ON THE AUTOMORPHISMS OF THE SET OF SIMPLE VECTORS

By

Kakutarô MORINAGA and Takayuki NÔNO

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§ 1. Introduction.

In this paper we shall consider the automorphisms of the set of simple r-vectors $v_{i_1 i_2 \dots i_r}$ in the real or complex vector space V_n of n dimensions. Under an r-vector we understand a skew-symmetric tensor $v_{i_1 i_2 \dots i_r} = v_{[i_1 i_2 \dots i_r]} \neq 0$. An r-vector $v_{i_1 i_2 \dots i_r}$ is said to be simple when it can be written by some r linearly independent vectors $\overset{1}{v}, \overset{2}{v}, \dots, \overset{r}{v}$ as follows:

$$v_{i_1 i_2 \dots i_r} = \overset{1}{v}_{[i_1} \overset{2}{v}_{i_2} \dots \overset{r}{v}_{i_r]} \ . \tag{1.1}$$

And a necessary and sufficient condition that an r-vector $v_{i_1 i_2 \dots i_r}$ shall be simple is that

$$v_{[i_1 i_2 \dots i_r} v_{j_1] j_2 \dots j_r} = 0 \, .^{(1)} \tag{1.2}$$

In the following we shall prove a

THEOREM. *Any automorphisms of the set of simple r-vectors:*

$$'w_{j_1 j_2 \dots j_r} = p^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r} w_{i_1 i_2 \dots i_r} \tag{1.3}$$

is reduced to the following:

(1) *In the case where $2r \neq n$*

$$p^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r} = v^{i_1}_{[j_1} v^{i_2}_{j_2} \dots v^{i_r}_{j_r]} \ . \tag{1.4}$$

(2) *In the case where $2r = n$*

$$p^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r} = v^{i_1}_{[j_1} v^{i_2}_{j_2} \dots v^{i_r}_{j_r]} \tag{1.4}$$

or

$$= \varepsilon^{k_1 k_2 \dots k_r l_1 l_2 \dots l_r} g_{l_1 j_1} g_{l_2 j_2} \dots g_{l_r j_r} v^{i_1}_{[k_1} v^{i_2}_{k_2} \dots v^{i_r}_{k_r]} \ . \tag{1.5}$$

The theorem means that any automorphisms of the set of simple r-vectors:

$$'w_{j_1 j_2 \dots j_r} = p^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r} w_{i_1 i_2 \dots i_r}$$

(1) Cf. J. A. Schouten: Der Ricci-Kalkul (1924) Berlin. p. 53.

can be reduced to an automorphisms of the vector space V_n: $'w_j = v^i_j w_i$, or an automorphisms: $'w_j = v^i_j w_i$ and ε-mapping: $'\overline{w}_{j_1\dots j_r} = \varepsilon^{k_1\dots k_r l_1\dots l_r} g_{l_1 j_1} \dots g_{l_r j_r} \overline{w}_{k_1\dots k_r}$.
Moreover an $(r-1)$-dimensional linear subspace S_{r-1} of the real or complex projective space S_{n-1} of $n-1$ dimensions is considered as a simple r-vector in the space V_n. (Prücker's coordinates). Therefore our theorem is stated as follows:

THEOREM. *Any automorphisms of the set of S_{r-1} in the S_{n-1} can be reduced to a generalized*[1] *point transformation of S_{n-1}.*

§ 2. The classification of automorphisms.

If an r-vector $v_{i_1 i_2\dots i_r}$ is simple, its dual $(n-r)$-vector $v^{i_{r+1}\dots i_n} = \varepsilon^{i_1\dots i_n} v_{i_1 i_2\dots i_r}$ is simple, and conversely. So[2] it is enough to prove this theorem for the case where $r \leqq n-r$ i.e. $2r \leqq n$.

Let us take as the simple r-vector $v_{i_1\dots i_r}$ in (1.3), specially

$$\overset{1}{v}_{i_1 i_2\dots i_r} = \overset{1}{e}_{[i_1}\overset{2}{e}_{i_2}\dots\overset{r}{e}_{i_r]} \quad\text{and}\quad \overset{2}{v}_{i_1 i_2\dots i_r} = \overset{1}{e}_{[i_1}\dots\overset{r-1}{e}_{i_{r-1}}\overset{r+1}{e}_{i_r]}$$

where $\overset{\lambda}{e}_i$ denotes the covariant mass vector, then

$$\underset{1}{a}\overset{1}{v}_{i_1\dots i_r} + \underset{2}{a}\overset{2}{v}_{i_1\dots i_r} = \overset{1}{e}_{[i_1}\dots\overset{r-1}{e}_{i_{r-1}}\,\overset{r}{'e}_{i_r]}, \qquad (\overset{r}{'e}_{i_r} = \underset{1}{a}\overset{r}{v}_{i_r} + \underset{2}{a}\overset{r+1}{v}_{i_r})$$

that is, $\underset{1}{a}\overset{1}{v}_{i_1\dots i_r} + \underset{2}{a}\overset{2}{v}_{i_1\dots i_r}$ is a simple r-vector, accordingly $\underset{1}{a}p^{1\dots r-1\,r}_{j_1\dots j_r} + \underset{2}{a}p^{1\dots r-1\,r+1}_{j_1\dots j_r}$ is a simple r-vector for all $\underset{1}{a}, \underset{2}{a}$. By applying the condition (1.2) for this simple r-vector, we obtain

$$p^{12\dots r}_{[j_1 j_2\dots j_r} p^{12\dots r-1\,r+1}_{k_1]\,k_2\dots k_r} + p^{1\dots r-1\,r+1}_{[j_1\dots j_r} p^{12\dots r}_{k_1]\,k_2\dots k_r} = 0. \tag{2.1}$$

From this condition (2.1) we know that the simple r-vectors $p^{12\dots r}$[3] and $p^{12\dots r-1\,r+1}$ have a simple $(r-1)$-vector in common. In fact, let us suppose that these two simple r-vectors have a simple t-vector $(t \leqq r-2)$ in common only, then by a coördinate transformation we can take as follows:

$$\begin{aligned} p^{12\dots r}_{j_1 j_2\dots j_r} &= \overset{1}{e}_{[j_1}\overset{2}{e}_{j_2}\dots\overset{r}{e}_{j_r]} \\ p^{12\dots r-1\,r+1}_{j_1 j_2\dots j_{r-1} j_r} &= \overset{1}{e}_{[j_1}\dots\overset{t}{e}_{j_t}\overset{r+1}{e}_{j_{t+1}}\dots\overset{2r-t}{e}_{j_r]}. \end{aligned} \tag{2.2}$$

Since $t \leqq r-2$; $2r-t,\ 2r-t-1 \neq 1, 2, \dots, r$. If we substitute (2.2) for (2.1) and take $2r-t$ and $2r-t-1$ as k_1 and k_2 respectively, then we have from

(1) We shall denote (1.4) together with (1.5) as a generalized point transformation.

(2) Any automorphisms of the set of simple r-vectors corresponds to one and only one automorphisms of the set of simple $(n-r)$-vectors, hence we have only to consider the case for min $(r, n-r)$.

(3) For the sake of simplicity, we write $p^{12\dots r}$ for $p^{12\dots r}_{j_1 j_2\dots j_r}$ in the case where any confusion does not occur.

$p^{12\cdots r-1\,r}_{l_1 l_2\ldots l_{r-1}\,2r-t}=0$ and $p^{12\cdots r}_{l_1\,2r-t-1\,l_3\ldots l_r}=0$ (for all l's)

$$\overset{1}{e}_{[j_1}\overset{2}{e}_{j_2}\ldots\overset{r}{e}_{j_r]}\cdot\overset{1}{e}_{[2r-t}\overset{2}{e}_{2r-t-1}\overset{3}{e}_{k_3}\ldots\overset{t}{e}_{k_t}\overset{r+1}{e}_{k_{t+1}}\ldots\overset{2r-t}{e}_{k_r]}=0. \tag{2.3}$$

But we see that (2.3) is not true, by putting $(j_1\ldots j_r)=(12\ldots r)$ and $(k_3\ldots k_r)=(1,\ldots,t,r+1,\ldots,2r-t-2)$. Hence we obtain $r-1\leqq t\leqq r$. On the other hand we deduce $p^{1\cdots r}_{j_1\ldots j_r}\not\approx p^{1\cdots r-1\,r+1}_{j_1\ldots j_r}$ [1] from $\overset{1}{v}_{i_1\ldots i_r}\not\approx\overset{2}{v}_{i_1\ldots i_r}$ by the automorphisms (1.3), consequently we have $t<r$. Therefore we get $t=r-1$.

