, let $\mathfrak{T}$ be a subset of $L/\mathfrak{a}$ which satisfies (α_1) and (β_1) of ($\mathbf{T}_1$). If we take out one element a such that $A_a = A$ for each $A \in \mathfrak{T}$, then set T of such an element a satisfies (α_2) and (β_2) of ($\mathbf{T}_2$), and T and $\mathfrak{T}$ have the same power. Hence L satisfies the $\aleph$-independence condition relative to $\mathfrak{a}$ if, and only if, $L/\mathfrak{a}$ satisfies the $\aleph$-independence condition. Consequently the present theorem follows from Theorem 4.3 and Definition 4.6.

THEOREM 4.7. *When a generalized $\aleph$-Boolean algebra L has zero element* 0, *the $\aleph$-independence condition relative to $\mathfrak{a}$ is equivalent to the following condition*:

($\mathbf{T}_3$) *For every set $T \subset L$ such that*

(α_3) *$a \in T$ implies $a \notin \mathfrak{a}$,*

(β_3) *$a, b \in T$, $a \neq b$ implies $a \wedge b = 0$,*

the power of T is $< \aleph$.[(2)]

PROOF. It is evident that when L satisfies ($\mathbf{T}_2$), then L satisfies ($\mathbf{T}_3$). Hence we shall show that when L satisfies ($\mathbf{T}_3$), then L satisfies ($\mathbf{T}_2$). Let T be a subset of L which satisfies (α_2) and (β_2) of ($\mathbf{T}_2$). Now, if the power of $T \geqq \aleph$, then there is a subset T' of T with power $\aleph$. Let Ω be the smallest ordinal of power $\aleph$, and write $T' = (a_\alpha;\ \alpha < \Omega)$. Then for every $\alpha < \Omega$, $b_\alpha = \sum(a_\gamma;\ \gamma < \alpha)$ exists. Now,

$$a_\alpha \wedge b_\alpha = a_\alpha \wedge \sum(a_\gamma;\ \gamma < \alpha) = \sum(a_\alpha \wedge a_\gamma;\ \gamma < \alpha).$$

By the property of T, $a_\alpha \wedge a_\gamma \in \mathfrak{a}$. Hence

(1) For, if $A_a = A_b$, then $a \vee u = b \vee v$ $(u, v \in \mathfrak{a})$. And $a = a \wedge (b \vee v) = (a \wedge b) \vee (a \wedge v) \in \mathfrak{a}$, which is absurd.

(2) J. v. Neumann introduced this condition ($\mathbf{T}_3$), and proved that, if an $\aleph$-Boolean algebra L satisfies ($\mathbf{T}_3$), then $L/\mathfrak{a}$ is a continuous Boolean algebra. (Cf. J. v. Neumann [2], 11.) By Theorem 4.6, we may say that Theorem 4.5 is a generalization of J. v. Neumann's theorem.

$$a_\alpha \wedge b_\alpha \in \mathfrak{a} \quad \text{for all} \quad \alpha < \Omega .$$

Since $a_\alpha \notin \mathfrak{a}$, $a_\alpha \wedge b_\alpha \neq a_\alpha$. Now let $\bar{a}_\alpha$ be such that

$$(a_\alpha \wedge b_\alpha) \vee \bar{a}_\alpha = a_\alpha , \qquad (a_\alpha \wedge b_\alpha) \wedge \bar{a}_\alpha = 0 . \tag{1}$$

Since $a_\alpha \wedge b_\alpha \neq a_\alpha$, $\bar{a}_\alpha \neq 0$. Therefore the power of $(\bar{a}_\alpha ;\ \alpha < \Omega)$ is $\aleph$. If $\alpha < \beta < \Omega$, then, since $\bar{a}_\alpha \subset a_\alpha \subset b_\beta$, $\bar{a}_\beta \subset a_\beta$, we have, by (1),

$$\bar{a}_\alpha \wedge \bar{a}_\beta \subset b_\beta \wedge \bar{a}_\beta \wedge a_\beta = 0 .$$

Next we shall show that $\bar{a}_\alpha \notin \mathfrak{a}$ for all $\alpha < \Omega$. For if $\bar{a}_\alpha \in \mathfrak{a}$, then, since $a_\alpha \wedge b_\alpha \in \mathfrak{a}$, we have, by (1), $a_\alpha \in \mathfrak{a}$, which contradicts the assumption of T. Consequently $(\bar{a}_\alpha ;\ \alpha < \Omega)$ satisfies conditions (α_3) and (β_3) of (T_3). Hence, by assumption, the power of $(\bar{a}_\alpha ;\ \alpha < \Omega)$ is $< \aleph$, which is absurd; and the proof is complete.

THEOREM 4.8. *Let $\mathfrak{a}$, $\mathfrak{b}$ be $\aleph$-ideals in a generalized $\aleph$-Boolean algebra L, such that $\mathfrak{a} \subset \mathfrak{b}$. If L satisfies the $\aleph$-chain condition relative to $\mathfrak{a}$, then L satisfies, also, the $\aleph$-chain condition relative to $\mathfrak{b}$.*[1]

PROOF. If L satisfies the $\aleph$-chain condition relative to $\mathfrak{a}$, then, by Definition 4.6, $L/\mathfrak{a}$ satisfies the $\aleph$-chain condition. Hence, by Theorem 4.3, $L/\mathfrak{a}$ has zero element and satisfies the $\aleph$-independence condition, that is, (T_1). Hence $L/\mathfrak{a}$ satisfies (T_3) with respect to the $\aleph$-ideal $\mathfrak{B}$, $\mathfrak{B}$ being the image of $\mathfrak{b}$ under the homomorphism $L \to L/\mathfrak{a}$. Therefore, by Theorem 4.7, $L/\mathfrak{a}$ satisfies the $\aleph$-independence condition relative to $\mathfrak{B}$. Hence, by Theorem 4.6 and Definition 4.6, $(L/\mathfrak{a})/\mathfrak{B}$ satisfies the $\aleph$-chain condition. Therefore by Theorem 3.3 $L/\mathfrak{b}$ satisfies $\aleph$-chain condition. Conseqnently, by Definition 4.6, L satisfies the $\aleph$-chain condition relative to $\mathfrak{b}$.

THEOREM 4.9. *If a generalized $\aleph$-Boolean algebra L satisfies the $\aleph$-chain condition relative to an $\aleph$-ideal $\mathfrak{a}$, then class $\mathfrak{Y}$ of all $\aleph$-ideals which contain $\mathfrak{a}$ is an $\aleph$-Boolean algebra.*

PROOF. From Theorem 3.5, $\mathfrak{Y}$ is isomorphic to class $\mathfrak{J}$ of all $\aleph$-ideals in $L/\mathfrak{a}$ with respect to the join of any subset and to the meet of subsets of power $< \aleph$. But, by Theorem 4.5, every $\aleph$-ideal in $L/\mathfrak{a}$ is principal; hence, by Theorem 2.2, $\mathfrak{J}$ is a generalized $\aleph$-Boolean algebra. Obviously, $\mathfrak{J}$ has zero and unit elements. Consequently $\mathfrak{Y}$ is an $\aleph$-Boolean algebra.

If we extract the complementedness of the $\aleph$-Boolean algebra $\mathfrak{Y}$, then, from Theorem 4.9, we have the following theorem, $\mathfrak{a}$ being zero element of $\mathfrak{Y}$.

THEOREM 4.10. *If a generalized $\aleph$-Boolean algebra L satisfies the $\aleph$-chain condition relative to an $\aleph$-ideal $\mathfrak{a}$, then, for any $\aleph$-ideals $\mathfrak{b}$, $\mathfrak{c}$ such that $\mathfrak{a} \subset \mathfrak{b} \subset \mathfrak{c}$, there exists an $\aleph$-ideal $\mathfrak{d}$ such that*

$$\mathfrak{b} \vee \mathfrak{d} = \mathfrak{c} , \qquad \mathfrak{b} \wedge \mathfrak{d} = \mathfrak{a} .$$

---

(1) If L has zero element, then this theorem is evident from Theorems 4.6 and 4.7.

THEOREM 4.11. *In a generalized $\aleph$-Boolean algebra L, class $\mathfrak{P}_{\aleph}$ of all basic $\aleph$-ideals of the $\aleph$-chain condition is a generalized $\aleph$-Boolean algebra with unit element $\mathfrak{e}$. And $\mathfrak{a} \in \mathfrak{P}_{\aleph}$, $\mathfrak{b} \in \mathfrak{J}$, $\mathfrak{b} \supset \mathfrak{a}$ implies $\mathfrak{b} \in \mathfrak{P}_{\aleph}$.*

PROOF. If $\mathfrak{a} \in \mathfrak{P}_{\aleph}$ and $\mathfrak{b} \supset \mathfrak{a}$, then, by Theorem 4.8, $\mathfrak{p} \in \mathfrak{P}_{\aleph}$. Hence $\mathfrak{P}_{\aleph}$ being the subset of $\mathfrak{J}$, by Theorem 2.1, $\mathfrak{P}_{\aleph}$ is closed with respect to the join of any subset. Next I shall show that $\mathfrak{P}_{\aleph}$ is closed with respect to the meet of subsets of power $<\aleph$. Let $\mathfrak{S}$ be any subset of $\mathfrak{P}_{\aleph}$ of power $<\aleph$. Let Ω be the first ordinal corresponding to the power of $\mathfrak{S}$, and write $\mathfrak{S}=(\mathfrak{a}_{\alpha}; \alpha<\Omega)$. Since $\mathfrak{S}$ is the subset of $\mathfrak{J}$, by Theorem 2.1 $\mathfrak{a}=\Pi(\mathfrak{a}_{\alpha}; \alpha<\Omega)$ exists. Let T be any subset of L which satisfies (α_2) and (β_2) of (T_2) with respect to $\mathfrak{a}$. Denote by T_{α} the set of all elements of T which is not contained in $\mathfrak{a}_{\alpha}$. Since T_{α} satisfies (α_2) and (β_2) of (T_2) with respect to $\mathfrak{a}_{\alpha}$, the power of T_{α} is $<\aleph$. Since T is the set-theoretical sum of T_{α} $(\alpha<\Omega)$, the power of T is $<\aleph\cdot\aleph=\aleph$. Consequently L satisfies the $\aleph$-independence condition relative to $\mathfrak{a}$, and $\mathfrak{a}$ is a basic $\aleph$-ideal of the $\aleph$-chain condition. That is, $\mathfrak{a} \in \mathfrak{P}_{\aleph}$.

