On Stress-functions in General Coordinates.

By

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§ 1. Stress equations of motion and Hooke's law in tensor form.

In this paper we shall consider the stress-functions in the theory of elasticity. We begin rewriting the stress equations of motion in tensor forms. We introduce a general coordinates x^i into the three dimensional Euclidean space, in which the metric is given by

$$ds^2 = g_{ij}dx^i dx^j, \tag{1.1}$$

where we sum up for $i, j=1, 2, 3$; in the following we shall use the familiar notations in tensor analysis.[1]

The equations of motion for the body in motion with acceleration f^i under body force T^i and force $T^i_{(\varphi)}$ applied over any surface $\varphi=const.$ in the body are

$$\int \rho f^i d\tau = \int \rho T^i d\tau + \int T^i_{(\varphi)} d\sigma \;^{2)} \tag{1.2}$$

and

$$\int \rho v^{[i} f^{j]} d\tau = \int \rho v^{[i} T^{j]} d\tau + \int v^{[i} T^{j]}_{(\varphi)} d\sigma, \tag{1.3}$$

where v^i is any solution of $\nabla_i v^j = \delta^j_i$ (Kronecker's delta),[3] ∇_i denotes the covariant derivative with respect to g_{ij}. Since (1.2) and (1.3) are tensor equations, and in a Cartesian coordinates coincide with the ordinary equations of motion, where v^i becomes $(x+a, y+b, z+c)$, these are the equations of motion in the general coordinates.

Let T^{ij} be the force applied over the surfaces $x^i=const.$, then by projecting the force $T^i_{(\varphi)}$ on the normal line of the surface $\varphi=const.$, we get

$$T^i_{(\varphi)} = ((\nabla_h \varphi)/\sqrt{g^{kj}\nabla_k\varphi\nabla_j\varphi})T^{hi}, \tag{1.4}$$

1) As for tensor notations, see L. P. Eisenhart: Riemannian Geometry (1926).

2) $\int \ldots d\tau$ and $\int \ldots d\sigma$ denote the integrals with respect to the volumes and surfaces in the unstrained state respectively, and so in the following.

3) This system of equations is always integrable, since this satisfies the integrability conditions $\nabla_{[k}\delta^j_{i]}=0$.

where $\nabla_h\varphi$ is the normal vector to the surface $\varphi=const.$. Since (1.4) holds for the arbitrary surface $\varphi=const.$, by means of the for the quotients of of tensors, T^{hi} is a contravariant tensor of degree two.

Next from (1.2) and (1.4), it follows

$$\int \rho f^i d\tau = \int \rho T^i d\tau + \int T^{ij}(\nabla_j\varphi)d\sigma. \tag{1.5}$$

And by Gauss's theorem [1]:

$$\int T^{ij}(\nabla_j\varphi)d\sigma = \int \nabla_i T^{ij} d\tau, \tag{1.6}$$

(1.5) becomes

$$\int (\nabla_j T^{ij} + \rho T^i - \rho f^i)d\tau = 0.$$

Since these equations hold for any domain of integral, we have

$$\nabla_i T^{ij} + \rho T^i - \rho f^i = 0. \tag{1.7}$$

Moreover we know that the equations (1.3) are equivalent to

$$T^{ij} = T^{ji} \tag{1.8}$$

In the theory of elasticity, this symmetric contravariant tensor T^{ij} is called a stress tensor.[2]

Thus we have the following result:

RESULT: *The equation of motion in tensor from are reduced to*

$$\nabla_j T^{ij} + \rho T^i = \rho f^i \quad and \quad T^{ij} = T^{ji} \quad (i,\ j=1,\ 2,\ 3).$$

We shall call these equations the stress equations of motion in the following.

Furthermore the generalized Hooke's law is written in the following:

Let u_j be a displacement vector, e_{ij} and T^{kl} be the strain tensor and the stress tensor (corresponding to it) respectively, then there hold the relations:

$$\nabla_{(i}u_{j)} \equiv e_{ij} = c_{ijkl}T^{kl}, \tag{1.9}$$

where c_{ijkl} are called the elastic constants.