Thus, generally we know that *two simple r-vectors $p^{i_1\ldots i_{r-1}i_k}$ and $p^{i_1\ldots i_{r-1}i_s}$ have a simple $(r-1)$-vector in common.*

Now for the sake of simplicity we write

$$p^{1\cdots r-1\,s}_{j_1\cdots j_r}=p^s \qquad (s=r,\ r+1,\ldots,n). \tag{2.4}$$

We shall show that *the sequence p^s $(s=r, r+1,\ldots,n)$ has either of following two types*:

(I) *All p^s $(s=r,r+1,\ldots,n)$ have the same simple $(r-1)$-vector in common.*

(II) *All simple $(r-1)$-vectors which are the intersection of p^s and p^k $(s\neq k;\ s,\ k=r,\ r+1,\ldots,n)$ are distinct.*

And moreover *in the case of type* (II), we shall see that p^s $(s=r,r+1,\ldots,n)$ *are all contained in the same simple $(r+1)$-vector, and $2r=n$* [2].

If p^r, p^{r+1} and p^{r+2} have a simple $(r-1)$-vector in common, we can write as follows:

$$\left.\begin{aligned} p^r &= \overset{1}{v}_{[j_1}\overset{2}{v}_{j_2}\ldots\overset{r-1}{v}_{j_{r-1}}\overset{r}{v}_{j_r]}\\ p^{r+1} &= \overset{1}{v}_{[j_1}\overset{2}{v}_{j_2}\ldots\overset{r-1}{v}_{j_{r-1}}\overset{r+1}{v}_{j_r]}\\ p^{r+2} &= \overset{1}{v}_{[j_1}\overset{2}{v}_{j_2}\ldots\overset{r-1}{v}_{j_{r-1}}\overset{r+2}{v}_{j_r]}.\end{aligned}\right\} \tag{2.5}$$

Since p^r, p^{r+1} and p^{r+2} are independent,[3] $r+2$ vectors $\overset{1}{v}, \overset{2}{v},\ldots,\overset{r+2}{v}$ must be independent. Now let us suppose that p^{r+3} does not contain this simple $(r-1)$-vector, then we can write p^{r+3}, since p^r and p^{r+3} have a simple $(r-1)$-vector in common, as follows:

$$p^{r+3}={}'\overset{1}{v}_{[j_1}\ldots{}'\overset{r-1}{v}_{j_{r-1}}{}'\overset{r}{v}_{j_r]}$$

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(1) $A\approx B$ denotes that $A=\rho B$ where ρ is a proportional factor.

(2) Some parts of these results have been already obtained from the other point of view. But we shall explain by the tensor calculation.

(3) In this paper, "independence" means "linearly independence".

where $\qquad {}'\overset{r}{v} = \underset{1}{a}\overset{1}{v}+\dots+\underset{r}{a}\overset{r}{v}\ (\underset{r}{a}\neq 0);\ {}'\overset{1}{v},\dots,{}'\overset{r-2}{v}\in \mathfrak{L}(\overset{1}{v},\overset{2}{v},\dots,\overset{r-1}{v})$ [1].

By the similar consideration for the pairs p^{r+1}, p^{r+3} and p^{r+2}, p^{r+3}, we can write p^{r+3} as follows:

$$p^{r+3} = {}''\overset{1}{v}_{[j_1}{}''\overset{2}{v}_{j_2}\dots{}''\overset{r-1}{v}_{j_{r-1}}{}''\overset{r+1}{v}_{j_r]}$$

where $\qquad {}''v = \underset{1}{b}\overset{1}{v}+\dots+\underset{r-1}{b}\overset{r-1}{v}+\underset{r+1}{b}\overset{r+1}{v}\ (\underset{r+1}{b}\neq 0);\ {}''\overset{1}{v},\dots,{}''\overset{r-2}{v}\in \mathfrak{L}(\overset{1}{v},\overset{2}{v},\dots,\overset{r-1}{v})$;

and too

$$p^{r+3} = {}'''\overset{1}{v}_{[j_1}{}'''\overset{2}{v}_{j_2}\dots{}'''\overset{r-1}{v}_{j_{r-1}}{}'''\overset{r+2}{v}_{j_r]}$$

where $\qquad {}'''\overset{r+2}{v} = \underset{1}{c}\overset{1}{v}+\dots+\underset{r-1}{c}\overset{r-1}{v}+\underset{r+2}{c}\overset{r+2}{v}\ (\underset{r+2}{c}\neq 0);\ {}'''\overset{1}{v},\dots,{}'''\overset{r-2}{v}\in \mathfrak{L}(\overset{1}{v},\overset{2}{v},\dots,\overset{r-1}{v})$.

Hence p^{r+3} contains at least $(r+1)$ linearly independent vectors ${}'\overset{1}{v}, {}'\overset{2}{v},\dots, {}'\overset{r-2}{v}, {}''\overset{r+1}{v}$ and ${}'''\overset{r+2}{v}$. This contradicts the fact that p^{r+3} is simple. Therefore p^r, p^{r+1}, p^{r+2}, and p^{r+3} have the same simple $(r-1)$-vector in common.

Thus we can conclude that the sequence p^s $(s=r, r+1,\dots, n)$ has either of types (I) or (II).

Furthermore, in the case of type (II), if p^r and p^{r+1} are written as follows:

$$p^r = \overset{1}{v}_{[j_1}\dots\overset{r-1}{v}_{j_{r-1}}\overset{r}{v}_{j_r]}$$

$$p^{r+1} = \overset{1}{v}_{[j_1}\dots\overset{r-1}{v}_{j_{r-1}}\overset{r+1}{v}_{j_r]}$$

then p^k $(k\geqq r+2)$ does not contain the simple $(r-1)$-vector $\overset{1}{v}_{[j_1}\overset{2}{v}_{j_2}\dots\overset{r-1}{v}_{j_{r-1}]}$ and the intersections $p^k\bigwedge p^r$ and $p^k\bigwedge p^{r+1}$ are distinct simple $(r-1)$-vectors; accordingly p^k is written as

$$p^k = {}'\overset{1}{v}_{[j_1}\dots{}'\overset{r-2}{v}_{j_{r-2}}{}'\overset{r}{v}_{j_{r-1}}{}'\overset{r+1}{v}_{j_r]}$$

where $\quad {}'\overset{r}{v} = \underset{1}{a}\overset{1}{v}+\dots+\underset{r-1}{a}\overset{r-1}{v}+\underset{r}{a}\overset{r}{v},\ (\underset{r}{a}\neq 0);\ {}'\overset{r+1}{v} = \underset{1}{b}\overset{1}{v}+\dots+\underset{r-1}{b}\overset{r-1}{v}+\underset{r+1}{b}\overset{r+1}{v},\ (\underset{r+1}{b}\neq 0)$,

and ${}'\overset{1}{v},\dots,{}'\overset{r-2}{v}\in\mathfrak{L}(\overset{1}{v},\dots,\overset{r-1}{v})$. Hence p^s $(s=r, r+1,\dots, n)$ is written as

$$p^s_{j_1\dots j_r} = \overset{1}{w}_{[j_1}\dots\overset{r}{w}_{j_r]} \text{ where } \overset{\lambda}{w}\in\mathfrak{L}(\overset{1}{v}\dots\overset{r}{v}\,\overset{r+1}{v}),\ (\lambda = 1,\dots, r) \qquad (2.6)$$

and the number of independent simple r-vectors contained in $\mathfrak{L}(\overset{1}{v},\dots,\overset{r+1}{v})$ is

(1) Let $\mathfrak{L}(\overset{1}{v},\dots,\overset{r}{v})$ be the set of all linear combinations of $\overset{1}{v}, \overset{2}{v},\dots\overset{r}{v}$.

equal to $r+1$. On the other hand, $n-r+1$ simple r-vectors p^s $(s=r, r+1, \dots, n)$ are independent. Hence we have $n-r+1\leqq r+1$, i. e. $n\leqq 2r$.(1) Since we have only to prove for the case where $2r\leqq n$, we know that the type (II) occurs only in the case where $2r=n$.