Now the complementedness and the distributivity of $\mathfrak{P}_{\aleph}$ follow from Theorems 4.10 and 2.1 respectively. The proof is thus completed.

When L has zero element 0, and the $\aleph$-ideal $\mathfrak{o}$ belongs to $\mathfrak{P}_{\aleph}$, it is a very trivial case; for in this case $\mathfrak{P}_{\aleph}=\mathfrak{J}$, and, by Theorem 4.2, all $\aleph$-ideals in $\mathfrak{P}_{\aleph}$ are principal; hence, by Theorem 2.2, $\mathfrak{P}_{\aleph}$ is is isomorphic to L.

By Theorem 4.11, $\mathfrak{P}_{\aleph}$ is a generalized $\aleph$-Boolean algebra with elements $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots.$ Hence, in $\mathfrak{P}_{\aleph}$ we can define dual $\aleph$-ideals, which we denote by $\mathfrak{A}^*, \mathfrak{B}^*, \mathfrak{C}^*, \ldots.$ And let $\Phi^*_{\aleph}$ be the class of all dual $\aleph$-ideals in $\mathfrak{P}_{\aleph}$. Then we have the following theorems.

THEOREM 4.12. *Every dual $\aleph$-ideal $\mathfrak{B}^*$ in $\mathfrak{P}_{\aleph}$ contained in a principal dual $\aleph$-ideal $\mathfrak{A}^*(\mathfrak{a})$ in $\mathfrak{P}_{\aleph}$, is also principal.*

PROOF. Since $\mathfrak{a} \in \mathfrak{P}_{\aleph}$, by Theorem 4.11 $\mathfrak{A}^*(\mathfrak{a})$ is a class of all elements $\mathfrak{b}$ of $\mathfrak{J}$ such that $\mathfrak{b} \supset \mathfrak{a}$. Hence, by Theorem 3.5, $\mathfrak{A}^*(\mathfrak{a})$ is isomorphic to class $\mathfrak{J}$ of all $\aleph$-ideals in $L/\mathfrak{a}$ with respect to the join of any subset and to the meet of subsets of power $<\aleph$. Since $\mathfrak{a} \in \mathfrak{P}_{\aleph}$, L satisfies the $\aleph$-chain condition relative to $\mathfrak{a}$, therefore, by Theorem 4.5, $\mathfrak{J}$ is isomorphic to $L/\mathfrak{a}$ and every dual $\aleph$-ideal in $L/\mathfrak{a}$ is principal. Consequently, $\mathfrak{B}^*$ being a dual $\aleph$-ideal in $\mathfrak{A}^*(\mathfrak{a})$, $\mathfrak{B}^*$ is principal, as we wished to prove.

THEOREM 4.13. *Class $\Phi^*_{\aleph}$ of all dual $\aleph$-ideals in $\mathfrak{P}_{\aleph}$ is a continuous Boolean algebra.*

PROOF. By Theorem 4.11, $\mathfrak{P}_{\aleph}$ is a generalized $\aleph$-Boolean algebra with unit element $\mathfrak{e}$. Hence, applying Theorem 2.4* to $\mathfrak{P}_{\aleph}$ instead of L, by Theorem 4.12, the assertion of the present theorem holds good.

Generalized $\aleph$-Boolean Algebra of Measure Functions.

5. In what follows, let L be an $\aleph$-Boolean algebra, where $\aleph=\aleph_0$ or $\aleph_1$.

DEFINITION 5.1. If, to any element a of L, there corresponds a real (finite) number $\phi(a)$ such that

(i) Case $\aleph=\aleph_0$: $\phi(a)=n$, $n=0, 1, 2, \dots$,
Case $\aleph=\aleph_1$: $0 \leqq \phi(a)$;

(ii) $\phi(a)=\sum \phi(a_i)$ when $a=\sum_i \oplus a_i$ [1];

then we say that $\phi(a)$ is a *measure function* defined in L.

Denote by $\mathfrak{F}_\aleph$ the class of all measure functions defined in L, where $\aleph=\aleph_0$ or $\aleph_1$.

It is to be noted that (ii) implies $\phi(a)+\phi(b)=\phi(a\vee b)+\phi(a\wedge b)$. For let c be such that $(a\wedge b)\vee c=a$, $(a\wedge b)\wedge c=0$. Then $c\vee b=c\vee(a\wedge b)\vee b=a\vee b$, $c\wedge b=(a\wedge b)\wedge c=0$. Hence $\phi(a)=\phi(a\wedge b)+\phi(c)$ and $\phi(a\vee b)=\phi(c)+\phi(b)$. Consequently
$$\phi(a)+\phi(b)=\phi(a\vee b)+\phi(a\wedge b).$$

THEOREM 5.1. *Set $\mathfrak{a}_\phi$ of all elements a such that $\phi(a)=0$ is an $\aleph$-ideal in L.*[2]

PROOF. If $b\subset a$ and $\phi(a)=0$, then, since
$$\phi(a)=\phi(b)+\phi(c),$$
where c is an element such that $b\vee c=a$, $b\wedge c=0$, we have $\phi(b)=0$. Let $(a_i; i=1, 2, \dots)$ be a subset of $\mathfrak{a}_\phi$ of power $<\aleph$; then, since $\phi(\sum a_i)\leqq \sum\phi(a_i)$, $\phi(\sum a_i)=0$, that is, $\sum a_i \in \mathfrak{a}_\phi$. Consequently $\mathfrak{a}_\phi$ is an $\aleph$-ideal.

THEOREM 5.2. *L satisfies the $\aleph$-chain condition relative to $\mathfrak{a}_\phi$.*

PROOF. From Theorems 4.6 and 4.7, to prove the present theorem, it is sufficient to show that L satisfies the following condition:

(T_3) For every set $T\subset L$ such that
(α_3) $a\in T$ implies $\phi(a)>0$,
(β_3) $a, b\in T$, $a\neq b$ implies $a\wedge b=0$,
the power of T is $<\aleph$.

When $\aleph=\aleph_0$, then $\phi(a)\geqq 1$ for any $a\in T$. Since $\phi(1)$ is finite, by the additivity of $\phi(a)$, the power of T must be finite.

When $\aleph=\aleph_1$, by the complete additivity of $\phi(a)$, power of set S of those elements $a\in T$, such that $\frac{1}{2^n}\geqq\phi(a)>\frac{1}{2^{n+1}}$ is finite. Hence power of T is $\aleph_0$.

(1) We write $\sum_i \oplus a_i$ in place of $\sum(a_i;\ i=1, 2, \dots)$, provided $a_i\wedge a_j=0$ for all $i\neq j$.

(2) It has already been pointed out by A. Tarski that an $\aleph_0$-ideal $\mathfrak{a}$ is a prime ideal when, and only when, $\mathfrak{a}=\mathfrak{a}_\phi$, where $\phi(a)$ takes only two values 0 and 1. (Cf. A. Tarski, Fund. Math. **15** (1930), 42.)

If we denote by $\mathfrak{Q}_{\aleph}$ the set of all $\aleph$-ideals $\mathfrak{a}_{\phi}$ $(\phi \in \mathfrak{F}_{\aleph})$, then, by Theorem 5.2, $\mathfrak{Q}_{\aleph}$ is a subset of $\mathfrak{P}_{\aleph}$ $(\aleph = \aleph_0$ or $\aleph_1)$.

THEOREM 5.3. $\mathfrak{Q}_{\aleph_0} = \mathfrak{P}_{\aleph_0}$.

PROOF. Since $\mathfrak{Q}_{\aleph_0} \leqq \mathfrak{P}_{\aleph_0}$, we must prove that $\mathfrak{P}_{\aleph_0} \leqq \mathfrak{Q}_{\aleph_0}$. Let $\mathfrak{a} \in \mathfrak{P}_{\aleph_0}$; by Definition 4.6, $L/\mathfrak{a}$ satisfies the $\aleph_0$-chain condition, and hence, by Theorem 4.4, $L/\mathfrak{a}$ is lattice-isomorphic to the class of all subsets of a finite set. When $A \in L/\mathfrak{a}$, if n is the number of elements of the subset which corresponds to A, then define $\phi(A) = n$. Then $\phi(A)$ is a measure function defined in $L/\mathfrak{a}$. Now define $\phi(a) = \phi(A)$ when $a \in A$. Then $\phi(a) = 0$ when $a \in \mathfrak{a}$. Let $(a_i;\ i = 1, 2, \ldots, \nu)$ be any set such that $a_i \wedge a_j = 0$ when $i \neq j$. Then $A_{a_i} \wedge A_{a_j} = A_{a_i \wedge a_j}{}^{(1)} = A_0 = O$ when $i \neq j$. Now

$$\phi(\sum a_i) = \phi(A_{\sum a_i}) = \phi(\sum A_{a_i})^{(2)} = \sum \phi(A_{a_i}) = \sum \phi(a_i).$$

Consequently $\phi(a)$ is a measure function such that $\mathfrak{a} = \mathfrak{a}_{\phi}$. That is, $\mathfrak{a} \in \mathfrak{Q}_{\aleph_0}$, and $\mathfrak{P}_{\aleph_0} \leqq \mathfrak{Q}_{\aleph_0}$, as we wished to prove.

