MATROID LATTICES OF INFINITE LENGTH

By

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1. In a lattice L, consider the following conditions:

L7R. Given $a \leqq x \leqq b$, y exists with $x \cup y = b$, $x \cap y = a$.

L7R$'$. Every element of L is the join of points.

(ξ') If a and b cover c, and $a \neq b$, then $a \cup b$ covers a and b.

(η') If p, q are points, and $a < a \cup q \leqq a \cup p$, then $a \cup p = a \cup q$.

(η'') If p is a point, then either $p \leqq a$ or $a \cup p$ covers a.

(δ) For points $p_1, \dots, p_n$, the condition that $(p_1 \cup \dots \cup p_k) \cup p_{k+1} = 0$ for $k = 1, \dots, n-1$ is invariant under all permutations of the p_i.

A *relatively complemented* lattice is a lattice which satisfies L7R, and a lattice is called *semi-modular* if it satisfies (ξ').[1] A relatively complemented (or relatively atomic[2]) semi-modular lattice of finite length is called a *matroid lattice*. Then the following theorem is already known.[3]

THEOREM. *A lattice L of finite length is a matroid lattice, if and only if it satisfies* (A) L7R *or* L7R$'$, *and* (B) *either* (ξ') *or* (η') *or* (η'') *or* (δ).

The object of this paper is to extend this theorem to the case of infinite length. That is, defining a *matroid lattice* as a relatively atomic, upper continuous, semi-modular lattice, the above theorem holds also for an atomic, upper continuous lattice L. And MacLane's [1] "exchange lattices" are equivalent to this "matroid lattices".

2. In this section, we give some definitions and lemmas which are already known.

DEFINITION 1. For any element $a (\neq 0)$ of a lattice L with 0, if there exists a point p such that $p \leqq a$, then L is called *atomic*. If $a < b$ implies $a < a \cup p \leqq b$ for some point p, then L is called *relatively atomic*. It is evident that an atomic, relatively complemented lattice L is relatively atomic.

1) In Birkhoff [1, p. 100], a semi-modular lattice is defined as a lattice of finite length which satisfies (ξ'). But we shall delete the condition "of finite length". The numbers in square brackets refer to the list of references at the end of this paper.

2) Cf. Lemma 1.

3) Cf. Birkhoff [1] 106 Theorem 7.

LEMMA 1. *A lattice with 0 is relatively atomic, if and only if L satisfies* L7R′.

PROOF. Cf. Maeda [1] 88 Lemma 1·1.

DEFINITION 2. Let $\{a_\delta; \delta \in D\}$ be a directed set of a complete lattice L. When

$$a_\delta \uparrow a \quad \text{implies} \quad a_\delta \frown b \uparrow a \frown b,$$

we say that L is an *upper continuous* lattice.

LEMMA 2. *In a relativly atomic, compete lattice L, the following two propositions* (α) *and* (β) *are equivalent:*

(α) *L is upper continuous.*

(β) *Let p be a point and P a set of points in L. Then $p \leqq \bigvee(P)$ implies $p \leqq q_1 \smile \dots \smile q_n$ where each q_i is in P.*

PROOF. Cf. Maeda [1] 90 Lemma 1·3.

LEMMA 3. *Let p, r_i are points in a semi-modular lattice L with 0. Then*

$$(\eta_0''') \quad p \not\leqq \bigvee_{i=1}^{n} r_i \smile b \quad \textit{implies} \quad \bigvee_{i=1}^{n} r_i \frown b = (\bigvee_{i=1}^{n} r_i \smile p) \frown b.$$

PROOF. Cf. Sasaki and Fujiwara [1] Lemma 1.

3. DEFINITION 3. Let S be a system of elements of a complete lattice L. If $\bigvee(S_1) \frown \bigvee(S_2) = 0$ for any two disjoint subsets S_1 and S_2 of S, then we say that S is an *independent system* and we write $(a; a \in S) \perp$. Similarly in any lattice L with 0, we can define an independent system for finite subsets $\{a_1, \dots, a_n\}$, and we write $(a_1, \dots, a_n) \perp$.

An independent system P of points with $\bigvee(P) = a$ is called a *basis* of the element a of L.

LEMMA 4. *In an upper continuous lattice L, a set S of elements of L is an independent system, if and only if every finite subset ν of S is an independent system.*

PROOF. Necessity is evident. To prove the sufficiency, let S_1, S_2 be any two disjoint subsets S. Let ν_1 and ν_2 be any finite subsets of S_1 and S_2 respectively. If we put $a_{\nu_1} = \bigvee(\nu_1)$, $a_{\nu_2} = \bigvee(\nu_2)$, $a_{\nu_1} \uparrow \bigvee(S_1)$, $a_{\nu_2} \uparrow \bigvee(S_2)$. Since the set sum $\nu_1 \bigcup \nu_2$ is an independent system, we have $a_{\nu_1} \frown a_{\nu_2} = 0$ for all ν_2. L being upper continuous, we have $a_{\nu_1} \frown \bigvee(S_2) = 0$. Similarly we have $\bigvee(S_1) \frown \bigvee(S_2) = 0$, and S is an independent system.