Furthermore we shall prove that *the type* (I) *and* (II) *can not coexist for the system* $p^{i_1i_2\dots i_r}_{j_1j_2\dots j_r}$ $(i_1, i_2 \dots, i_r=1, 2, \dots, n)$; that is, if for a fixed indices $(i)=(1, 2, \dots, r-1)$, the sequence

$$p^{(i)r},\ p^{(i)r+1}, \dots, p^{(i)r+s}, \dots, p^{(i)n} \quad {}^{(2)} \tag{2.7}$$

has the type (I), then for the indices $((i))=(1, \dots, r-2, r+s)$, the sequence

$$p^{((i))r},\ p^{((i))r+1}, \dots, p^{((i))r-1}, \dots, p^{((i))n} \quad {}^{(3)} \tag{2.8}$$

has, too, the same type (I).

In fact, we can write $p^{(i)r}$, $p^{(i)r+1}$ and $p^{(i)r+s}$ as follows:

$$\left.\begin{aligned} p^{(i)r}_{j_1\dots j_r} &= \overset{1}{v}[j_1 \dots \overset{r-1}{v}_{j_{r-1}}\overset{r}{v}_{j_r}] \\ p^{(i)r+1}_{j_1\dots j_r} &= \overset{1}{v}[j_1 \dots \overset{r-1}{v}_{j_{r-1}}\overset{r+1}{v}_{j_r}] \\ p^{(i)r+s}_{j_1\dots j_r} &= \overset{1}{v}[j_1 \dots \overset{r-1}{v}_{j_{r-1}}\overset{r+s}{v}_{j_r}]. \end{aligned}\right\} \tag{2.9}$$

And the intersection $p^{(i)r}\bigwedge p^{((i))r}$ is a simple $(r-1)$-vector and is distinct from $\overset{1}{v}[j_1 \dots \overset{r-1}{v}_{j_{r-1}}]$. For, if $p^{((i))r}$ contains the simple $(r-1)$-vector $\overset{1}{v}[j_1 \dots \overset{r-1}{v}_{j_{r-1}}]$, then it must be the linear combination of (2.7), since the sequence (2.7) is a complete system of independent simple r-vectors containing the simple $(r-1)$-vector $\overset{1}{v}[j_1 \dots \overset{r-1}{v}_{j_{r-1}}]$; however, since the inverse images of (2.7) and $p^{((i))r}$ under our automorphisms are independent of each other, (2.7) and $p^{((i))r}$ are independent of each other. This contradicts the above.

And, since $p^{((i))}$ and $p^{((i))r-1}$ in the sequence (2.8) have a simple $(r-1)$-vector in common and $p^{(i)r+s}_{j_1\dots j_1}=-p^{((i))(r-1)}_{j_1\dots j_r}$, so $p^{((i))r}$ and $p^{(i)r+s}$ have a simple $(r-1)$-vector in common, hence by the similar consideration we know that the intersection $p^{(i)r+s}\bigwedge p^{((i))r}$ is a simple $(r-1)$-vector different from $\overset{1}{v}[j_1 \dots \overset{r-1}{v}_{j_{r-1}}]$. Therefore, by the above two facts, we can write $p^{((i))r}$ as follows:

$$p^{((i))r}_{j_1\dots j_r} = {}'\overset{1}{v}[j_1 \dots {}'\overset{r-2}{v}_{j_{r-2}}{}'\overset{r}{v}_{r-1}{}'\overset{r+s}{v}_{j_r}] \tag{2.10}$$

(1) This mdthod will be used later on.

(2) $p^{(i)r}$ denotes $p^{1\dots r-1\,r}_{j_1\dots j_{r-1}\,j_r}$.

(3) $p^{((i))r+s}=0$, so in this sequence we take $p^{((i))r-1}$ corresponding to $p^{(i)r+s}$ in 2.7.

where $\quad {}'\overset{r}{v} = a\underset{1}{\overset{1}{v}} + \dots + a\underset{r}{\overset{r}{v}}\ (\underset{r}{a} \neq 0),\ {}'\overset{r+s}{v} = b\underset{1}{\overset{1}{v}} + \dots + b\underset{r-1}{\overset{r-1}{v}} + b\underset{r+s}{\overset{r+s}{v}}\ (\underset{r+s}{b} \neq 0),$

and ${}'\overset{1}{v}, \dots, {}'\overset{r}{v} \overset{2}{\in} \mathfrak{L}(\overset{1}{v}, \dots, \overset{r-1}{v})$. Similarly we have

$$p^{((i))r+1}_{j_1 \dots j_r} = {}''\overset{1}{v}_{[j_1} \dots {}''\overset{r-2}{v}_{j_{r-2}} {}''\overset{r+1}{v}_{j_{r-1}} {}''\overset{r+s}{v}_{j_r]} \tag{2.11}$$

where $\quad {}''\overset{r+1}{v} = c\underset{1}{\overset{1}{v}} + \dots + c\underset{r-1}{\overset{r-1}{v}} + c\underset{r+1}{\overset{r+1}{v}}(\underset{r+1}{c} \neq 0),\ {}''\overset{r+s}{v} = d\underset{1}{\overset{1}{v}} + \dots + d\underset{r-1}{\overset{r-1}{v}} + d\underset{r+s}{\overset{r+s}{v}}(\underset{r+s}{d} \neq 0).$

and ${}''\overset{1}{v}, \dots, {}''\overset{r-2}{v} \in \mathfrak{L}(\overset{1}{v}, \dots, \overset{r-1}{v})$. Hence the joint $p^{((i))r} \bigcup p^{((i))r+1} \bigcup p^{((i))r-1}$ contains at least $r+2$ linearly independent vectors $\overset{1}{v}, \dots, \overset{r-1}{v}, {}'\overset{r}{v}, {}'\overset{r+s}{v}$ and ${}''\overset{r+1}{v}$. However, if the sequence (2.8) has the type (II), as we have seen in the above (2.6), then the joint $p^{((i))r} \bigcup p^{((i))r+1} \bigcup p^{((i))r-1}$ contains exactly $r+1$ linearly independent vectors. This contracts the above, hence the sequence (2.8) has the type (I).

Repeating this procedure from (i) to $((i))$, we can attain to any indices $(k)=(k_1, k_2, \dots, k_{r-1})$ from any indices $(i)=(i_1, i_2 \dots i_{r-1})$. Hence, when the sequence $p^{(i)s}\ (s \notin (i))$ has the type (I), the sequence $p^{(k)t}\ (t \notin (k))$ has also the same type (I).

Thus we obtain the classification of automorphisms (1.3) *into the type* (I) *and type* (II).

§ 3. The type (I).

First we shall show that *the intersection of linearly independent* $(r-1)$ *simple* $(r-1)$*-vector in* r*-dimensional vector space is a vector* (1).

The intersection of two simple $(r-1)$-vectors p and $\bar{p}$ in r-dimensional vector space contains a simple $r-2$-vector. In fact, suppose that this intersection is a simple $(r-t)$-vector, then the simple $(r-1)$-vectors p and $\bar{p}$ contain $(t-1)$ linearly independent vectors which are not contained in this intersection respectively, hence we have $(r-t)+(t-1)+(t-1) \leq r$, i. e., $r-2 \leq r-t$.

Similarly, the intersection of a simple $(r-1)$-vector and a simple $(r-2)$-vector in a simple r-vector contains a simple $(r-3)$-vector, and repeating this process we know that the intersection of $(r-1)$ simple $(r-1)$-vectors in the r-dimensional vector contains a vector.