THEOREM 5.4. *$\mathfrak{Q}_{\aleph_1}$ is a dual $\aleph_1$-ideal in $\mathfrak{P}_{\aleph_1}$; hence $\mathfrak{Q}_{\aleph_1}$ is a generalized $\aleph_1$-Boolean algebra with unit element. And class $\Psi^*_{\aleph_1}$ of all dual $\aleph_1$-ideals in $\mathfrak{Q}_{\aleph_1}$ is a continuous Boolean algebra.*

PROOF. (i) Let $S = (\mathfrak{a}_{\phi_i};\ i = 1, 2, \ldots)$ be a denumerable subset of $\mathfrak{Q}_{\aleph_1}$. Put

$$\phi(a) = \sum_i \frac{\phi_i(a)}{2^i \phi_i(1)}.$$

Then $\phi(a)$ is a measure function. Since $\phi(a) = 0$ when, and only when, $\phi_i(a) = 0$ for all i, we have $\mathfrak{a}_{\phi} = \Pi(\mathfrak{a}_{\phi_i};\ i = 1, 2, \ldots)$. Hence $\Pi(\mathfrak{a}_{\phi_i};\ i = 1, 2, \ldots) \in \mathfrak{Q}_{\aleph_1}$.

(ii) Let $\mathfrak{a} \in \mathfrak{Q}_{\aleph_1}$, $\mathfrak{b} \in \mathfrak{P}_{\aleph_1}$ and $\mathfrak{b} \supset \mathfrak{a}$. Let $\phi(a)$ be a measure function such that $\mathfrak{a} = \mathfrak{a}_{\phi}$. By Theorem 4.10, there exists an $\aleph_1$-ideal $\mathfrak{c}$ such that

$$\mathfrak{b} \vee \mathfrak{c} = \mathfrak{e}, \qquad \mathfrak{b} \wedge \mathfrak{c} = \mathfrak{a}.$$

By Theorem 2.1, every element $a \in \mathfrak{e}$ is expressed in the form $a = b \vee c$ where $b \in \mathfrak{b}$, $c \in \mathfrak{c}$. Since $b \wedge c \in \mathfrak{b} \wedge \mathfrak{c} = \mathfrak{a}$, and $\phi(b \wedge c) = 0$, we have

$$\phi(a) = \phi(b) + \phi(c).$$

The value of $\phi(c)$ is uniquely determined for a definite element a. For, if $a = b_1 \vee c_1$ where $b_1 \in \mathfrak{b}$, $c_1 \in \mathfrak{c}$, then

$$c = c \wedge a = c \wedge (b_1 \vee c_1) = (c \wedge b_1) \vee (c \wedge c_1)$$

and $c \wedge b_1 \in \mathfrak{c} \wedge \mathfrak{b} = \mathfrak{a}$; hence we have $\phi(c) = \phi(c \wedge c_1)$. Similarly $\phi(c_1) = \phi(c \wedge c_1)$. Consequently $\phi(c) = \phi(c_1)$.

Now, put $\psi(a) = \phi(c)$.

(a) $\psi(a) = 0$ when $a \in \mathfrak{b}$, and $\psi(a) \geqq 0$.

(1) By (2) in the proof of Theorem 3.1.

(2) By (1) in the proof of Theorem 3.1.

(β) When $a=\sum_i \oplus a_i$, put

$$a_i=b_i \vee c_i, \qquad b_i \in \mathfrak{b},\ c_i \in \mathfrak{c},$$

for all i. Then $a=\sum_i \oplus b_i \vee \sum_i \oplus c_i$, and $\sum_i \oplus b_i \in \mathfrak{b}$, $\sum_i \oplus c_i \in \mathfrak{c}$. Hence

$$\psi(a)=\phi(\sum_i \oplus c_i)=\sum_i \phi(c_i)=\sum_i \psi(a_i).$$

(γ) $\psi(a) \neq 0$ when $a \notin \mathfrak{b}$. For, when $a \notin \mathfrak{b}$, let $a=b \vee c$, $b \in \mathfrak{b}$, $c \in \mathfrak{c}$. If $c \in \mathfrak{a}$, then $c \in \mathfrak{a} \subset \mathfrak{b}$ and $a=b \vee c \in \mathfrak{b}$, which is absurd. Hence $c \notin \mathfrak{a}$, and $\phi(c) \neq 0$.

Thus $\psi(a)$ is a measure function such that $\mathfrak{b}=\mathfrak{a}_\psi$. Consequently $\mathfrak{b} \in \mathfrak{Q}_{\aleph_1}$.

From (i) and (ii), $\mathfrak{Q}_{\aleph_1}$ is a dual $\aleph_1$-ideal in $\mathfrak{P}_{\aleph_1}$.

(iii) By Theorem 4.13, class $\Phi^*_{\aleph_1}$ of all dual $\aleph_1$-ideals in $\mathfrak{P}_{\aleph_1}$ is a continuous Boolean algebra, $\Psi^*_{\aleph_1}$ being the set of all dual $\aleph_1$-ideals $\mathfrak{A}^*$ in $\mathfrak{P}_{\aleph_1}$ such that $\mathfrak{A}^* \subset \mathfrak{Q}_{\aleph_1}$, $\Psi^*_{\aleph_1}$ is also a continuous Boolean algebra.

6. Let $\phi(a)$ and $\psi(a)$ be two measure functions in $\mathfrak{F}_{\aleph_1}$. If $\psi(a)=0$ for all a such that $\phi(a)=0$, then we say, as in the theory of set functions, that $\psi(a)$ is absolutely continuous with respect to $\phi(a)$, and write $\psi \prec \phi$ or $\phi \succ \psi$.[1] To indicate that the relations $\psi \prec \phi$, $\phi \prec \psi$ both hold good, we write $\psi \sim \phi$.

Since $\psi \prec \phi$ when, and only when, $\mathfrak{a}_\psi \supset \mathfrak{a}_\phi$, if we use the relation $\prec$ as the order in the lattice theory, then $\mathfrak{F}_{\aleph_1}$ is dual-isomorphic to $\mathfrak{Q}_{\aleph_1}$. Hence, from Theorem 5.4, we have

THEOREM 6.1.[2] *System $\mathfrak{F}_{\aleph_1}$ of all measure functions is a generalized $\aleph_1$-Boolean algebra with zero element.*[3] *And class $\Psi_{\aleph_1}$ of all $\aleph$-ideals in $\mathfrak{F}_{\aleph_1}$ is a continuous Boolean algebra.*

Of course, the zero element of $\mathfrak{F}_{\aleph_1}$ is the function $\phi(a)$ such that $\phi(a)=0$ for all $a \in L$.

Application to the Spectral Theory.

7. Let $\mathfrak{H}$ be a complete complex Euclidean space, with elements $f, g, \ldots.$, that is, a space which satisfies all the axioms of Hilbert space except the axiom of separability. Let $E(a)$ be a projection defined for all elements a of an $\aleph_1$-Boolean algebra L. If $E(a)$ satisfies the following conditions, then we say that $E(a)$ is a *resolution of identity* in the generalized sense.[4]

(1) This definition is a generalization of that used by M. H. Stone. Cf. M. H. Stone, *Linear Transformations in Hilbert Space*, (1932), 214.

(2) Direct proof of this theorem, using the functional property of $\phi(a)$, is given in Sec. 9.

(3) From Theorem 5.4, $\mathfrak{F}_{\aleph_1}$ is closed with respect to the operation of meet of any subclass of $\mathfrak{F}_{\aleph_1}$.

(4) This is a generalization of the ordinary defined resolution of identity $E(\lambda)$. (Cf. F. Maeda [1], 78; K. Friedrichs [1], 54–58; F. Wecken [1], 443.)

(α) $\boldsymbol{E}(a)\boldsymbol{E}(b)=0$ when $a\wedge b=0$;
(β) $\boldsymbol{E}(a)=\boldsymbol{E}(a_1)+\boldsymbol{E}(a_2)+\cdots\cdots+\boldsymbol{E}(a_i)+\cdots\cdots$ when $a=\sum_i\oplus a_i$,
(γ) $\boldsymbol{E}(1)=1$.

Let a, b be any elements in L. And let a_1, b_1 be the inverses of $a\wedge b$ in a and b respectively. That is,

$$a=(a\wedge b)\oplus a_1, \qquad b=(a\wedge b)\oplus b_1.$$

From (β), we have

$$\boldsymbol{E}(a)=\boldsymbol{E}(a\wedge b)+\boldsymbol{E}(a_1), \qquad \boldsymbol{E}(b)=\boldsymbol{E}(a\wedge b)+\boldsymbol{E}(b_1).$$

Since $a_1\wedge b_1=0$, from (α) we have

$$(\alpha') \qquad \boldsymbol{E}(a)\boldsymbol{E}(b)=\boldsymbol{E}(a\wedge b).$$

(α') shows the permutability of $\boldsymbol{E}(a)$.