1) J. A. Schouten: der Riccl-kalkul, (1924). p. 95.
2) A. E. H. Love: A Treatise on the Mathematical Theory of Elasticity. (1934).
K. Morinaga and T. Nôno: An Expression of the Theory of Elatsicity in General Coordinates. Reports for National Research Council of Japan, (in Japanese), (1946).

§ 2. General solutions of the stress equations of equilibrium under no body force in the two dimensional space.

In the two dimensional Euclidean space, we shall consider the stress equations

$$\nabla_b T^{ab}=0 \quad and \quad T^{ab}=T^{ba} \quad (a,\ b=1,\ 2).^{1)} \tag{2.1}$$

We shall prove the theorem:

THEOREM 1. *The general solution of*

$$\nabla_b T^{ab}=0 \quad and \quad T^{ab}=T^{ba} \quad (a,\ b=1,\ 2)$$

is given by

$$T^{ab}=\varepsilon^{ac}\varepsilon^{bd}\nabla_c\nabla_d v,$$

where ε^{ac} denotes the Eddington's symbol in the two dimensional space,[2] *and v is an arbitrary function.*

PROOF. Froom (2.1) we have $T^{ab}=\varepsilon^{ac}\nabla_c v^b$, $\nabla_c v^c=0$.
Since $\nabla_c v^c=0$, we have $v^b=\varepsilon^{bd}\nabla_d v$, accordingly

$$T^{ab}=\varepsilon^{ac}\varepsilon^{bd}\nabla_c\nabla_d v. \qquad \text{q. e. d.}$$

REMARK. If we take a Cartesian coordinates, then we get the well-known G. B. Airy's stress-functions[3]; and moreover, let $\overset{0}{v}$ be a stress-function, then

$$v=\overset{0}{v}+\overset{1}{v},$$

where $\varepsilon^{ac}\varepsilon^{bd}\nabla_c\nabla_d\overset{1}{v}=0$, accordingly $\overset{1}{v}=\alpha_1x_1+\alpha_2x_2+\alpha_3$, that is, $v=\overset{0}{v}+\alpha_1x_1+\alpha_2x_2+\alpha_3$. ($\alpha_1$, α_2, α_3 *are arbitrary constants*)

§ 3. General solution of the stress equations of equilibrium under no body forces in the three dimensional space.

We shall consider the body in equilibrium under forces applied over their surfaces only, and later in § 4 we shall investigate the stress equations of motion in the general case. The equations of equilibrium, that

1) We can easily see that the stress equations for the two dimensional space are written by (2.1).

2) $\varepsilon^{i_1 i_2 \cdots i_n}$ denotes the Eddington's symbol in the n-dimensional space. Cf. A. S. Eddington; The Mathematical Theory of Relativity, (1937). p. 107.

3) A. E. H. Love; ibid., p. 88.

case when there are no body force and no acceleration, are

$$\nabla_j T^{ij} = 0 \tag{3.1}$$

and

$$T^{ij} = T^{ji}. \tag{3.2}$$

It is easily seen that the equations (3.1) and (3.2) are satisfied by

$$T^{ij} = \varepsilon^{ipk}\varepsilon^{jql}\nabla_p\nabla_q v_{kl}, \quad v_{kl} = v_{lk}. \tag{3.3}$$

Moreover we can write (3.3) symbolically (as matrix represntation) as follows:

$$\boldsymbol{t} = X\boldsymbol{v}, \quad (\boldsymbol{t} = (T^{ij}),\ \boldsymbol{v} = (v_{ij})), \tag{3.4}$$

where

$$X \equiv \varepsilon^{ipk}\varepsilon^{jql}\nabla_p\nabla_q$$

As for this we shall prove the theorem:

Theorem 2. *The general solution of*

$$\nabla_j T^{ij} = 0 \text{ and } T^{ij} = T^{ji} \quad (i, j = 1, 2, 3)$$

is given by

$$T^{ij} = \varepsilon^{ipk}\varepsilon^{jql}\nabla_p\nabla_q v_{kl}, \text{ symbolically } \boldsymbol{t} = X\boldsymbol{v},$$

where v_{kl} is an arbitrary symmetric tensor.