If this intersection contains simple 2-vector, there exist $(r-2)$ linearly

(1) See footnote (2) p. 13.

independent vectors which are independent for these two vectors. Hence the number of independent all simple $(r-1)$-vectors which contain this simple 2-vector is equal to $(r-2)$. This contradicts our assumption.

After these preparations we shall return to our problem. In the type (I) we can write $p^{i_1\dots i_{r-2} i_r s}$, $p^{i_1\dots i_{r-3} i_{r-1} i_r s}, \dots, p^{i_2\dots i_{r-1} i_r s}$ and $p^{i_1\dots i_{r-1} s}$ $(s \neq i_1, \dots, i_r)$ as follows:[1]

$$\left.\begin{aligned} p^{i_1\dots i_{r-2} i_r s}_{j_1\dots j_r} &= p^{i_1\dots i_{r-2} i_r}_{[j_1\dots j_{r-1}} \overset{\langle r-1\rangle}{v}{}^{s}_{j_r]} \\ p^{i_1\dots i_{r-3} i_{r-1} i_r s}_{j_1\dots j_r} &= p^{i_1\dots i_{r-3} i_{r-1} i_r}_{[j_1\dots j_{r-1}} \overset{\langle r-2\rangle}{v}{}^{s}_{j_r]} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ p^{i_2\dots i_r s}_{j_1\dots j_r} &= p^{i_2\dots i_r}_{[j_1\dots j_{r-1}} \overset{\langle 1\rangle}{v}{}^{s}_{j_r]} \\ p^{i_1\dots i_{r-1} s}_{j_1\dots j_r} &= p^{i_1\dots i_{r-1}}_{[j_1\dots j_{r-1}} \overset{\langle r\rangle}{v}{}^{s}_{j_r]} \ . \end{aligned}\right\} \tag{3.1}$$

And the number of independent simple r-vectors varying $s \neq i_a$ $(a=1 \dots r)$ for any fixed indices $i_1 \dots i_r$ in the left member of (3.1) is equal to $(n-r)\,r$. On the other hand, in the right member of (3.1) if there is $r-t$ $(t \geqq 0)$ independent simple $(r-1)$-vectors among $p^{i_1\dots i_{r-2} i_r}$, $p^{i_1\dots i_{r-3} i_{r-1} i_r}, \dots, p^{i_2\dots i_r}$ and $p^{i_1\dots i_{r-1}}$, then the number of independent simple r-vectors is equal to $(r-t)(n-r)$ at most, accordingly we have $(n-r)\,r \leqq (r-t)(n-r)$ i. e. $t=0$. Therefore $p^{i_1\dots i_{r-2} i_r}$, $p^{i_1\dots i_{r-3} i_{r-1}, i_r}, \dots, p^{i_2\dots i_r}$ and $p^{i_1\dots i_{r-1}}$ are mutually independent.

Hence, by the above preparation and the above results, we know that the $(r-1)$ independent simple $(r-1)$-vectors $p^{i_1\dots i_{r-2} i_r}$, $p^{i_1\dots i_{r-3} i_{r-1} i_r}, \dots, p^{i_2\dots i_r}$ in a simple r-vector $p^{i_1\dots i_r}$ have only a vector $\overset{(i)}{v}{}^{i_r}_{j}$ in common.[2] By taking $i_a\,(a=1, \dots, r)$ in place of i_r, similarly we obtain the vectors $\overset{(i)}{v}{}^{i_a}_{j}$ which are contained in the simple r-vector $p^{i_1\dots i_r}$.

And we shall show that these r-vectors $\overset{(i)}{v}{}^{i_a}$ are mutually independent. The r simple $(r-1)$-vectors $p^{i_1\dots i_{r-1}}$, $p^{i_1\dots i_{r-2} i_r}$, $p^{i_1\dots i_{r-3} i_{r-1} i_r}, \dots, p^{i_2\dots i_{r-1} i_r}$ are mutually independent, however on the other hand the number of independent simple $(r-1)$-vectors in the simple r-vector $p^{i_1\dots i_r}$ which contain a vector $\overset{(i)}{v}{}^{i_r}$ is equal to $(r-1)$. Hence the simple $(r-1)$-vector $p^{i_1\dots i_{r-1}}$ can not contain this vector $\overset{(i)}{v}{}^{i_r}$. Contrary to this, the vectors $\overset{(i)}{v}{}^{i_a}\,(a \neq r)$ are contained in this simple $(r-1)$-vector $p^{i_1\dots i_{r-1}}$. Therefore we have $\overset{(i)}{v}{}^{i_r} \notin \mathfrak{L}(\overset{(i)}{v}{}^{i_a},\ a \neq r)$, and similarly $\overset{(i)}{v}{}^{i_b} \notin \mathfrak{L}(\overset{(i)}{v}{}^{i_a},\ a \neq b)$. Thus we can say that r vectors $\overset{(i)}{v}{}^{i_a}\,(a=1, 2, \dots, r)$ are mutually independent.

---

(1) $\langle a\rangle$ denotes $(i_1 \dots i_{a-1} i_{a+1} \dots i_r)$ for any fixed indices $i_1 \dots i_r$ and any $s \neq i_a$ $(a=1, \dots, r)$.
(2) The notation (i) means $(i_1 \dots i_r)$ in this place.

So we can write $p^{i_1\dots i_r}_{j_1\dots j_r}$ *as follows*:

$$p^{i_1\dots i_r}_{j_1\dots j_r} \approx \overset{(i)}{v}{}^{i_1}_{[j_1} \cdots \overset{(i)}{v}{}^{i_r}_{j_r]} \ . \tag{3.2}$$

Moreover, since $\overset{(i)}{v}{}^{i_a} \in p^{i_1\dots i_{b-1}i_{b+1}\dots i_r}$ $(i_a \neq i_b \ 1\leq a,\ b\leq r)$ and $\overset{(i)}{v}{}^{i_a}$ are mutually independent, *we have*

$$p^{i_1\dots i_{b-1}i_{b+1}\dots i_r}_{j_1\dots j_{b-1}j_{b+1}\dots j_r} \approx \overset{(i)}{v}{}^{i_1}_{[j_1} \overset{(i)}{v}{}^{i_{b-1}}_{j_{b-1}} \cdots \overset{(i)}{v}{}^{i_{b+1}}_{j_{b+1}} \cdots \overset{(i)}{v}{}^{i_r}_{j_r]} \ .$$

Furthermore we must prove that *if we write*

$$p^{i_1\dots i_{r-2}i_s i_r}_{j_1\dots j_r} \approx \overset{((i))}{v}{}^{i_1}_{[j_1} \cdots \overset{((i))}{v}{}^{i_{r-2}}_{j_{r-2}} \overset{((i))}{v}{}^{i_s}_{j_s} \overset{((i))}{v}{}^{i_r}_{j_r]} \tag{3.4}$$

where $((i))=(i_1 \dots i_{r-2} i_s i_r)$, *then we have*

$$\overset{(i)}{v}{}^{r} = \overset{((i))}{v}{}^{r} \ . \tag{3.5}$$

To do this first we shall show that *the intersection of the simple* r*-vectors* $p^{i_1\dots i_{r-2}i_s i_t}$ $(i_1 \dots i_{r-2}$ *are fixed*, i_s *and* i_t *are arbitrary*) *is a simple* $(r-2)$*-vector.*

By using (3.1) and (3.3) we can write the simple r-vectors $p^{i_1\dots i_{r-2}i_s i_t}$, $p^{i_1\dots i_{r-2}i_s i_u}$, $p^{i_1\dots i_{r-2}i_v i_t}$, $p^{i_1\dots i_{r-2}i_v i_u}$ as follows:

$$\left.\begin{aligned} p^{i_1\dots i_{r-2}i_s i_t}_{j_1\dots j_r} &= \overset{(i)}{v}{}^{1}_{[j_1} \cdots \overset{(i)}{v}{}^{r-1}_{j_{r-1}} \overset{(i)}{v}{}^{a}_{j_r]} \\ p^{i_1\dots i_{r-2}i_s i_u}_{j_1\dots j_r} &= \overset{(i)}{v}{}^{1}_{[j_1} \cdots \overset{(i)}{v}{}^{r-1}_{j_{r-1}} \overset{(i)}{v}{}^{b}_{j_r]} \\ p^{i_1\dots i_{r-2}i_v i_t}_{j_1\dots j_r} &= \overset{((i))}{v}{}^{1}_{[j_1} \cdots \overset{((i))}{v}{}^{r-1}_{j_{r-1}} \overset{((i))}{v}{}^{c}_{j_r]} \\ p^{i_1\dots i_{r-2}i_v i_u}_{j_1\dots j_r} &= \overset{((i))}{v}{}^{1}_{[j_1} \cdots \overset{((i))}{v}{}^{r-1}_{j_{r-1}} \overset{((i))}{v}{}^{d}_{j_r]} \ , \end{aligned}\right\} \tag{3.6}$$

where $\quad (i) = (i_1, \dots, i_{r-2}, i_s, i_t)$ and $((i)) = (i_1, \dots, i_{r-2}, i_v, i_t)$.