Let f be any element in $\mathfrak{H}$. Then, from (α'),

$$\big(\boldsymbol{E}(a)f, \boldsymbol{E}(b)f\big)=\|\boldsymbol{E}(a)\boldsymbol{E}(b)f\|^2=\|\boldsymbol{E}(a\wedge b)f\|^2=\sigma(a\wedge b),$$

where $\sigma(a)=\|\boldsymbol{E}(a)f\|^2$. Hence

$$\big(\boldsymbol{E}(a)f,\ \boldsymbol{E}(b)f\big)=0 \quad \text{when} \quad a\wedge b=0. \tag{1}$$

From (β), when $a=\sum_i\oplus a_i$,

$$\boldsymbol{E}(a)f=\boldsymbol{E}(a_1)f+\boldsymbol{E}(a_2)f+\cdots\cdots+\boldsymbol{E}(a_i)f+\cdots\cdots \tag{2}$$

From (1) and (2), we have

$$\sigma(a)=\sigma(a_1)+\sigma(a_2)+\cdots\cdots+\sigma(a_i)+\cdots\cdots$$

Then $\sigma(a)$ is a measure function as defined in Sec. 5.

Let f be any element in $\mathfrak{H}$, and let $\mathfrak{a}_f$ be the class of all elements $a\in L$ such that $\boldsymbol{E}(a)f=0$. If we put $\|\boldsymbol{E}(a)f\|^2=\sigma(a)$, then $\sigma(a)$ is a measure function, and $\mathfrak{a}_f=\mathfrak{a}_\sigma$. Hence $\mathfrak{a}_f$ is an $\aleph_1$-ideal in L contained in $\mathfrak{Q}_{\aleph_1}$ and therefore in $\mathfrak{P}_{\aleph_1}$.

Denote by $\mathfrak{M}(f)$ the closed linear manifold determined by $\boldsymbol{E}(a)f$ when a runs over L.

Lemma 7.1. *If $g\in\mathfrak{M}(f)$, then $\mathfrak{a}_f\subset\mathfrak{a}_g$.*

Proof. If $g\in\mathfrak{M}(f)$, then g is a limit of a sequence of elements of the form

$$h=\alpha_1\boldsymbol{E}(a_1)f+\alpha_2\boldsymbol{E}(a_2)f+\cdots\cdots+\alpha_n\boldsymbol{E}(a_n)f,$$

where α's are complex numbers. Hence, by (α'), $\boldsymbol{E}(b)g$ is the limit of the sequence of elements of the form

$$\boldsymbol{E}(b)h=\alpha_1\boldsymbol{E}(b\wedge a_1)f+\alpha_2\boldsymbol{E}(b\wedge a_2)f+\cdots\cdots+\alpha_n\boldsymbol{E}(b\wedge a_n)f.$$

But if $b\in\mathfrak{a}_f$, then $\boldsymbol{E}(b\wedge a_i)f=0$ $(i=1,2,\ldots\ldots,n)$ and $\boldsymbol{E}(b)h=0$. Hence $\boldsymbol{E}(b)g=0$. Consequently $\mathfrak{a}_f\subset\mathfrak{a}_g$.

Lemma 7.2. *If $\mathfrak{a}_f \vee \mathfrak{a}_g = \mathfrak{e}$, then $\mathfrak{M}(f) \perp \mathfrak{M}(g)$.*

Proof. Let $g = g_1 + g_2$, where $g_1 \in \mathfrak{M}(f)$ and $g_2 \perp \mathfrak{M}(f)$. Since $E(a)g = E(a)g_1 + E(a)g_2$ and $\big(E(a)g_1, E(a)g_2\big) = \big(E(a)g_1, g_2\big) = 0$,

we have $$\|E(a)g\|^2 = \|E(a)g_1\|^2 + \|E(a)g_2\|^2.$$

Hence $$\mathfrak{a}_g \subset \mathfrak{a}_{g_1}.$$

Since $g_1 \in \mathfrak{M}(f)$, by Lemma 7.1 we have $\mathfrak{a}_f \subset \mathfrak{a}_{g_1}$. Consequently $\mathfrak{a}_{g_1} \supset \mathfrak{a}_f \vee \mathfrak{a}_g = \mathfrak{e}$. That is, $E(a)g_1 = 0$ for all $a \in L$. Therefore $g_1 = 0$, and $g \perp \mathfrak{M}(f)$. That is, $\mathfrak{M}(g) \perp \mathfrak{M}(f)$.[1]

Lemma 7.3. *Let $\mathfrak{b}$ be any $\aleph_1$-ideal contained in $\mathfrak{P}_{\aleph_1}$ such that $\mathfrak{b} \supset \mathfrak{a}_f$. Then there exists an element $c \in L$ such that*

$$\mathfrak{b} = \mathfrak{a}_g \quad \textit{where} \quad g = E(c)f.$$

Proof. By Theorem 4.10, there exists an $\aleph_1$-ideal $\mathfrak{c}$ in $\mathfrak{P}_{\aleph_1}$ such that

$$\mathfrak{b} \vee \mathfrak{c} = \mathfrak{e}, \qquad \mathfrak{b} \wedge \mathfrak{c} = \mathfrak{a}_f.$$

By Theorem 2.1, since $1 \in \mathfrak{e}$, we have

$$1 = b \vee c \quad \text{where} \quad b \in \mathfrak{b},\ c \in \mathfrak{c}.$$

Now, put $g = E(c)f$. If $a \in \mathfrak{b}$, then $E(a)g = 0$. For

$$E(a)g = E(a)E(c)f = E(a \wedge c)f$$

by (a'), and $a \wedge c \in \mathfrak{b} \wedge \mathfrak{c} = \mathfrak{a}_f$.

If $a \notin \mathfrak{b}$, then $E(a)g \neq 0$. For if not, then $E(a \wedge c)f = E(a)g = 0$ and $a \wedge c \in \mathfrak{a}_f \subset \mathfrak{b}$. Since

$$a = a \wedge 1 = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

and $a \wedge b \in \mathfrak{b}$, we have $a \in \mathfrak{b}$, which is absurd.

Thus $a \in \mathfrak{b}$ when, and only when, $E(a)g = 0$. That is, $\mathfrak{b} = \mathfrak{a}_g$.

Theorem 7.1. *Class $\big(\mathfrak{a}_g;\ g \in \mathfrak{M}(f)\big)$ is a principal dual $\aleph_1$-ideal $\mathfrak{A}^*(\mathfrak{a}_f)$ in $\mathfrak{P}_{\aleph_1}$ (or in $\mathfrak{Q}_{\aleph_1}$).*

Proof. (i) If $g \in \mathfrak{M}(f)$, then, by Lemma 7.1, $\mathfrak{a}_f \subset \mathfrak{a}_g$.

(ii) Next, let $\mathfrak{b}$ be any $\aleph_1$-ideal in $\mathfrak{P}_{\aleph_1}$ (or in $\mathfrak{Q}_{\aleph_1}$) such that $\mathfrak{b} \supset \mathfrak{a}_f$. Then, by Lemma 7.3, there exists an element $g \in \mathfrak{M}(f)$ such that $\mathfrak{b} = \mathfrak{a}_g$.

From (i) and (ii), the theorem is proved.

Theorem 7.2. *Class $\mathfrak{R}_E$ of all $\mathfrak{a}_f$ such that $f \in \mathfrak{H}$ is a dual $\aleph_1$-ideal in $\mathfrak{P}_{\aleph_1}$[2]. Especially when $\mathfrak{H}$ is separable, $\mathfrak{R}_E$ is a principal dual $\aleph_1$-ideal in $\mathfrak{P}_{\aleph_1}$.*

Proof. There exists a system $(f_i;\ f_i \in \mathfrak{H},\ i \in I)$ such that the closed linear manifolds $\mathfrak{M}(f_i)$ are mutually orthogonal and

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(1) For, since $(g, E(a)f) = 0$ for all $a \in L$, $(E(b)g, E(a)f) = (g, E(b \wedge a)f) = 0$ for all $a, b \in L$.

(2) Or in $\mathfrak{Q}_{\aleph_1}$.

$$\mathfrak{H}=\sum_{i\in I}\oplus\mathfrak{M}(f_i)^{(1)}. \tag{1}$$

Since $\mathfrak{A}^*(\mathfrak{a}_{f_i})$ is a dual $\aleph_1$-ideal in $\mathfrak{P}_{\aleph_1}$, if we apply Theorem 2.1* to $\mathfrak{P}_{\aleph_1}$ instead of L, then join $\mathfrak{B}^*=\sum\left(\mathfrak{A}^*(\mathfrak{a}_{f_i});\ i\in I\right)$ is a dual $\aleph_1$-ideal in $\mathfrak{P}_{\aleph_1}$.

Let g be any element in $\mathfrak{H}$, and g_i be the component of g in $\mathfrak{M}(f_i)$. Then $g_i=0$ for all i except a denumerable subset I_1 of I. And $g=\sum_{i\in I_1}g_i$. Then $\|E(a)g\|^2=\sum_{i\in I_1}\|E(a)g_i\|^2$. Hence $\mathfrak{a}_g=\Pi(\mathfrak{a}_{g_i};\ i\in I_1)$. Since, by Theorem 7.1, $\mathfrak{a}_{g_i}\in\mathfrak{A}^*(\mathfrak{a}_{f_i})\subset\mathfrak{B}^*$, we have $\mathfrak{a}_g\in\mathfrak{B}^*$. Consequently

$$\mathfrak{R}_E\subset\mathfrak{B}^*.$$

Next, let $\mathfrak{a}$ be any element in $\mathfrak{B}^*$. Then, by Theorem 2.1*,

$$\mathfrak{a}=\Pi\left(\mathfrak{a}_{g_i};\ \mathfrak{a}_{g_i}\in\mathfrak{A}^*(\mathfrak{a}_{f_i}),\ i\in I_1\right),$$

where I_1 is a denumerable subset of I. Put $g=\sum_{i\in I_1}\alpha_i g_i$, where $(\alpha_i;\ i\in I_1)$ is a system of complex numbers such that $\sum_{\alpha\in I_1}\alpha_i g_i$ converges. Then, since $\|E(a)g\|^2=\sum_{i\in I_1}|\alpha_i|^2\|E(a)g_i\|^2$, we have $\mathfrak{a}_g=\Pi(\mathfrak{a}_{g_i};\ i\in I_1)$. That is, $\mathfrak{a}=\mathfrak{a}_g$. Hence

$$\mathfrak{B}^*\subset\mathfrak{R}_E.$$

Consequently $\mathfrak{R}_E=\mathfrak{B}^*$, and the first part of the theorem is proved.