PROOF. Any stress tensor T^{ij} expressed by

$$T^{ij} = \varepsilon^{ipk}\nabla_p V_k^{\cdot j} \tag{3.5}$$

since the integrability condition of (3.5),[1] i. e.,

$$2!\ \nabla_{[p} V_{k]}^{\cdot j} = \varepsilon_{ipk} T^{ij} \tag{3.6}$$

is

$$\nabla_{[l}\varepsilon_{|i|pk]} T^{ij} = 0, \text{ i. e., } \nabla_i T^{ij} = 0.$$

Contracting for j and p in (3.6) and making use of $T^{ij} = T^{ji}$, we have

$$\nabla_j V_k^{\cdot j} = \nabla_k V_j^{\cdot j}. \tag{3.7}$$

If we put

$$V_k^{j} = V_k'^{\cdot j} + \nabla_k V^j, \tag{3.8}$$

1) As for the integrability condition of the system of differential equations, see J. A. Schouten: ibid., Kap. III. p.p. 104-126.

where V^j is an arbitrary vector satisfying $\nabla_j V^j = V^{\cdot j}_{j}$,[1]
then we get

$$T^{ij} = \varepsilon^{ipk} \nabla_p V'^{\cdot j}_{k}, \tag{3.9}$$

where

$$\nabla_j V'^{\cdot j}_{k} = 0. \tag{3.10}$$

Since (3.10) is the integrability condition of

$$V'^{\cdot j}_{k} = \varepsilon^{jql} \nabla_q v_{kl}, \tag{3.11}$$

$V'^{\cdot j}_{k}$ which satisfies (3.10) is expressed by (3.9). From (3.7) and (3.9), we have

$$T^{ij} = \varepsilon^{ipk} \varepsilon^{jql} \nabla_p \nabla_q v_{kl} \tag{3.12}$$

Since $T^{ij} = T^{ji}$, it follows[2]

$$v_{kl} = v_{lk}. \qquad \text{q. e. d.} \tag{3.13}$$

REMARKS. The above tensor v_{kl} is what we call the system of stress-functions in the general coordinates. If we write v_{kl} in a Cartesian coordinates as follows:

$$\boldsymbol{v} = \| v_{kl} \| = \begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix} + \begin{pmatrix} 0 & \psi_3 & \psi_2 \\ \psi_3 & 0 & \psi_1 \\ \psi_2 & \psi_1 & 0 \end{pmatrix},$$

then χ_1, χ_2, χ_3 are the socalled Maxwell's stress-functions, and ψ_1, ψ_2, ψ_3 are the socalled Morera's stress-functions.[3]

Moreover, let $\overset{0}{v}_{kl}$ be a system of stress-functions corresponding to a given stress tnesor T^{ij}, then the general system of stress-functions v_{kl} corresponding to the same T^{ij} is given by

$$v_{kl} = \overset{0}{v}_{kl} + \overset{1}{v}_{kl}, \tag{3.14}$$

where

$$\varepsilon^{ipk} \varepsilon^{jql} \nabla_p \nabla_q \overset{1}{v}_{kl} = 0, \tag{3.15}$$

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1) $\nabla_j V^j = V^{\cdot j}_{j}$ is always integrable.

2) Since $T^{ij} = T^{ji}$ it follows $T^{ij} = T^{(ij)} = \varepsilon^{(i|pk|}\varepsilon^{j)ql} \nabla_p \nabla_q v_{kl} = \varepsilon^{ipk} \varepsilon^{jql} \nabla_p \nabla_q v_{(kl)}$, where $v_{(kl)} = \frac{1}{2}(v_{kl} + v_{lk})$.