Since the intersection of $p^{i_1\dots i_{r-2}i_s i_t}$ and $p^{i_1\dots i_{r-2}i_v i_t}$ is a simple $(r-1)$-vector and, as we have seen in the above (3.1), $\overset{(i)}{v}{}^{1}_{[j_1} \cdots \overset{(i)}{v}{}^{r-1}_{j_{r-1}]} \not\approx \overset{((i))}{v}{}^{1}_{[j_1} \cdots \overset{((i))}{v}{}^{r-1}_{j_{r-1}]}$, we may take $\overset{(i)}{v}{}^{a}$ and $\overset{((i))}{v}{}^{c}$ such that are contained in this intersection. Accordingly, the intersection $\overset{(i)}{v}{}^{1}_{[j_1} \cdots \overset{(i)}{v}{}^{r-1}_{j_{r-1}]} \bigwedge \overset{((i))}{v}{}^{1}_{[j_1} \cdots \overset{((i))}{v}{}^{r-1}_{j_{r-1}]}$ contains at least a simple $(r-3)$-vector $\overset{0}{v}{}^{1}_{[j_1} \cdots \overset{0}{v}{}^{r-3}_{j_{r-3}]}$.

Therefore we can write these four simple r-vectors as follows:

$$\left.\begin{aligned} p^{i_1\dots i_{r-2}i_s i_t}_{j_1\dots j_r} &= \overset{0}{v}{}^{1}_{[j_1} \cdots \overset{0}{v}{}^{r-3}_{j_{r-3}} v^{r-2}_{j_{r-2}} v^{r-1}_{j_{r-1}} v^{a}_{j_r]} \\ p^{i_1\dots i_{r-2}i_s i_u}_{j_1\dots j_r} &= \overset{0}{v}{}^{1}_{[j_1} \cdots \overset{0}{v}{}^{r-3}_{j_{r-3}} v^{r-2}_{j_{r-2}} v^{r-1}_{j_{r-1}} v^{b}_{j_r]} \\ p^{i_1\dots i_{r-2}i_v i_t}_{j_1\dots j_r} &= \overset{0}{v}{}^{1}_{[j_1} \cdots \overset{0}{v}{}^{r-3}_{j_{r-3}} {}'v^{r-2}_{j_{r-2}} {}'v^{r-1}_{j_{r-1}} v^{c}_{j_r]} \\ p^{i_1\dots i_{r-2}i_v i_u}_{j_1\dots j_r} &= \overset{0}{v}{}^{1}_{[j_1} \cdots \overset{0}{v}{}^{r-3}_{j_{r-3}} {}'v^{r-2}_{j_{r-2}} {}'v^{r-1}_{j_{r-1}} v^{d}_{j_r]} \ . \end{aligned}\right\} \tag{3.7}$$

And the intersections $p^{i_1\cdots i_{r-2}i_si_t} \bigwedge p^{i_1\cdots i_{r-2}i_vi_t}$ and $p^{i_1\cdots i_{r-2}i_si_u} \bigwedge p^{i_1\cdots i_{r-2}i_vi_u}$ are the simple $(r-1)$-vectors respectively, (for $r-1$ indices are common). So, in the above (3.7) we can take $\overset{r-2}{v}$, $\overset{r-1}{v}$, $\overset{a}{v}$, $'\overset{r-2}{v}$, $'\overset{r-1}{v}$ and $\overset{b}{v}$ such that the intersections

$$\left(v^{r-2}_{[j_{r-2}}v^{r-1}_{j_{r-1}}v^{a}_{j_r]}\right)\bigwedge\left('v^{r-2}_{[j_{r-2}}{}'v^{r-1}_{j_{r-1}}v^{c}_{j_r]}\right) \text{ and } \left(v^{r-2}_{[j_{r-2}}v^{r-1}_{j_{r-1}}v^{b}_{j_r]}\right)\bigwedge\left('v^{r-2}_{[j_{r-2}}{}'v^{r-1}_{j_{r-1}}v^{d}_{j_r]}\right) \tag{3.8}$$

are the simple 2-vectors respectively. Eliminating v^a and v^b, we have

$$\left(v^{r-2}_{[j_{r-2}}v^{r-1}_{j_{r-1}]}\right)\bigwedge\left('v^{r-2}_{[j_{r-2}}{}'v^{r-1}_{j_{r-1}}v^{c}_{j_r]}\right) \neq \phi \tag{3.9}$$

and

$$\left(v^{r-2}_{[j_{r-2}}v^{r-1}_{j_{r-1}]}\right)\bigwedge\left('v^{r-2}_{[j_{r-2}}{}'v^{r-1}_{j_{r-1}}v^{d}_{j_r]}\right) \neq \phi \tag{3.10}$$

where ϕ denotes the null set.

(i): In the case where these intersections (3.9) and (3.10) have a vector w in common, it follows

$$\left.\begin{aligned} w &= l_1v^{r-2}+l_2v^{r-1} \\ &= l_1^1{}'v^{r-2}+l_2^1{}'v^{r-1}+l_3^1v^c \\ &= l_1^2{}'v^{r-2}+l_2^2{}'v^{r-1}+l_3^2v^d \,. \end{aligned}\right\} \tag{3.11}$$

Since the vectors $'v^{r-2}$, $'v^{r-1}$. v^c and v^d are independent, l_3^1 and l_3^2 must be zero, and $l_1^1=l_1^2$, $l_2^1=l_2^2$. Hence we have

$$\begin{aligned} w &= l_1v^{r-2}+l_2v^{r-1} \\ &= 'l_1{}'v^{r-2}+'l_2{}'v^{r-1} \,. \end{aligned}$$

Accordingly this vector w is contained in the four simple r-rector (3.7). Thus[(1)] the intersection of all $p^{i_1\cdots i_{r-2}i_si_t}$ (where i_s and i_t varied arbitrarily) contains simple $(r-2)$-vector.

(ii): In the case where the intersection of the intersections (3.9) and (3.10) is empty, in (3.9) and (3.10) we have two distinct vectors $'v^c$ and $'v^d$ respectively such that

$$'v^c = m_1^1v^{r-2}+m_2^1v^{r-1} = 'l_1^1{}'v^{r-2}+'l_2^1{}'v^{r-1}+'l_3^1v^c \tag{3.12}$$

$$'v^d = m_1^2v^{r-2}+m_2^2v^{r-1} = 'l_1^2{}'v^{r-2}+'l_2^2{}'v^{r-1}+'l_2^2v^d \,. \tag{3.13}$$

Here $'l_3^1\neq 0$ and $'l_3^2\neq 0$, for if $'l_3^1=0$ for example, then the vector $'v^c$ will be

(1) Any two of $(n-r+2)$ simple $(r-1)$-vector $p^{i_1\cdots i_{r-2}i_s}$ $(s=r-1, \dots n)$, being mutnally independent, have a simple $(r-2)$-vector in common, hence all $p^{i_1\cdots i_{r-2}i_s}$ $(s=r-1, \dots, n)$ have the same simple $(r-2)$-vector in common. For, otherwise, by using the previous method about $r-1$ in place of r (footnote (1), p. 15) we obtain $2(r-1)\geqq n$, this contradicts to $n\geqq 2r$. Moreover here we should remark that $p^{i_1\cdots i_{r-2}i_si_t}_{j_1\cdots\cdots j_r} = p^{i_1\cdots i_{r-2}i_s}_{(j_1\cdots j_{r-1}}v^{i_t}_{j_r)}$.

contained in (3, 9) and (3. 10); now this is impossible.