Especially when $\mathfrak{H}$ is separable, then I in (1) is denumerable. Hence put $f=\sum_{i\in I}\alpha_i f_i$, where $(\alpha_i;\ i\in I)$ is a system of complex numbers such that $\sum_{i\in I}\alpha_i f_i$ converges. Then, since $\|E(a)f\|^2=\sum_{i\in I}|\alpha_i|^2\|E(a)f_i\|^2$, we have $\mathfrak{a}_f=\Pi(\mathfrak{a}_{f_i};\ i\in I)$. And $\mathfrak{B}^*=\mathfrak{A}^*(\mathfrak{a}_f)$.

From Theorem 7.2, $\mathfrak{R}_E$ is a generalized $\aleph_1$-Boolean subalgebra of $\mathfrak{P}_{\aleph_1}$ with unit element $\mathfrak{e}$. And especially when $\mathfrak{H}$ is separable, $\mathfrak{R}_E$ is an $\aleph_1$-Boolean subalgebra of $\mathfrak{P}_{\aleph_1}$ with zero and unit elements.

Let Θ_E^* be the class of all dual $\aleph_1$-ideals $\mathfrak{A}^*$ in $\mathfrak{R}_E$. Since $\mathfrak{R}_E$ is a dual $\aleph_1$-ideal in $\mathfrak{P}_{\aleph_1}$, and, by Theorem 4.13, class $\Phi_{\aleph_1}^*$ of all dual $\aleph_1$-ideals in $\mathfrak{P}_{\aleph_1}$ is a continuous Boolean algebra, Θ_E^* being the set of all dual $\aleph$-ideals $\mathfrak{A}^*$ such that $\mathfrak{A}^*\subset\mathfrak{R}_E$, Θ_E^* is also a continuous Boolean algebra.

THEOREM 7.3. *Let $\mathfrak{A}^*\in\Theta_E^*$, and $\mathfrak{M}_{\mathfrak{A}^*}$ be the class of all elements $f\in\mathfrak{H}$ such that $\mathfrak{a}_f\in\mathfrak{A}^*$. Then $\mathfrak{M}_{\mathfrak{A}^*}$ is a closed linear manifold, and*

$$\mathfrak{M}_{\mathfrak{A}^*}\perp\mathfrak{M}_{\mathfrak{A}^{*\backprime}},\qquad \mathfrak{M}_{\mathfrak{A}^*}\oplus\mathfrak{M}_{\mathfrak{A}^{*\backprime}}=\mathfrak{H},$$

where $\mathfrak{A}^{*\backprime}$ is the inverse of $\mathfrak{A}^*$.

PROOF. (i) Let $f\in\mathfrak{M}_{\mathfrak{A}^*}$, $g\in\mathfrak{M}_{\mathfrak{A}^{*\backprime}}$. Then $\mathfrak{a}_f\in\mathfrak{A}^*$, $\mathfrak{a}_g\in\mathfrak{A}^{*\backprime}$. Since

(1) Give $\mathfrak{H}$ a well-order type. We find the elements of $P=(f_i;\ f_i\in\mathfrak{H},\ i\in I)$ by transfinite induction as follows: g belongs to P when, and only when, $\|g\|>0$ and g is orthogonal to $\mathfrak{M}(f)$ for all $f\in P$ which have lower rank than g in the well-order type. Since $(E(a)f, E(b)g)=(E(a\wedge b)f, g)=0$, $\mathfrak{M}(f)$ $(f\in P)$ are mutually orthogonal. ($\sum_{i\in I}\oplus$ means the closed linear sum.)

$\mathfrak{A}^* \wedge \mathfrak{A}^{*\prime} = \mathfrak{O}^*$, we have, by Theorem 2.1*, $\mathfrak{a}_f \vee \mathfrak{a}_g = \mathfrak{e}$. Hence, by Lemma 7.2, $f \perp g$. Consequently $\mathfrak{M}_{\mathfrak{A}^*} \perp \mathfrak{M}_{\mathfrak{A}^{*\prime}}$.

(ii) Let g be any element in $\mathfrak{H}$ such that $g \perp \mathfrak{M}_{\mathfrak{A}^{*\prime}}$. Then $\mathfrak{A}^{*\prime} \wedge \mathfrak{A}^*(\mathfrak{a}_g) = \mathfrak{O}^*$. For if not, by Theorem 4.12 there exists $\mathfrak{b} \in \mathfrak{P}_{\aleph}$ such that

$$\mathfrak{A}^{*\prime} \wedge \mathfrak{A}^*(\mathfrak{a}_g) = \mathfrak{A}^*(\mathfrak{b}).$$

Since $\mathfrak{A}^*(\mathfrak{b}) \subset \mathfrak{A}^*(\mathfrak{a}_g)$, we have $\mathfrak{b} \supset \mathfrak{a}_g$. Then, by Lemma 7.3, there exists an element $g_1 = \boldsymbol{E}(c)g$ such that $\mathfrak{b} = \mathfrak{a}_{g_1}$. Let f be any element in $\mathfrak{M}_{\mathfrak{A}^{*\prime}}$. Then, since $\boldsymbol{E}(c)f \in \mathfrak{M}_{\mathfrak{A}^{*\prime}}$, we have

$$(g_1, f) = \big(\boldsymbol{E}(c)g, f\big) = \big(g, \boldsymbol{E}(c)f\big) = 0.$$

Hence $g_1 \perp \mathfrak{M}_{\mathfrak{A}^{*\prime}}$. But, since $\mathfrak{A}^*(\mathfrak{b}) \subset \mathfrak{A}^{*\prime}$, we have $\mathfrak{a}_{g_1} = \mathfrak{b} \in \mathfrak{A}^{*\prime}$. Hence $g_1 \in \mathfrak{M}_{\mathfrak{A}^{*\prime}}$, which is absurd.

Consequently $$\mathfrak{A}^*(\mathfrak{a}_g) \subset \mathfrak{A}^*,$$

that is, $\mathfrak{a}_g \in \mathfrak{A}^*$. Hence, from (i), $\mathfrak{M}_{\mathfrak{A}^*}$ is composed of all elements g such that $g \perp \mathfrak{M}_{\mathfrak{A}^{*\prime}}$. Therefore $\mathfrak{M}_{\mathfrak{A}^*}$ is a closed linear manifold.

(iii) Let h be any element in $\mathfrak{H}$. And put

$$h = f + g \quad \text{where} \quad f \in \mathfrak{M}_{\mathfrak{A}^*},\ g \perp \mathfrak{M}_{\mathfrak{A}^*}.$$

By (ii), $g \in \mathfrak{M}_{\mathfrak{A}^{*\prime}}$. Hence $\mathfrak{H} = \mathfrak{M}_{\mathfrak{A}^*} \oplus \mathfrak{M}_{\mathfrak{A}^{*\prime}}$.

THEOREM 7.4. *$\mathfrak{M}_{\mathfrak{A}^*}$ defined for all $\mathfrak{A}^* \in \Theta_E^*$ has the following properties:*

(i) $\mathfrak{M}_{\mathfrak{A}^*} \perp \mathfrak{M}_{\mathfrak{B}^*}$ *when* $\mathfrak{A}^* \wedge \mathfrak{B}^* = \mathfrak{O}^*$,

(ii) $\mathfrak{M}_{\mathfrak{A}^*} = \prod_{\alpha \in I} \mathfrak{M}_{\mathfrak{A}^*_\alpha}$ *when* $\mathfrak{A}^* = \Pi(\mathfrak{A}^*_\alpha;\ \alpha \in I)$,

(iii) $\mathfrak{M}_{\mathfrak{A}^*} = \sum_{\alpha \in I} \oplus \mathfrak{M}_{\mathfrak{A}^*_\alpha}$ *when* $\mathfrak{A}^* = \sum(\mathfrak{A}^*_\alpha;\ \alpha \in I)$,

(iv) $\mathfrak{M}_{\mathfrak{O}^*} = 0$[1] *and* $\mathfrak{M}_{\mathfrak{E}^*} = \mathfrak{H}$,

where $\mathfrak{O}^*$ and $\mathfrak{E}^*$ are zero and unit elements of Θ_E^* respectively.

PROOF. (i) If $\mathfrak{A}^* \wedge \mathfrak{B}^* = \mathfrak{O}^*$, then $\mathfrak{B}^* \subset \mathfrak{A}^{*\prime}$ and $\mathfrak{M}_{\mathfrak{B}^*} \subset \mathfrak{M}_{\mathfrak{A}^{*\prime}}$. Since, by Theorem 7.3, $\mathfrak{M}_{\mathfrak{A}^*} \perp \mathfrak{M}_{\mathfrak{A}^{*\prime}}$, we have $\mathfrak{M}_{\mathfrak{A}^*} \perp \mathfrak{M}_{\mathfrak{B}^*}$.