3) A. E. H. Love; ibid., p. 88.

accordingly [1]

$$\overset{1}{v}_{kl}=\nabla_{(k}u_{l)} \quad (u_l \text{ is an arbitrary vector}) \tag{3.16}$$

In a Cartesian coordinates, by considering the integrability conditions (3.15) of (3.16), we know that the system of equations

$$\nabla_{(k}u_{l)}=v_{kl}-\overset{0}{v}_{kl}$$

is not solvable under the condition such that functions of v_{kl} equal to zero, but under the condition such that three function of v_{kl} equal to zero, is a always solvable only under the condition that $v_{a1}=v_{b2}=v_{c3}=0$, $(a, b, c=1, 2, 3)$. Therefore, by varying the vector u_l arbitraily, the system of stress-functions can take any each form following:

$$\begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & \psi_3 & \psi_2 \\ \psi_3 & 0 & \psi_1 \\ \psi_2 & \psi_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \kappa_1 \\ 0 & \kappa_2 & 0 \\ \kappa_1 & 0 & \kappa_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & \sigma_1 & 0 \\ \sigma_1 & 0 & \sigma_2 \\ 0 & \sigma_2 & \sigma_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tau_1 & \tau_2 \\ 0 & \tau_2 & \tau_3 \end{pmatrix}.$$

Furthermore, we can verify for each case that these five forms are not transformable one another by a coordinate transformation, that is, by a transformation:

$$v'_{\alpha\beta}=v_{ij}p^i_\alpha p^j_\beta$$

where $\| p^i_\alpha \|$ is an regular constant matrix. Moreover R. V. Southwell has already proved that the Mexwell's stress-functions and the Morera's stress-functions are able to express any stress tensor respectively.[2]

§ 4. General solution of the stress equations in the three dimensional space.

We shall investigate the general solution of the stress equations in the three dimensional space:

$$\nabla_j T^{ij}+\rho T^i=\rho f^i, \quad T^{ij}=T^{jk}, \quad (i, j=1, 2, 3)\,^{3} \tag{4.1}$$

Let $\overset{0}{T}{}^{ij}$ be determined by

$$\overset{0}{T}{}^{ij}=0 \quad for \ i \neq j$$

1) See §6 in this paper.

2) R. V. Southwell: Castigliano's Drinciple of Main Strain Energy. "Stephen Timoschenko 60th Aniversary volume." p. 211-217 (1938).,

3) As for integrability conditions of these equations, see J. A. Schouten, ibid., p. 119.

and

$$\left.\begin{aligned}\nabla_1\overset{0}{T}{}^{11}&=\rho T^1-\rho f^1\\ \nabla_2\overset{0}{T}{}^{22}&=\rho T^2-\rho f^2\\ \nabla_3\overset{0}{T}{}^{33}&=\rho T^3-\rho f^3\end{aligned}\right\} \tag{4.2}$$

then $\overset{0}{T}{}^{ij}$ is a special solution of (4.1). If we put $T^{ij}-\overset{0}{T}{}^{ij}=\overset{1}{T}{}^{ij}$, from (4.1) and (4.2), we have

$$\nabla_j\overset{1}{T}{}^{ij}=0,$$

accordingly we obtain

$$T^{ij}=\varepsilon^{ipk}\varepsilon^{jql}\nabla_p\nabla_q v_{kl}+\overset{0}{T}{}^{ij}. \tag{4.3}$$

Thus we have the theorem:

THEOREM 3. *The general solution of*

$$\nabla_j T^{ij}+\rho T^i=\rho f^i \quad \textit{and} \quad T^{ij}=T^{ji} \quad (i,\ j=1,2,3),$$

is given by

$$T^{ij}=\varepsilon^{ipk}\varepsilon^{jql}\nabla_p\nabla_q v_{kl}+\overset{0}{T}{}^{ij}$$

where v_{kl} *is any symmetric tensor, and* $\overset{0}{T}{}^{ij}$ *is determined by*

$$\left.\begin{aligned}\overset{0}{T}{}^{ij}&=0 \quad \textit{for } i\neq j\\ \nabla_1\overset{0}{T}{}^{11}&=\rho T^1-\rho f^1\\ \nabla_2\overset{0}{T}{}^{22}&=\rho T^2-\rho f^2\\ \nabla_3\overset{0}{T}{}^{33}&=\rho T^3-\rho f^3\end{aligned}\right\}$$

§ 5. General solution of the stress equations in the *n* dimensional space.