Hence from (3. 12) and (3. 13) it holds that

$$'v^{r-2}_{[j_{r-2}}\,'v^{r-1}_{j_{r-1}}v^{c}_{j_r]} \approx {}'v^{r-2}_{[j_{r-2}}\,'v^{r-1}_{j_{r-1}}\,'v^{c}_{j_r]}$$

$$'v^{r-2}_{[j_{r-2}}\,'v^{r-1}_{j_{r-1}}v^{d}_{j_r]} \approx {}'v^{r-2}_{[j_{r-2}}\,'v^{r-1}_{j_{r-1}}\,'v^{d}_{\ r]}$$

and

$$v^{r-2}_{[j_{r-2}}v^{r-1}_{j_{r-1}]} \approx {}'v^{c}_{[j_{r-2}}\,'v^{d}_{j_{r-1}]}$$

From these facts and (3. 8) we can say that

$$\left('v^{c}_{[j_{r-2}}\,'v^{d}_{j_{r-1}}v^{a}_{j_r]}\right) \bigwedge \left('v^{r-2}_{[j_{r-2}}\,'v^{r-1}_{j_{r-1}}\,'v^{c}_{j_r]}\right)$$

and

$$\left('v^{c}_{[j_{r-2}}\,'v^{d}_{j_{r-1}}v^{b}_{j_r]}\right) \bigwedge \left('v^{r-2}_{[j_{r-2}}\,'v^{r-1}_{j_{r-1}}\,'v^{d}_{j_r]}\right)$$

are the simple 2-vectors. Eliminating $'v^c$ and $'v^d$ as similar as the above, we have

$$\left('v^{c}_{[j_{r-2}}\,'v^{d}_{j_{r-1}}v^{a}_{j_r]}\right) \bigwedge \left('v^{r-2}_{[j_{r-2}}\,'v^{r-1}_{j_{r-1}]}\right) \neq \phi \tag{3.14}$$

and

$$\left('v^{c}_{[j_{r-2}}\,'v^{d}_{j_{r-1}}v^{b}_{j_r]}\right) \bigwedge \left('v^{r-2}_{[j_{r-2}}\,'v^{r-1}_{j_{r-1}]}\right) \neq \phi\,. \tag{3.15}$$

Since, if (3. 14) and (3. 15) have a vector in common, we can consider as similar as in case (i), now we have only to restrict ourselves to the case where (3. 14) and (3. 15) have not a vector in common. Then, in (3. 14) and (3. 15) we take two distinct vector $'v^a$, $'v^b$, respectively, such that

$$'v^{a} = {}'m^{1}_{1}\,'v^{r-2} + {}'m^{1}_{2}\,'v^{r-2} = h^{1}_{1}\,'v^{c} + h^{1}_{2}\,'v^{d} + h^{1}_{3}v^{a} \tag{3.16}$$

$$'v^{b} = {}'m^{2}_{1}\,'v^{r-2} + {}'m^{2}_{2}\,'v^{r-2} = h^{2}_{1}\,'v^{c} + h^{2}_{2}\,'v^{d} + h^{2}_{3}v^{b} \tag{3.17}$$

where $h^1_3 \neq 0$ and $h^2_3 \neq 0$, similarly as we have seen in the above. Hence we have

$$'v^{c}_{[j_{r-2}}\,'v^{d}_{j_{r-1}}v^{a}_{j_r]} \approx {}'v^{c}_{[j_{r-2}}\,'v^{d}_{j_{r-1}}\,'v^{a}_{j_r]}$$

$$'v^{c}_{[j_{r-2}}\,'v^{d}_{j_{r-1}}v^{b}_{j_r]} \approx {}'v^{c}_{[j_{r-2}}\,'v^{d}_{j_{r-1}}\,'v^{b}_{j_r]}$$

and

$$'v^{r-2}_{[j_{r-2}}\,'v^{r-1}_{j_{r-1}]} \approx {}'v^{a}_{[j_{r-2}}\,'v^{b}_{j_{r-1}]}\,.$$

Therefore we can rewrite (3.7) as follows:

$$p^{i_1\ldots i_{r-2}i_s i_t}_{j_1\ldots j_r} \approx \overset{0}{v}{}^{1}_{[j_1} \ldots \overset{0}{v}{}^{r-3}_{j_{r-3}}\,'v^{c}_{j_{r-2}}\,'v^{d}_{j_{r-1}}\,'v^{a}_{j_r]} \tag{3.18}$$

$$p^{i_1\ldots i_{r-2}i_s i_u}_{j_1\ldots j_r} \approx \overset{0}{v}{}^{1}_{[j_1} \ldots \overset{0}{v}{}^{r-3}_{j_{r-3}}\,'v^{c}_{j_{r-2}}\,'v^{d}_{j_{r-1}}\,'v^{b}_{j_r]} \tag{3.19}$$

$$p^{i_1\ldots i_{r-2}i_v i_t}_{j_1\ldots j_r} \approx \overset{0}{v}{}^{1}_{[j_1} \ldots \overset{0}{v}{}^{r-3}_{j_{r-3}}\,'v^{a}_{j_{r-2}}\,'v^{b}_{j_{r-1}}\,'v^{c}_{j_r]} \tag{3.20}$$

and

$$p^{i_1\dots i_{r-2}i_v i_u}_{j_1\dots j_r} \approx \overset{0}{v}{}^{1}_{[j_1}\dots\overset{0}{v}{}^{r-3}_{j_{r-3}}\,{}'v^{a}_{j_{r-2}}\,{}'v^{b}_{j_{r-1}}\,{}'v^{d}_{j_r]}\ . \tag{3.21}$$

From (3.18) and (3.21), we see that $p^{i_1\dots i_{r-2}i_s i_t}_{j_1\dots j_r}+p^{i_1\dots i_{r-2}i_v i_u}_{j_1\dots j_r}$ is a simple r-vector. But the inverse image of this r-vector under our automorphisms, i. e.

$$\overset{1}{e}_{[j_1}\overset{2}{e}_{j_2}\dots\overset{r-2}{e}_{j_{r-2}}\overset{s}{e}_{j_{r-1}}\overset{t}{e}_{j_r]}+\overset{1}{e}_{[j_1}\overset{2}{e}_{j_2}\dots\overset{r-2}{e}_{j_{r-1}}\overset{v}{e}_{j_{r-2}}\overset{u}{e}_{j_r]}\quad ,$$

as we know from the conditions for simple vector (1.2), is not simple. This contradicts our assumption. Hence the case (ii) does not happen. Thus we have proved our assertion.

By using (3.2) and (3.3) we can write as follows:

$$\left.\begin{aligned}
p^{i_1\dots i_{r-3}i_r i_{r-2}i_{r-1}}_{j_1\dots j_{r-3}j_r j_{r-2}j_{r-1}} &= \overset{(t)}{v}{}^{i_1}_{[j_1}\dots\overset{(t)}{v}{}^{i_{r-3}}_{j_{r-3}}\overset{(t)}{v}{}^{i_r}_{j_r}\overset{(t)}{v}{}^{i_{r-2}}_{j_{r-2}}\overset{(t)}{v}{}^{i_r}_{j_r]}\\
p^{i_1\dots i_{r-3}i_r i_{r-2}}_{j_1\dots j_{r-3}j_r j_{r-2}} &= \overset{(t)}{v}{}^{i_1}_{[j_1}\dots\overset{(t)}{v}{}^{i_{r-3}}_{j_{r-3}}\overset{(t)}{v}{}^{i_r}_{j_r}\overset{(t)}{v}{}^{i_{r-2}}_{j_{r-2}]}\\
p^{i_1\dots i_{r-3}i_r i_{r-1}}_{j_1\dots j_{r-3}j_r j_{r-1}} &= \overset{(t)}{v}{}^{i_1}_{[j_1}\dots\overset{(t)}{v}{}^{i_{r-3}}_{j_{r-3}}\overset{(t)}{v}{}^{i_r}_{j_r}\overset{(t)}{v}{}^{i_{r-1}}_{j_{r-1}]}\\
p^{i_1\dots i_{r-3}i_r i_{r-2}i_s}_{j_1\dots j_{r-3}j_r j_{r-2}j_{r-1}} &= p^{i_1\dots i_{r-3}i_r i_{r-2}}_{[j_1\dots j_{r-3}j_r j_{r-2}}\,{}'v^{i_s}_{j_{r-1}]}\\
\text{and}\qquad & \\
p^{i_1\dots i_{r-3}i_r i_{r-1}i_s}_{j_1\dots j_{r-3}j_r j_{r-1}j_{r-2}} &= p^{i_1\dots i_{r-3}i_r i_{r-1}}_{[j_1\dots j_{r-3}j_r j_{r-1}}\,{}''v^{i_s}_{j_{r-2}]}\ .
\end{aligned}\right\} \tag{3.22}$$