(ii) If $\mathfrak{A}^* = \Pi(\mathfrak{A}^*_\alpha;\ \alpha \in I)$, then $\mathfrak{A}^* \subset \mathfrak{A}^*_\alpha$ and $\mathfrak{M}_{\mathfrak{A}^*} \subset \mathfrak{M}_{\mathfrak{A}^*_\alpha}$ for all $\alpha \in I$. Hence

$$\mathfrak{M}_{\mathfrak{A}^*} \subset \prod_{\alpha \in I} \mathfrak{M}_{\mathfrak{A}^*_\alpha}. \tag{1}$$

Let $f \in \mathfrak{M}_{\mathfrak{A}^*_\alpha}$ for all $\alpha \in I$; that is, $\mathfrak{a}_f \in \mathfrak{A}^*_\alpha$ $(\alpha \in I)$. Then $\mathfrak{a}_f \in \mathfrak{A}^*$. Consequently $f \in \mathfrak{M}_{\mathfrak{A}^*}$. Therefore

$$\mathfrak{M}_{\mathfrak{A}^*} \supset \prod_{\alpha \in I} \mathfrak{M}_{\mathfrak{A}^*_\alpha}. \tag{2}$$

From (1) and (2), (ii) is proved.

(iii) If $\mathfrak{A}^* = \sum(\mathfrak{A}^*_\alpha;\ \alpha \in I)$, then $\mathfrak{A}^* \supset \mathfrak{A}^*_\alpha$ and $\mathfrak{M}_{\mathfrak{A}^*} \supset \mathfrak{M}_{\mathfrak{A}^*_\alpha}$ for all $\alpha \in I$. Hence

$$\mathfrak{M}_{\mathfrak{A}^*} \supset \sum_{\alpha \in I} \oplus \mathfrak{M}_{\mathfrak{A}^*_\alpha}. \tag{3}$$

Let f be any element orthogonal to all $\mathfrak{M}_{\mathfrak{A}^*_\alpha}$ $(\alpha \in I)$. Then, by Theorem 7.3, $f \in \mathfrak{M}_{\mathfrak{A}^{*\prime}_\alpha}$ for all $\alpha \in I$. Since $\mathfrak{A}^{*\prime} = \Pi(\mathfrak{A}^{*\prime}_\alpha;\ \alpha \in I)$, by (ii) $\mathfrak{M}_{\mathfrak{A}^{*\prime}} = \prod_{\alpha \in I} \mathfrak{M}_{\mathfrak{A}^{*\prime}_\alpha}$. Hence $f \in \mathfrak{M}_{\mathfrak{A}^{*\prime}}$. Therefore, by Theorem 7.3, $f \perp \mathfrak{M}_{\mathfrak{A}^*}$. Consequently

(1) This means that $\mathfrak{M}_{\mathfrak{O}^*}$ is composed of only one element 0 in $\mathfrak{H}$.

$$\mathfrak{M}_{\mathfrak{A}^*} \subset \sum_{\alpha \in I} \oplus \mathfrak{M}_{\mathfrak{A}^*_\alpha} . \tag{4}$$

From (3) and (4), (iii) is proved.

(iv) This is evident from the definition of $\mathfrak{M}_{\mathfrak{A}^*}$.

8. By Theorem 7.4 $\mathfrak{M}_{\mathfrak{A}^*}$ satisfies the following conditions:

(α) $\mathfrak{M}_{\mathfrak{A}^*} \perp \mathfrak{M}_{\mathfrak{B}^*}$ when $\mathfrak{A}^* \wedge \mathfrak{B}^* = \mathfrak{O}^*$,

(β) $\mathfrak{M}_{\mathfrak{A}^*} = \sum_n \oplus \mathfrak{M}_{\mathfrak{A}^*_n}$ when $\mathfrak{A}^* = \sum_n \oplus \mathfrak{A}^*_n$,

(γ) $\mathfrak{M}_{\mathfrak{E}^*} = \mathfrak{H}$.

That is, $(\mathfrak{M}_{\mathfrak{A}^*};\ \mathfrak{A}^* \in \theta^*_E)$ is a complete orthogonal system of closed linear manifolds in $\mathfrak{H}$ whose index is the element of the continuous Boolean algebra θ^*_E.(1) Let $F(\mathfrak{A}^*)$ be the projection of $\mathfrak{H}$ on $\mathfrak{M}_{\mathfrak{A}^*}$; then $F(\mathfrak{A}^*)$ satisfies the following conditions:

(α) $F(\mathfrak{A}^*)F(\mathfrak{B}^*) = 0$ when $\mathfrak{A}^* \wedge \mathfrak{B}^* = \mathfrak{O}^*$,

(β) $F(\mathfrak{A}^*) = F(\mathfrak{A}^*_1) + F(\mathfrak{A}^*_2) + \cdots\cdots + F(\mathfrak{A}^*_n) + \cdots\cdots$ when $\mathfrak{A}^* = \sum_n \oplus \mathfrak{A}^*_n$,

(γ) $F(\mathfrak{E}^*) = 1$.

As defined in Sec. 7, $F(\mathfrak{A}^*)$ is a resolution of identity in the generalized sense defined in θ^*_E. Thus, *from a resolution of identity $E(a)$ defined in L, we obtain a resolution of identity $F(\mathfrak{A}^*)$ defined in θ^*_E.* In what follows, I shall investigate the relation between $E(a)$ and $F(\mathfrak{A}^*)$.

THEOREM 8.1. *Let $\mathfrak{A}^*_b$ be the class of all $\mathfrak{a}_f \in \mathfrak{R}_E$ such that $\mathfrak{a}_f \supset \mathfrak{a}(b')$, where b' is the inverse of b. Then $\mathfrak{A}^*_b \in \theta^*_E$, and the system $(\mathfrak{A}^*_b;\ b \in L)$ is an $\aleph_1$-Boolean algebra which is isomorphic to L with respect to the join and the meet of subsets of power $< \aleph_1$.*

PROOF. (i) It is evident that $\mathfrak{A}^*_b$ is a dual $\aleph_1$-ideal in $\mathfrak{R}_E$, that is, $\mathfrak{A}^*_b \in \theta^*_E$.

(ii) If $b \supset c$, then $\mathfrak{a}(b') \subset \mathfrak{a}(c')$; hence $\mathfrak{A}^*_b \supset \mathfrak{A}^*_c$.

(iii) Let $b = \sum(c_i;\ i=1, 2, \ldots.)$; then $b' = \Pi(c'_i;\ i=1, 2, \ldots.)$. Since $b \supset c_i$, by (ii) $\mathfrak{A}^*_b \supset \mathfrak{A}^*_{c_i}$ for all i. Hence

$$\mathfrak{A}^*_b \supset \sum(\mathfrak{A}^*_{c_i};\ i=1, 2, \ldots.). \tag{1}$$

Next, let $\mathfrak{a}_f$ be any element in $\mathfrak{A}^*_b$. Then, by Theorem 2.2,

$$\mathfrak{a}_f \supset \mathfrak{a}(b') = \Pi\big(\mathfrak{a}(c'_i);\ i=1, 2, \ldots.\big).$$

Put $\mathfrak{b}_i = \mathfrak{a}_f \vee \mathfrak{a}(c'_i)$, and $E(c_i)f = g_i$; then, since $E(a)g_i = E(a)E(c_i)f = E(a \wedge c_i)f$, we have $\mathfrak{a}_{g_i} = \mathfrak{b}_i \in \mathfrak{A}^*_{c_i}$. Now

$$\begin{aligned} \mathfrak{a}_f &= \mathfrak{a}_f \vee \Pi\big(\mathfrak{a}(c'_i);\ i=1, 2, \ldots.\big) = \Pi\big(\mathfrak{a}_f \vee \mathfrak{a}(c'_i);\ i=1, 2, \ldots.\big)^{(2)} \\ &= \Pi(\mathfrak{a}_{g_i};\ i=1, 2, \ldots.). \end{aligned}$$

Hence
$$\mathfrak{A}^*_b \subset \sum(\mathfrak{A}^*_{c_i};\ i=1, 2, \ldots.)^{(3)}. \tag{2}$$

(1) Cf. F. Maeda [2], 111.
(2) By Theorem 2.1.
(3) By Theorem 2.1.*

From (1) and (2), we have

$$\mathfrak{A}_b^* = \sum(\mathfrak{A}_{c_i}^*;\ i=1, 2, \ldots\ldots).$$

(iv) Let $b = \Pi(c_i;\ i=1, 2, \ldots\ldots)$; then $b' = \sum(c_i';\ i=1, 2, \ldots\ldots)$. Since $b \subset c_i$, by (ii) $\mathfrak{A}_b^* \subset \mathfrak{A}_{c_i}^*$ for all i. Hence

$$\mathfrak{A}_b^* \subset \Pi(\mathfrak{A}_{c_i}^*;\ i=1, 2, \ldots\ldots). \tag{3}$$

Next, let $\mathfrak{a}_f$ be any element in $\Pi(\mathfrak{A}_{c_i}^*;\ i=1, 2, \ldots\ldots)$. Then $\mathfrak{a}_f \supset \mathfrak{a}(c_i')$ for all i. Hence

$$\mathfrak{a}_f \supset \sum\big(\mathfrak{a}(c_i');\ i=1, 2, \ldots\ldots\big) = \mathfrak{a}(b').$$

Therefore
$$\mathfrak{A}_b^* \supset \Pi(\mathfrak{A}_{c_i}^*;\ i=1, 2, \ldots\ldots). \tag{4}$$

From (3) and (4),
$$\mathfrak{A}_b^* = \Pi(\mathfrak{A}_{c_i}^*;\ i=1, 2, \ldots\ldots).$$

(ii), (iii), (iv) show that the correspondence $b \leftrightarrow \mathfrak{A}_b^*$ between L and $(\mathfrak{A}_b^*;\ b \in L)$ is an isomorphism with respect to the join and the meet of subsets of power $< \aleph_1$.