We shall first consider the general solution of

$$\nabla_\mu T^{\lambda\mu}=0 \quad \textit{and} \quad T^{\lambda\mu}=T^{\mu\lambda} \quad (\lambda,\ \mu=1,\ 2,\ 3,\ 4) \tag{5.1}$$

in the four diminsional Euclidean space, where the metric is given by

$$ds^2=g_{\lambda\mu}dx^\lambda dx^\mu \quad (\lambda,\ \mu=1,\ 2,\ 3,\ 4) \tag{5.2}$$

We can similarly prove the following theorem as in § 3.

THEOREM 4. *The general solution of*

$$\nabla_\mu T^{\lambda\mu}=0 \quad \textit{and} \quad T^{\lambda\mu}=T^{\mu\lambda} \quad (\lambda,\ \mu=1,\ 2,\ 3,\ 4)$$

is given by

$$T^{\lambda\mu}=\varepsilon^{\lambda\lambda_1\lambda_2\lambda_3}\varepsilon^{\mu\mu_1\mu_2\mu_3}\nabla_{\lambda_1}\nabla_{\mu_1}v_{\lambda_2\lambda_3\mu_2\mu_3},$$

where $v_{\lambda_2\lambda_3\mu_2\mu_3}=v_{\mu_2\mu_3\lambda_2\lambda_3}$.

PROOF. The tensor $T^{\lambda\mu}$ satisfying (5.1) is expressed by

$$T^{\lambda\mu}=\varepsilon^{\lambda\lambda_1\lambda_2\lambda_3}\nabla_{\lambda_1}V_{\lambda_2\lambda_3}^{\cdot\cdot\mu}. \tag{5.3}$$

Since the integrability condition of (5.3), i. e.,

$$3!\ \nabla_{[\lambda_1}V_{\lambda_2\lambda_3]}^{\cdot\cdot\mu}=\varepsilon_{\lambda\lambda_1\lambda_2\lambda_3}T^{\lambda\mu} \tag{5.4}$$

is (5.1). Contract'ng for λ_1 and μ in (5.4), and making use of $T^{\lambda\mu}=T^{\mu\lambda}$, we have

$$\nabla_\nu V_{[\lambda_2\lambda_3]}^{\cdot\cdot\nu}=-2!\ \nabla_{[\lambda_2}V_{\lambda_3]\nu}^{\cdot\cdot\nu} \tag{5.5}$$

If we put

$$V_{[\lambda_2\lambda_3]}^{\cdot\cdot\mu}=V_{\lambda_2\lambda_3}^{\prime\cdot\cdot\mu}+\nabla_{[\lambda_3}V_{\lambda_3]}^{\cdot\mu},\ (\textit{where } V_{\lambda_3}^{\cdot\nu}\ \textit{satisfies}\ \nabla_\nu V_{\lambda_3}^{\cdot\nu}=-V_{\lambda_3\nu}^{\cdot\cdot\nu}), \tag{5.6}$$

then we get

$$T^{\lambda\mu}=\varepsilon^{\lambda\lambda_1\lambda_2\lambda_3}\nabla_{\lambda_1}V_{\lambda_2\lambda_3}^{\prime\cdot\cdot\mu}, \tag{5.7}$$

where

$$\nabla_\nu V_{\lambda_2\lambda_3}^{\prime\cdot\cdot\nu}=0. \tag{5.8}$$

Since (5.8) is the integrability condition of

$$V_{[\lambda_2\lambda_3]}^{\prime\cdot\cdot\mu}=\varepsilon^{\mu\mu_1\mu_2\mu_3}\nabla_{\mu_1}v_{[\lambda_2\lambda_3]\mu_2\mu_3} \tag{5.9}$$