As we know from (3.22), the intersection $p^{i_1\dots i_{r-3}i_r i_{r-2}i_{r-1}}\bigcap p^{i_1\dots i_{r-3}i_r i_{r-2}i_s}\bigcap p^{i_1\dots i_{r-3}i_r i_{r-1}i_s}$ contains the simple $(r-2)$-vector $\overset{(t)}{v}{}^{i_1}_{[j_1}\dots\overset{(t)}{v}{}^{i_{r-3}}_{j_{r-3}}\overset{(t)}{v}{}^{i_r}_{j_r]}$. And since the intersection $\pi\equiv\bigcap_{i_s,i_l}p^{i_1\dots i_{r-3}i_s i_l}$ is a simple $(r-2)$-vector contained in $\overset{(t)}{v}{}^{i_1}_{[j_1}\dots\overset{(t)}{v}{}^{i_{r-3}}_{j_{r-3}}\overset{(t)}{v}{}^{i_r}_{j_r]}$ (1), we can conclude that

$$\pi\equiv\left(\overset{(t)}{v}{}^{i_1}_{[j_1}\dots\overset{(t)}{v}{}^{i_{r-3}}_{j_{r-3}}\overset{(t)}{v}{}^{i_r}_{j_r]}\right).$$

Hence we have

$$\overset{(t)}{v}{}^{i_r}\in p^{i_1\dots i_{r-3}i_r i_s i_l}_{j_1\dots j_r}\equiv p^{i_1\dots i_{r-3}i_r i_s}_{[j_1\dots j_{r-1}}\overset{*}{v}{}^{i_l}_{j_r]}$$

where $\overset{*}{v}{}^{i_l}$ has the freedom of $(n-r+1)$. Consequently we get

$$\overset{(t)}{v}{}^{i_r}\in p^{i_1\dots i_{r-3}i_r i_s}\ .^{(2)} \tag{3.23}$$

Similarly we know

(1) The intersection $p^{i_1\dots i_{r-3}i_r i_{r-2}i_{r-1}}\cap p^{i_1\dots i_{r-3}i_r i_{r-2}i_s}\cap p^{i_1\dots i_{r-3}i_r i_{r-1}i_s}$ does not contain a simple $(r-1)$-vector. For, the complete system of independent simple r-vector containing the simle $(r-1)$-vector has the form (2.7).

(2) (3.23) is obviously obtained from the footnote (1) in p. 19.

$$\left.\begin{array}{l} \overset{(i)}{v}{}^{i_r} \in p^{i_1 \ldots i_{r-4} i_{r-2} i_r i_s} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \overset{(i)}{v}{}^{i_r} \in p^{i_2 \ldots i_{r-2} i_r i_s} \\ \overset{(i)}{v}{}^{i_r} \in p^{i_1 \ldots i_{r-2} i_r} \quad . \end{array}\right\} \tag{3.24}$$

Since $\overset{((i))}{v}{}^{i_r}$ is the intersection of $(r-1)$ linearly independent simple $(r-1)$-vectors $p^{i_1 \ldots i_{r-3} i_r i_s}, \ldots, p^{i_2 \ldots i_{r-2} i_r i_s}$ and $p^{i_1 \ldots i_{r-2} i_r}$ in the r-dimensional vector space $p^{i_1 i_2 \ldots i_{r-2} i_r i_s}$, $\overset{(i)}{v}{}^{i_r}$ must coincide with $\overset{((i))}{v}{}^{i_r}$.

By repeating this procedure from (i) to $((i))$, any indices (k) containing i_r can be attained from (i). Therefore, if we write

$$p_{j_1 \ldots j_r}^{k_1 \ldots k_{r-1} i_r} \approx \overset{(k)}{v}{}^{k_1}_{[j_1} \overset{(k)}{v}{}^{k_2}_{j_2} \ldots \overset{(k)}{v}{}^{i_r}_{j_r]} \quad \text{where} \quad (k) = (k_1, k_2 \ldots k_{r-1} i_r),$$

then we have $\overset{(i)}{v}{}^{i_r} \approx \overset{(k)}{v}{}^{i_r}$. Since this is true for every index i_a instead of i_r, we can obtain

$$p_{j_1 \ldots j_r}^{i_1 \ldots i_r} \approx v_{[j_1}^{i_1} v_{j_2}^{i_2} \ldots v_{j_r]}^{i_r}. \tag{3.25}$$

Moreover, since we have not considered the proportional factor, exactly *we must write*

$$p_{j_1 j_2 \ldots j_r}^{i_1 i_2 \ldots i_r} = a^{i_1 i_2 \ldots i_r} v_{[j_1}^{i_1} v_{j_2}^{i_2} \ldots v_{j_r]}^{i_r} \tag{3.26}$$

where we do not sum for indices $i_1, \ldots, i_r$.

§ 4. The type (II).

In this case, we have seen that the sequences $p_{j_1 j_2 \ldots j_r}^{i_1 i_2 \ldots i_{r-1} i_s}$ ($i_1, i_2, \ldots, i_{r-1}$ are any fixed indices; $s=r, r+1, \ldots, n$) are such that

$$\left(p_{j_1 \ldots j_r}^{i_1 \ldots i_{r-1} i_s}\right) \subset \overset{(i)}{v}{}^{1}_{[j_1} \ldots \overset{(i)}{v}{}^{r+1}_{j_{r+1}]} \quad (s = r, r+1, \ldots, n). \tag{4.1}$$

So we have

$$\begin{aligned} &\left(\varepsilon^{j_1 \ldots j_r j_{r+1} \ldots j_{2r}} g_{j_{r+1} k_{r+1}} \ldots g_{j_{2r} k_{2r}} p_{j_1 \ldots j_r}^{i_1 \ldots i_{r-1} i_s}\right) \\ &\supset \left(\varepsilon^{j_1 \ldots j_r j_{r+1} \ldots j_{2r}} g_{j_{r+1} k_{r+1}} \ldots g_{j_{2r} k_{2r}} \overset{(i)}{v}{}^{1}_{[j_1} \ldots \overset{(i)}{v}{}^{r+1}_{j_{r+1}]}\right) . \end{aligned} \tag{4.2}$$

Since the right member of (4.2) is a simple $(r-1)$-vector, the sequence

$$\varepsilon^{j_1 \ldots j_r \ldots j_{2r}} g_{j_{r+1} k_{r+1}} \ldots g_{j_{2r} k_{2r}} p_{j_1 \ldots j_r}^{i_1 \ldots i_{r-1} i_s} \tag{4.3}$$

has the type (I).

Hence by the above result for the case of type (I) and (4.3) *we have*

$$p^{i_1\dots i_r}_{j_1\dots j_r} = a^{i_1\dots i_r} v^{i_1}_{[k_1} \dots v^{i_r}_{k_r]} \varepsilon^{\dots\dots k_1 k_2 \dots k_r}_{j_1\dots j_r} \tag{4.4}$$

where we do not sum for the indices $i_1, i_2, \dots i_n$.

§ 5. Proportional factors.

Finally we investigate the proportional factor $a^{i_1 i_2 \dots i_r}$.

Case of the type (I).