THEOREM 8.2. *$E(a)$ is imbedded in $F(\mathfrak{A}^*)$ in such a manner that*

$$E(a) = F(\mathfrak{A}_a^*) \quad \textit{for all} \quad a \in L.$$

PROOF. Let $E(b)$ be the projection of $\mathfrak{H}$ on the closed linear manifold $\mathfrak{N}_b$. If $f \in \mathfrak{N}_b$, then $E(b)f = f$ and

$$E(a)f = E(a)E(b)f = E(a \wedge b)f.$$

Hence $\mathfrak{a}_f \supset \mathfrak{a}(b')$, that is, $f \in \mathfrak{M}_{\mathfrak{A}_b^*}$. Consequently

$$\mathfrak{N}_b \subset \mathfrak{M}_{\mathfrak{A}_b^*}. \tag{1}$$

Next, if $f \in \mathfrak{M}_{\mathfrak{A}_b^*}$, then $\mathfrak{a}_f \supset \mathfrak{a}(b')$. Hence $E(b')f = 0$, and $f \in \mathfrak{N}_b$. Consequently

$$\mathfrak{M}_{\mathfrak{A}_b^*} \subset \mathfrak{N}_b. \tag{2}$$

From (1) and (2), we have $\mathfrak{N}_b = \mathfrak{M}_{\mathfrak{A}_b^*}$, which shows that $E(b) = F(\mathfrak{A}_b^*)$.

From Theorem 8.2, we may say that $F(\mathfrak{A}^*)$ *is an extension of* $E(a)$.[1]

Direct Proof of Theorem 6.1.

9. In Sec. 6, we have Theorem 6.1 as a simple application of the general theory of ideals in a Boolean algebra. In what follows, I shall give a direct proof of this theorem, using the functional property of $\phi(a)$. For this purpose I shall first prove the following lemmas:

LEMMA 9.1. *Let $\phi(a) \in \mathfrak{F}_{\aleph_1}$, and let $\mathfrak{b}$ be an $\aleph_1$-ideal such that $\mathfrak{b} \supset \mathfrak{a}_\phi$. Then there exists an element $b \in \mathfrak{b}$ such that, if we put $\phi^\times(a) = \phi(a \wedge b')$, $\phi^{\times\times}(a) = \phi(a \wedge b)$, b' being the inverse of b, then*

(1) When L is a σ-field of Borel sets in the space of real numbers, $E(a)$ is a resolution of identity of a self-adjoint operator A. And in this case the extended resolution of identity $F(\mathfrak{A}^*)$ is useful for the investigation of the unitary invariance of A. (Cf. F. Wecken [1].)

$$\phi(a)=\phi^{\times}(a)+\phi^{\times\times}(a)$$

and $$\mathfrak{b}=\mathfrak{a}_{\phi^{\times}}, \qquad \mathfrak{a}_{\phi^{\times}}\vee\mathfrak{a}_{\phi^{\times\times}}=\mathfrak{e}.$$

Proof. Let α be the least upper bound of $\phi(a)$ for all a such that $a\in\mathfrak{b}$. Then there exists a sequence of elements $(a_i;\ i=1,2,\dots.)$ such that $a_i\in\mathfrak{b}$ and $\lim_{i\to\infty}\phi(a_i)=\alpha$. Put $b=\sum(a_i;\ i=1,2,\dots.)$. Then $b\in\mathfrak{b}$. Hence $\phi(b)\leqq\alpha$. Since $b\supset a_i$, $\phi(b)\geqq\phi(a_i)$. Hence $\phi(b)\geqq\lim_{i\to\infty}\phi(a_i)=\alpha$. Consequently $\phi(b)=\alpha$.

Let b' be the inverse of b. And put

$$\phi^{\times}(a)=\phi(a\wedge b') \quad \text{and} \quad \phi^{\times\times}(a)=\phi(a\wedge b). \tag{1}$$

Then it is evident that $\phi^{\times}(a)$, $\phi^{\times\times}(a)$ belong to $\mathfrak{F}_{\aleph_1}$, and

$$\phi(a)=\phi^{\times}(a)+\phi^{\times\times}(a). \tag{2}$$

If $a_0\in\mathfrak{b}$, then, since $a_0\vee b\in\mathfrak{b}$, we have $\phi(a_0\vee b)\leqq\alpha$. But $\alpha=\phi(b)\leqq\phi(a_0\vee b)$. Hence $\phi(a_0\vee b)=\phi(b)$. Now

$$\phi(a_0\vee b)+\phi(a_0\wedge b)=\phi(a_0)+\phi(b).$$

Hence $\phi(a_0\wedge b)=\phi(a_0)$. And, by (1), $\phi^{\times\times}(a_0)=\phi(a_0)$. Consequently, from (2), $\phi^{\times}(a_0)=0$. Hence

$$\mathfrak{b}\subset\mathfrak{a}_{\phi^{\times}}. \tag{3}$$

Next we shall prove that $\mathfrak{b}\supset\mathfrak{a}_{\phi^{\times}}$. For this purpose, let $\phi^{\times}(a_0)=0$; then, from (1), $\phi(a_0\wedge b')=0$; hence $a_0\wedge b'\in\mathfrak{a}_{\phi}\subset\mathfrak{b}$. Since $a_0\wedge b\in\mathfrak{b}$, we have $a_0=(a_0\wedge b')\vee(a_0\wedge b)\in\mathfrak{b}$. Consequently

$$\mathfrak{b}\supset\mathfrak{a}_{\phi^{\times}}. \tag{4}$$

From (3) and (4), $$\mathfrak{b}=\mathfrak{a}_{\phi^{\times}}.$$

Let a be any element in L; then

$$a=(a\wedge b)\vee(a\wedge b').$$

Since, by (1), $a\wedge b\in\mathfrak{a}_{\phi^{\times}}$, $a\wedge b'\in\mathfrak{a}_{\phi^{\times\times}}$, we have $a\in\mathfrak{a}_{\phi^{\times}}\vee\mathfrak{a}_{\phi^{\times\times}}$.

Hence $$\mathfrak{e}=\mathfrak{a}_{\phi^{\times}}\vee\mathfrak{a}_{\phi^{\times\times}}.$$

Lemma 9.2. *When $\psi<\phi$, there exists an element b_ψ such that $b_\psi\in\mathfrak{a}_\psi$ and $\phi(b_\psi)$ is the least upper bound of $\phi(a)$ for all $a\in\mathfrak{a}_\psi$; and if we put*

$$\phi_\psi^{\times}(a)=\phi(a\wedge b'_\psi); \qquad \phi_\psi^{\times\times}(a)=\phi(a\wedge b_\psi),$$

then $$\phi(a)=\phi_\psi^{\times}(a)+\phi_\psi^{\times\times}(a),\text{[1]}$$

and $$\psi\sim\phi_\psi^{\times}, \qquad \phi_\psi^{\times}\wedge\phi_\psi^{\times\times}\sim 0.\text{[2]}$$

Proof. Since $\mathfrak{a}_\psi\supset\mathfrak{a}_\phi$, if we put $\mathfrak{a}_\psi$ instead of $\mathfrak{b}$, then the present lemma follows from Lemma 9.1.

(1) $\phi_\psi^{\times}(a)$ and $\phi_\psi^{\times\times}(a)$ correspond to the "Regularitätsfunktion" and "Singularitätsfunktion" of $\phi(a)$ with respect to $\psi(a)$ in the theory of set functions. Cf. H. Hahn, *Theorie der reellen Funktionen* I (1921), 421.

(2) 0 is the zero element in $\mathfrak{F}_{\aleph_1}$.

DEFINITION 9.1. When $\psi < \phi$, the element b_ψ in Lemma 9.2 is called *a basic element of ψ with respect to ϕ.*[1]

LEMMA 9.3. *The basic element b_ψ of ψ with respect to ϕ has the following properties:*

(i) *If $\psi_1 < \psi_2$, then $\phi(b_{\psi_1}) \geqq \phi(b_{\psi_2})$.*

(ii) *$\psi_1 < \psi_2$ is equivalent to $\phi(b'_{\psi_1} \wedge b_{\psi_2}) = 0$.*

(iii) *If $\psi_1 < \psi_2$ and $\phi(b_{\psi_1}) = \phi(b_{\psi_2})$, then $\psi_1 \sim \psi_2$.*

PROOF. (i) is evident, since $\phi(b_\psi)$ is the least upper bound of $\phi(a)$ for all $a \in \mathfrak{a}_\psi$.

(ii) When $\psi_1 < \psi_2$, put $c = b'_{\psi_1} \wedge b_{\psi_2}$. Since $b_{\psi_1} \wedge c \subset b_{\psi_1} \wedge b'_{\psi_1} = 0$, we have

$$\phi(b_{\psi_1} \vee c) = \phi(b_{\psi_1}) + \phi(c).$$

Since $c \in \mathfrak{a}_{\psi_2} \subset \mathfrak{a}_{\psi_1}$, we have $b_{\psi_1} \vee c \in \mathfrak{a}_{\psi_1}$. Hence, $\phi(b_{\psi_1})$ being the least upper bound of $\phi(a)$ for all $a \in \mathfrak{a}_{\psi_1}$,

$$\phi(b_{\psi_1} \vee c) = \phi(b_{\psi_1}).$$

Consequently

$$\phi(c) = 0.$$

Next, assume that $\phi(b'_{\psi_1} \wedge b_{\psi_2}) = 0$. By Lemma 9.2,

$$\psi_1(a) \sim \phi^{\times}_{\psi_1}(a) = \phi(a \wedge b'_{\psi_1}), \qquad \psi_2(a) \sim \phi^{\times}_{\psi_2}(a) = \phi(a \wedge b'_{\psi_2}).$$

Now,

$$\begin{aligned} \phi(a \wedge b'_{\psi_1}) &= \phi(a \wedge b'_{\psi_1} \wedge b_{\psi_2}) + \phi(a \wedge b'_{\psi_1} \wedge b'_{\psi_2}) \\ &\leqq \phi(b'_{\psi_1} \wedge b_{\psi_2}) + \phi(a \wedge b'_{\psi_2}) = \phi(a \wedge b'_{\psi_2}). \end{aligned}$$

Consequently, if $\psi_2(a) = 0$, then $\psi_1(a) = 0$. That is, $\psi_1 < \psi_2$.