$V_{(\lambda_2\lambda_3)}^{\prime\cdot\cdot\mu}$ is expressed by (5.9). Therefore from (5.3) and (5 9) we obtain

$$T^{\lambda\mu}=\varepsilon^{\lambda\lambda_1\lambda_2\lambda_3}\varepsilon^{\mu\mu_1\mu_2\mu_3}\nabla_{\lambda_1}\nabla_{\mu_1}v_{\lambda_2\lambda_3\mu_2\mu_3} \tag{5.10}$$

and moreover from $T^{\lambda\mu}=T^{\mu\lambda}$ it follows

$$v_{\lambda_2\lambda_3\mu_2\mu_3}=v_{\mu_2\mu_3\lambda_2\lambda_3}. \qquad q.\ e.\ d.$$

If we now use $v_{\lambda_2\mu_2}g_{\lambda_3\mu_3}$ $(v_{\lambda_2\mu_2}=v_{\mu_2\lambda_2})$ in place of $v_{\lambda_2\lambda_3\mu_2\mu_3}$ in (5.9), we have

$$T^{\lambda\mu}=\varepsilon^{\lambda\lambda_1\lambda_2\nu}\varepsilon^{\mu\mu_1\mu_2}_{\cdot\cdot\cdot\nu}\nabla_{\lambda_1}\nabla_{\mu_1}v_{\lambda_2\mu_2} \tag{5.11}$$

which is a solution of (5.1). Here we can prove the following theorem :

THEOREM 5. *The general solution of*

$$\nabla_\mu T^{\lambda\mu}=0 \quad and \quad T^{\lambda\mu}=T^{\mu\lambda} \quad (\lambda,\ \mu=1, 2, 3, 4)$$

is given by

$$T^{\lambda\mu}=\varepsilon^{\lambda\lambda_1\lambda_2\nu}\varepsilon^{\mu\mu_1\mu_2}_{\cdot\cdot\cdot\nu}\nabla_{\lambda_1}\nabla_{\mu_1}v_{\lambda_2\mu_2},$$

where $v_{\lambda_2\mu_2}=v_{\mu_2\lambda_2}$.

PROOF. Since (5.11) is written in tensor form, it is sufficient to prove this theorem in a Cartesian coordinates.

The case where $\nabla_\mu T^{4\mu}=0$: The general solution of (5.1) is expressed by (5.11). To prove this, we shall show that the next equations are solvable for $v_{\lambda_2\mu_2}$,

$$T^{4\mu}=\varepsilon^{4\lambda_1\lambda_2\nu}\varepsilon^{\mu\mu_1\mu_2}_{\cdot\cdot\cdot\nu}\nabla_{\lambda_1}\nabla_{\mu_1}v_{\lambda_2\mu_2}. \tag{5.12}$$

i. e.,

$$T^{4}_{\cdot\cdot}=\nabla_{i_1}\nabla^{i_1}v^{i_2}_{i_2}-\nabla_{i_2}\nabla^{i_2}v^{i_1}_{i_2}, \quad (i_1,\ i_2=1,\ 2,\ 3) \tag{5.13}$$

$$T^{4}_{\cdot j}=(\nabla_j\nabla^{i_2}v^{\cdot 4}_{i_2}-\nabla^j\nabla^4 v^{i_2}_{i})+(\nabla^4\nabla_{i_1}v^{i_1}_{j}-\nabla_{i_1}\nabla^{i_1}v^{\cdot 4}_{j}), \ (j,\, i_1,\, i_2=1, 2, 3). \tag{5.14}$$

Under the conditions

$$\nabla_j v^j_i=0, \tag{5.15}$$

We shall solve (5.13) and (5.14). From (5.13) and (5.15) we have

$$\Delta v^i_i=T^{\cdot}_{\cdot}, \quad (\Delta=\nabla_i\nabla^i), \tag{5.16}$$

and, from (5.15) we get

$$v^{\cdot j}_{i}=\varepsilon^{jhk}\nabla_h\phi_{ki}, \tag{5.17}$$

accordingly,

$$v^i_i=\varepsilon^{ihk}\nabla_h\phi_{ki}=\frac{3\,!}{\sqrt{g}}\nabla_{[1}\phi_{23]}, \quad (g=det|g_{ij}|). \tag{5.18}$$

As a solution of (5.13), we obtain $\overset{0}{\phi}_{ki}$, and then $\overset{0}{v}{}^{\cdot j}_{i}$ by (5.17).