LEMMA 5. *Let $p_1, \dots, p_n$ be points of a semi-modular lattice L with 0. Then $(p_1, \dots, p_n) \perp$ if and only if*

$$(p_1 \smile \dots \smile p_k) \frown p_{k+1} = 0 \quad \textit{for} \quad k = 1, \dots, n-1.$$

PROOF. Cf. Sasaki and Fujiwara [1] Lemma 2.

LEMMA 6. *If P is an independent system of points in an upper continuous, semi-modular lattice L, and if q is a point with $q\frown\bigvee(p\,;p\in P)=0$, then the set obtained by adjoining q to P is an independent spstem.*[1]

PROOF. For any finite subset ν of P, by Lemma 5 the set union $\nu\bigcup\{q\}$ is an independent system. Hence by Lemma 4 $P\bigcup\{q\}$ is an independent system.

LEMMA 7. *If L is a relatively atomic, upper continuous, semi-modular lattice and if P is any independent system of points with $\bigvee(P)\leqq a$, then there is a set $Q\supset P$ which is a basis of a. In particular, every element of L has a basis.*[2]

PROOF. Let S be the set of all points contained in a. Then by Lemma 1 $a=\bigvee(S)$. By Zorn's lemma there exists a maximal independent subset Q of S with $Q\supset P$. If there exists a point $p\in S$ such that $\bigvee(Q)\frown p=0$, then by Lemma 6 $Q\bigcup\{p\}$ is an independent subset of S such that $Q\bigcup\{p\}\supset P$, which contradicts the property of Q. Hence $p\leqq\bigvee(Q)$ for all $p\in S$. Therefore $\bigvee(Q)=a$.

LEMMA 8. *Let P be a finite independent system of points p_i in a semi-modular lattice L with 0. Then the p_i generate a sublattice isomorphic with the field of all subsets of P.*[3]

PROOF. Let S and T be any subsets of P. Then it is evident

$$\bigvee(S)\smile\bigvee(T)=\bigvee(S\bigcup T).$$

To prove

$$\bigvee(S)\frown\bigvee(T)=\bigvee(S\bigcap T), \tag{1}$$

let $S-S\bigcap T$ be a set of points $p_1,\dots,p_n$. When $n=0$, since $S\subset T$, (1) is evident. Now assume that (1) holds when $n=k$, that is

$$\left\{\bigvee(S\bigcap T)\smile p_1\smile\dots\smile p_k\right\}\frown\bigvee(T)=\bigvee(S\bigcap T). \tag{2}$$

Put $a=\bigvee(S\bigcap T)\smile p_1\smile\dots\smile p_k$, $b=\bigvee(T)$, since $p_{k+1}\nleqq a\smile b$, by Lemma 3 $a\frown b=(a\smile p_{k+1})\frown b$. Hence from (2) we have

$$\bigvee(S\bigcap T)=\left\{\bigvee(S\bigcap T)\smile p_1\smile\dots\smile p_{k+1}\right\}\frown\bigvee(T).$$

That is, (2) holds for $n=k+1$. Therefore by induction we have (1). Associate with each subset S of P, $\bigvee(S)$, then $\bigvee(S)$ form a sublattice

1) Cf. MacLane [1] 458 Corollary.

2) Cf. MacLane [1] 458 Theorem 3.

3) This lemma is proved by Birkhoff [1, p. 104 Theorem 5], when L is a semi-modular lattice of finite length using the dimension function $d[a]$.

homomorphic to the field of all subsets of P. If there exists a point $p\in P$ such that $p\in S$, $p\notin T$, then $p\leqq\bigvee(S)$ and $p\frown\bigvee(T)=0$, hence $S\neq T$ implies $\bigvee(S)\neq\bigvee(T)$. Therefore the above homomorphism is an isomorphism.

THEOREM 1. *Let P be a (finite or infinite) independent system of points p_ι in a relatively atomic, upper continuous, semi-modular lattice L. Then the p_ι generate a sublattice isomorphic with the field of all subsets of P.*

PROOF. Let S and T be any two subsets of P. Then it is evident

$$\bigvee(S)\smile\bigvee(T)=\bigvee(S\cup T).$$

When $p\leqq\bigvee(S)\frown\bigvee(T)$, since $p\leqq\bigvee(S)$ and $p\leqq\bigvee(T)$, by Lemma 2 there are finite subsets ν and μ of S and T respectively, and $p\leqq\bigvee(\nu)$ and $p\leqq\bigvee(\mu)$. Hence $p\leqq\bigvee(\nu)\frown\bigvee(\mu)$. Then by lemma 8, we have $p\leqq\bigvee(\nu\cap\mu)$. Since $\nu\cap\mu$ is a finite subset of $S\cap T$, we have $\bigvee(S)\frown\bigvee(T)\leqq\bigvee(S\cap T)$. But $\bigvee(S)\frown\bigvee(T)\geqq\bigvee(S\cap T)$ is evident, we have $\bigvee(S)\frown\bigvee(T)=\bigvee(S\cap T)$. Hence associating with each subset S of P, $\bigvee(S)$, as Lemma 8, $\bigvee(S)$ form a sublattice isomorphic to the field of all subsets of P.