From (1.3) and (3.26) we get

$$'w_{j_1\dots j_r} = p^{i_1\dots i_r}_{j_1\dots j_r} w_{i_1\dots i_r} = \sum_i a^{i_1\dots i_r} v^{i_1}_{[j_1} \dots v^{i_r}_{j_r]} w_{i_1\dots i_r} \tag{5.1}$$

Let $\bar{v}^j_k$ be determined to be such that $v^i_j \bar{v}^j_k = \delta^i_k$. Multiplying $\bar{v}^{j_1}_{k_1} \dots \bar{v}^{j_r}_{k_r}$ on the both member of (5.1) and summing up for $j_1, j_2, \dots, j_r$, then it follows

$$\begin{aligned} '\bar{w}_{k_1 k_2 \dots k_r} &\equiv \bar{v}^{j_1}_{k_1} \dots \bar{v}^{j_r}_{k_r} {}'w_{j_1\dots j_r} = \sum_i a^{i_1\dots i_r} \delta^{i_1}_{[k_1} \dots \delta^{i_r}_{k_r]} w_{i_1\dots i_r} \\ &= a^{k_1\dots k_r} w_{k_1\dots k_r} = a^{k_1 k_2 \dots k_r} \overset{1}{w}_{[k_1} \dots \overset{r}{w}_{k_r]} \end{aligned} \tag{5.2}$$

Since $'\bar{w}_{k_1\dots k_r} \equiv \bar{v}^{j_1}_{k_1} \dots \bar{v}^{j_r}_{k_r} {}'w_{j_1\dots j_r}$ is a simple r-vector, $a^{i_1\dots i_r}$ must be symmetric for indices $i_1, \dots, i_r$, and moreover from the condition (1.2) of simplicity of r-vector, it satisfies the condition

$$\left(\overset{1}{w}_{[i_1} \dots \overset{s}{w}_{i_s} \overset{s+1}{w}_{k_{s+1}} \dots \overset{r}{w}_{k_r} a^{(i_1\dots i_s k_{s+1}\dots k_r}\right)\left(\overset{[1}{w}_{j_1]} \dots \overset{s}{w}_{j_s} \overset{s+1}{w}_{k_{s+1}} \dots \overset{r]}{w}_{k_r} a^{j_1)\dots j_s k_{s+1}\dots k_r}\right) = 0 \tag{5.3}$$

where $i_1, \dots, i_s$, $j_1, \dots, j_s$, $k_{s+1}, \dots, k_r$ (donot sum up) are all distinct and $\overset{\nu}{w}_i\ (\nu=1, \dots, r)$ are arbitrary vectors.(1) By putting in (5.3)

$$\overset{\lambda}{w}_i = \overset{\lambda}{a} e^{i_\lambda}_i + \overset{\lambda}{b} e^{j_\lambda}_i\ (\lambda=1, \dots, s); \quad \overset{\mu}{w}_i = \overset{\mu}{c} e^{k_\mu}_i\ (\mu=s+1, \dots, r),$$

we obtain the condition (2)

$$a^{i_1\dots i_s k_{s+1}\dots k_r} a^{j_1\dots j_s k_{s+1}\dots k_r} = a^{j_1 i_2 \dots i_s k_{s+1}\dots k_r} a^{i_1 j_2 \dots j_s k_{s+1}\dots k_r} \tag{5.4}$$

Furthermore from (5.4), it follows that

$$a^{i_1\dots i_r} a^{j_1\dots j_r} = a^{k_1\dots k_r} a^{l_1\dots l_r} \tag{5.5}$$

for $(i_1 \dots i_r j_1 \dots j_r) = (k_1 \dots k_r, l_1 \dots l_r)$.

(1) In this case only, two symbols ( ) and [ ] denote operations operating for the same indices at the same time.

(2) By the substitution into (5.3) we obtain

$$\overset{1}{a} \dots \overset{s}{a}\, \overset{1}{b} \dots \overset{s}{b} \left(\overset{s+1}{c} \dots \overset{r}{c}\right)^2 \left\{a^{i_1\dots i_s k_{s+1}\dots k_r} a^{j_1\dots j_s k_{s+1}\dots k_r} - a^{j_1 i_2 \dots i_s k_{s+1}\dots k_r} a^{i_1 j_2 \dots j_s k_{s+1}\dots k_r}\right\} = 0, \quad \text{consequently} \quad (5.4).$$

And we remark that $a^{i_1\dots i_r}\neq 0$ where $i_1\dots i_r$ are all distinct. For, if $a^{i_1 i_2\dots i_r}=0$ for any fixed index $i_1, i_2\dots i_r$, then we have $'\overline{w}_{i_1\dots i_r}=0$ when $w^1_{[i_1} \dots w^r_{i_r]}\neq 0$.

From (5.5) we get

$$a^{i_1\dots i_t\dots i_r}/a^{j_1\dots j_t i_{t+1}\dots i_r} = a^{i_1\dots i_t k_{t+1}\dots k_r}/a^{j_1\dots j_t k_{t+1}\dots k_r}. \tag{5.6}$$

Hence we have

$$a^{i_1\dots i_t k_{t+1}\dots k_r} = b^{i_1\dots i_t} a^{k_{t+1}\dots k_r} . \tag{5.7}$$

Considering that $a^{(i_1\dots i_r)}=a^{i_1\dots i_r}$, we obtain

$$a^{i_1\dots i_r} = a^{i_1}\dots a^{i_r}. \tag{5.8}$$

Case of the type (II).

In this case, also similarly, we can investigate $a^{i_1\dots i_r}$. From (1.3) and (4.4) we get

$$'w_{j_1\dots j_r} = \sum_i a^{i_1\dots i_r} v^{i_1}_{[h_1} \dots v^{i_r}_{h_r]} \varepsilon^{\dots\dots\dots h_1\dots h_r}_{j_1\dots j_r} w_{i_1\dots i_r}. \tag{5.9}$$

Multiplying $\varepsilon^{j_1\dots j_r}_{\dots\dots\dots l_1\dots l_r} \bar{v}^{l_1}_{k_1}\dots \bar{v}^{l_r}_{k_r}$ on the both members of (5.9) and then summing up for $j_1, \dots, j_r$, then it follows

$$\left.\begin{aligned} '\overline{w}_{k_1\dots k_r} &= \sum_i \varepsilon^{j_1\dots j_r}_{\dots\dots\dots l_1\dots l_r} \bar{v}^{l_1}_{k_1}\dots \bar{v}^{l_r}_{k_r}\, 'w_{j_1\dots j_r} \\ &= \sum_i a^{i_1\dots i_r} v^{i_1}_{[h_1}\dots v^{i_r}_{k_r]} \delta^{h_1}_{[l_1}\delta^{h_2}_{l_2}\dots \delta^{h_r}_{l_r]} \bar{v}^{l_1}_{k_1}\dots \bar{v}^{l_r}_{k_r} w_{i_1\dots i_r} \\ &= a^{k_1\dots k_r} w_{k_1\dots k_r} \\ &= a^{k_1\dots k_r} w^1_{[k_1}\dots w^r_{k_r]} \quad . \end{aligned}\right\} \tag{5.10}$$

Similarly as in the case (I), from $\overline{w}_{k_1\dots k_r}$ eing an r-simple vector we obtain

$$a^{i_1\dots i_r} = a^{i_1}\dots a^{i_r} .$$

From the property proportional factor $a^{i_1\dots i_r}$ in (3.26) and (4.4) we obtain the following result, writing v^i in place of $a^i v^i$ (donot sum for i): *we have*

for the case where $n\neq 2r$: $p^{i_1\dots i_r}_{j_1\dots j_r} = v^{i_1}_{[j_1}\dots v^{i_r}_{j_r]}$,

for the case where $n=2r$:

$$p^{i_1\dots i_r}_{j_1\dots j_r} = v^{i_1}_{[j_1}\dots v^{i_r}_{j_r]} \quad or \quad \varepsilon^{\dots\dots\dots h_1\dots h_r}_{j_1\dots j_r} v^{i_1}_{[h_1}\dots v^{i_r}_{k_r]} .$$

Thus we have completely proved our theorem.

Mathematical Institute, Hiroshima University