(iii) If $\psi_1 < \psi_2$ and not $\psi_1 \sim \psi_2$, then there exists an element a such that $\psi_1(a) = 0$ but $\psi_2(a) \neq 0$, and $a \wedge b_{\psi_2} = 0$. Since $\psi_2 < \phi$, $\phi(a) \neq 0$, and since $\psi_1(a) = 0$, $\psi_1(b_{\psi_1} \vee a) = 0$. Hence, by the property of b_{ψ_1} we have

$$\phi(b_{\psi_1} \vee a) = \phi(b_{\psi_1}). \tag{1}$$

From $\psi_1 < \psi_2$, by (ii), $\phi(b'_{\psi_1} \wedge b_{\psi_2}) = 0$. Hence

$$\phi(b_{\psi_1} \wedge b_{\psi_2}) = \phi(b_{\psi_2}). \tag{2}$$

Now $\phi(a) \neq 0$ and $a \wedge b_{\psi_2} = 0$; we have, by (1) and (2),

$$\begin{aligned} \phi(b_{\psi_2}) < \phi(b_{\psi_2}) + \phi(a) &= \phi(b_{\psi_2} \vee a) \leqq \phi(b_{\psi_1} \vee b_{\psi_2} \vee a) \\ &= \phi(b_{\psi_2}) + \phi(b_{\psi_1} \vee a) - \phi\big(b_{\psi_2} \wedge (b_{\psi_1} \vee a)\big) \\ &= \phi(b_{\psi_2}) + \phi(b_{\psi_1}) - \phi(b_{\psi_2} \wedge b_{\psi_1})^{(2)} = \phi(b_{\psi_1}). \end{aligned}$$

Consequently, if $\phi(b_{\psi_1}) = \phi(b_{\psi_2})$, then it must follow that $\psi_1 \sim \psi_2$.

Direct Proof of Theorem 6.1. (i) It is evident that $\mathfrak{H}_{\aleph_1}$ is a partially ordered set with respect to the order $<$.

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(1) If L is a σ-field of sets E, then $\phi(E)$, $\psi(E)$ are completely additive set functions, and when $\psi < \phi$, there exists a point function $f(x)$ such that $\psi(E) = \int_E f(x) d\phi(E)$. In this case, b_ψ corresponds to the set of all points x such that $f(x) = 0$.

(2) For $b_{\psi_2} \wedge (b_{\psi_1} \vee a) = (b_{\psi_2} \wedge b_{\psi_1}) \vee (b_{\psi_2} \wedge a) = b_{\psi_2} \vee b_{\psi_1}$.

(ii) Let $S=(\phi_i;\ i=1,2,\ldots.)$ be a denumerable subset of $\mathfrak{F}_{\aleph_1}$. Put $\phi(a)=\sum_i \frac{\phi_i(a)}{2^i\phi_i(1)}$; then $\phi(a)\in\mathfrak{F}_{\aleph_1}$. And $\phi(a)=0$ when, and only when, $\phi_i(a)=0$ for all i. Hence

$$\mathfrak{a}_\phi=\Pi(\mathfrak{a}_{\phi_i};\ i=1,2,\ldots.) \quad \text{and} \quad \phi\sim\sum(\phi_i;\ i=1,2,\ldots.).$$

Especially $$\phi_1\vee\phi_2\sim\phi_1+\phi_2. \tag{1}$$

Let $S=(\phi_i;\ i=1,2,\ldots.)$ be a denumerable subset of $\mathfrak{F}_{\aleph_1}$. Then, by Theorem 2.1, $\mathfrak{b}=\sum(\mathfrak{a}_{\phi_i};\ i=1\,2,\ldots.)$ is an $\aleph_1$-ideal in L, and $\mathfrak{b}\supset\mathfrak{a}_{\phi_1}$. By Lemma 9.1, there exists a function $\psi\in\mathfrak{F}_{\aleph_1}$, such that $\mathfrak{b}=\mathfrak{a}_\psi$. And

$$\psi\sim\Pi(\phi_i;\ i=1,2,\ldots.).^{(1)}$$

(iii) Let $\phi\in\mathfrak{F}_{\aleph_1}$ and $S=(\phi_i;\ i=1,2,\ldots.)$ be any denumerable subset of $\mathfrak{F}_{\aleph_1}$. Then, by Theorem 2.1,

$$\mathfrak{a}_\phi\wedge\sum(\mathfrak{a}_{\phi_i};\ i=1,2,\ldots.)=\sum(\mathfrak{a}_\phi\wedge\mathfrak{a}_{\phi_i};\ i=1,2,\ldots.).$$

This means that $\phi\vee\Pi(\phi_i;\ i=1,2,\ldots.)\sim\Pi(\phi\vee\phi_i;\ i=1,2,\ldots.).^{(2)}$

Similarly, $\phi\wedge\sum(\phi_i;\ i=1,2,\ldots.)\sim\sum(\phi\wedge\phi_i;\ i=1,2,\ldots.)$.

Thus the distributive law in the generalized sense holds good.

(iv) Let ϕ, ψ be such that $\psi\prec\phi$. Then, by Lemma 9.2, there exist $\phi_\psi^\times(a)$, $\phi_\psi^{\times\times}(a)$ such that

$$\phi(a)=\phi_\psi^\times(a)+\phi_\psi^{\times\times}(a),$$

and $\phi_\psi^\times\sim\psi$, $\phi_\psi^\times\wedge\phi_\psi^{\times\times}\sim0$. By (1), we have

$$\psi\vee\phi_\psi^{\times\times}\sim\phi, \qquad \psi\wedge\phi_\psi^{\times\times}\sim0. \tag{2}$$

Hence $\phi_\psi^{\times\times}$ is the inverse of ψ in ϕ.

When $\chi\prec\psi\prec\phi$. Put $\xi=\phi_\psi^{\times\times}+\chi$. Then we have, from (2),

$$\phi\sim\phi\vee\chi\sim(\psi\vee\phi_\psi^{\times\times})\vee\chi\sim\psi\vee\xi,$$

and $$\chi\sim0\vee\chi\sim(\psi\wedge\phi_\psi^{\times\times})\vee\chi\sim\psi\wedge\xi, \qquad \text{by (iii)}.$$

Hence $\mathfrak{F}_{\aleph_1}$ is complemented in the generalized sense.

Thus $\mathfrak{F}_{\aleph_1}$ is a generalized $\aleph_1$-Boolean algebra with zero element.

(v) To prove that class $\Psi_{\aleph_1}$ of all $\aleph_1$-ideals in $\mathfrak{F}_{\aleph_1}$ is a continuous Boolean algebra, by Theorem 2.4 it is sufficient to show that every $\aleph_1$-ideal $\mathfrak{B}$ in $\mathfrak{F}_{\aleph_1}$ contained in a principal $\aleph_1$-ideal $\mathfrak{a}(\phi)$ is also principal.

If $\psi\in\mathfrak{B}$, then $\psi\prec\phi$. Let α be the greatest lower bound of $\phi(b_\psi)$ for all $\psi\in\mathfrak{B}$. Then there exists a sequence $(\psi_i;\ i=1,2,\ldots.)$ such that $\psi_i\in\mathfrak{B}$ and $\lim_{i\to\infty}\phi(b_{\psi_i})=\alpha$. Put $\sum(\psi_i;\ i=1,2,\ldots.)=\chi$. Then $\chi\in\mathfrak{B}$ and $\psi_i\prec\chi$. Hence $\phi(b_{\psi_i})\geqq\phi(b_\chi)$ for all i. Consequently $\alpha\geqq\phi(b_\chi)$. On ther other hand, since $\chi\in\mathfrak{B}$, by the definition of α $\phi(b_\chi)\geqq\alpha$. Therefore $\phi(b_\chi)=\alpha$.

(1) This proof holds good for any subset S of $\mathfrak{F}_{\aleph_1}$. Hence $\mathfrak{F}_{\aleph_1}$ is closed with respect to the meet of any subset of $\mathfrak{F}_{\aleph_1}$.

(2) This relation holds good when the power of S is $\geqq\aleph_1$. Cf. Theorem 2.1, and footnote above.

Next, let ψ be any element in $\mathfrak{B}$. And put $\xi(\alpha)=\chi(\alpha)+\psi(\alpha)$. From $\xi\in\mathfrak{B}$, we have $\phi(b_\xi)\geqq\alpha$. And from $\xi>\chi$, we have $\phi(b_\xi)\leqq\phi(b_\chi)=\alpha$. Consequently $\phi(b_\xi)=\phi(b_\chi)$. Hence, by Lemma 9.3 (iii), $\xi\sim\chi$. Since $\xi\sim\chi\vee\psi$, we have $\chi>\psi$. Therefore $\mathfrak{B}$ is a principal $\aleph_1$-ideal $\mathfrak{A}(\chi)$.

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