4. THEOREM 2. *In a lattice L with 0, consider the following propositions:*

(ξ') *If a and b cover c, and $a\neq b$, then $a\smile b$ covers a and b.*

(ξ'') *If a covers $a\frown b$, then $a\smile b$ covers b.*

(η') *If p,q are points, and $a<a\smile q\leqq a\smile p$, then $a\smile p=a\smile q$.*

(η'') *If p is a point, then either $p\leqq a$ or $a\smile p$ covers a.*

(η''') *If p is a point and $p\nleqq a\smile b$, then $a\frown b=(a\smile p)\frown b$.*

(δ) *For any points $p_1,\dots,p_n$ the condition that $(p_1\smile\dots\smile p_k)\frown p_{k+1}=0$ for $k=1,\dots,n-1$ is invariant under all permutations of the p_ι.*

In any lattice L, $(\xi'')\to(\xi')\to(\delta)$, $(\xi'')\to(\eta'')\to(\eta')$, $(\eta'')\rightleftarrows(\eta''')$.

In a relatively atomic lattice L, $(\xi''),(\eta'),(\eta''),(\eta''')$ are equivalent.

In a relatively atomic, upper continuous lattice L, $(\xi'),(\xi''),(\eta'),(\eta''),(\eta''')$, (δ) are all equivalent.

PROOF. (i) In any lattice L, if a and b cover c, and $a\neq b$, then $c\leqq a\frown b<a$ and $c=a\frown b$. Hence we have $(\xi'')\to(\xi')$. By Lemma 5 we have $(\xi')\to(\delta)$. When $p\nleqq a$, since p covers $p\frown a=0$, by (ξ'') $a\smile p$ covers a. Thus we have $(\xi'')\to(\eta'')$. $(\eta'')\to(\eta')$ is evident. $(\eta'')\rightleftarrows(\eta''')$ is proved by Menger [1, p. 460].

(ii) In a relatively atomic lattice L, we shall prove $(\eta')\to(\xi'')$. Since a covers $a\frown b$, there exists a point p such that $a=(a\frown b)\smile p$. Assume that there exists an element c such that $b<c<a\smile b$, then there exists a point q such that

$$b<b\smile q\leqq c<a\smile b=(a\frown b)\smile p\smile b=b\smile p,$$

which cnotradicts (η'). Hence $(\eta')\rightarrow(\xi'')$. Consequently, by (i), (ξ''), (η'), (η''), (η''') are equivalent in a relatively atomic lattice.

(iii) In a relatively atomic, upper continuous lattice L, we shall prove $(\delta)\rightarrow(\eta')$. Let $a<a\smile q\leqq a\smile p$. Since $q\leqq a\smile p$, by Lemma 1 and Lemma 2, there exist points $r_i\leqq a$, such that

$$q\leqq r_1\smile \dots \smile r_n\smile p\,. \tag{1}$$

If $r_{k+1}\leqq r_1\smile \dots \smile r_k$, we can delete r_{k+1} in (1). Hence we may assume that

$$(r_1\smile \dots \smile r_k)\frown r_{k+1}=0 \quad (k=1,\dots,n-1)\,.$$

Since $a\frown q=0$, we have $(r_1\smile \dots \smile r_n)\frown q=0$. If $(r_1\smile \dots \smile r_n\smile q)\frown p=0$ by (δ) we have $(r_1\smile \dots \smile r_n\smile p)\frown q=0$, which contradicts (1). Hence $p\leqq r_1\smile \dots \smile r_n\smile q\leqq a\smile q$. And we have $a\smile p=a\smile q$.

Consequently by (i), (ii) and (iii), in a relatively atomic, upper continuous lattice L, all six propositions are equivalent.

LEMMA 9. *If L is a relatively atomic, upper continuous lattice which satisfies (η'), then L is a relatively complemented lattice.*

PROOF. By Lemma 2, a relatively atomic, upper continuous lattice which satisfies (η') is equivalent to an "exchange lattice" defined by MacLane [1, 456], hence this lemma is MacLane [1], 458 Theorem 7.

DEFINITION 4. A relatively atomic, upper continuous, semi-modular lattice is called a *matroid lattice.*[1)]

THEOREM 3. *An atomic, upper continuous lattice L is a matroid lattice, if and only if it satisfies* (A) L7R *or* L7R′, *and* (B) *either* (ξ') *or* (ξ'') *or* (η') *or* (η'') *or* (η''') *or* (δ).

PROOF. By Theorem 2, in a relatively atomic (i.e. L7R′), upper continuous lattice, the six propositions in (B) are equivalent. Since an atomic, relatively complemented lattice is relatively atomic, in an atomic, relatively complemented (i.e. L7R), upper continuous lattice, the six propositions in (B) are equivalent. By Lemma 9, in an atomic, upper continuous lattice which satisfies (η'), L7R and L7R′ are equivalent, completing the proof.

1) By Theorem 3, "matroid lattice" and "exchange lattice" are equivalent. When L is of finite length, Definition 4 coincides with the definition given by Birkhoff [1, p. 106].

References.

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