Next, substituting this $\overset{0}{v}{}^{\cdot j}_{i}$ into (5.14) we have

$$\nabla_j\nabla^i v^4_i-\Delta v^4_j=T^4_{\cdot j}+\nabla^4\nabla_j\overset{0}{v}{}^i_i. \tag{5.19}$$

Similarly as above, under the condition

$$\nabla^i v_i^{\cdot}=0, \tag{5.20}$$

We shall have a solution v_i^4 of (5.19).

From (5.20) we get

$$v_i^4=\varepsilon_{ihk}\nabla^h\psi^k, \tag{5.21}$$

Substituting (5.21) for (5.19), we have

$$\Delta(\varepsilon_{jhk}\nabla^h\psi^k)=-(T^{4}_{\cdot j}+\nabla^4\nabla_j\overset{0}{v}{}^i_i). \tag{5.22}$$

Here we see that

$$\nabla^j(T^{4}_{\cdot j}+\nabla^4\nabla_j\overset{0}{v}{}^i_i)=\nabla^jT^{4}_{\cdot j}+V^4\nabla\overset{0}{v}{}^i_i=\nabla^jT^{4}_{\cdot j}+\nabla^4T^{\cdot 4}_{\cdot}=0,$$

hence we can obtain

$$-(T^{4}_{\cdot j}+\nabla^4\nabla_j\overset{0}{v}{}^i_i)=\varepsilon_{jhk}\nabla^h\Psi^k, \tag{5.23}$$

Therefore from (5.22) and (5.23), we can see that, if we determine $\overset{0}{\psi}{}^k$ by $\Delta\overset{0}{\psi}{}^k=\Psi^k$, then $\overset{0}{v}{}^4_i$ such that $\overset{0}{v}{}^4_i=\varepsilon_{ihk}\nabla^h\overset{0}{\psi}{}^k$ satisfies (5.19).

Thus we obtain $\overset{0}{v}_{\lambda_2\mu_2}$ satisfying (5.12); if we put

$$\overset{0}{T}{}^{\lambda\mu}=\varepsilon^{\lambda\lambda_1\lambda_2\nu}\varepsilon^{\mu\mu_1\mu_2}_{\cdot\cdot\cdot\nu}\nabla_{\lambda_1}\nabla_{\mu_1}\overset{0}{v}_{(\lambda_2\mu_2)}, \tag{5.24}$$

then we have

$$\overset{0}{T}{}^{4\mu}=T^{\ \mu} \tag{5.25}$$

and

$$\nabla_\mu\overset{0}{T}{}^{\lambda\mu}=0,\ \overset{0}{T}{}^{\lambda\mu}=\overset{0}{T}{}^{\mu\lambda} \tag{5.26}$$

The case where $\nabla_\mu T^{4\mu}\neq0$: if we put

$$\overset{1}{T}{}^{\lambda\mu}=T^{\lambda\mu}-\overset{0}{T}{}^{\lambda\mu} \tag{5.27}$$

then we have

$$\nabla_\mu\overset{1}{T}{}^{\lambda\mu}=0,\ \overset{1}{T}{}^{\lambda\mu}=\overset{1}{T}{}^{\mu\lambda} \tag{5.28}$$

and

$$\overset{1}{T}{}^{4\mu}=\overset{1}{T}{}^{\mu4}=0. \tag{5.29}$$

By (5.29), the equations (5.28) are reduced to

$$\nabla_j\overset{1}{T}{}^{ij}=0,\quad \overset{1}{T}{}^{ij}=\overset{1}{T}{}^{ji}\quad (i,\ j=1,\ 2,\ 3). \tag{5.30}$$

The general solution (5.30) is given by theorem 2 as follows:

$$\overset{1}{T}{}^{ij}=\varepsilon^{ii_1i_2}\varepsilon^{jj_1j_2}\nabla_{i_1}\nabla_{j_1}\overset{1}{v}_{i_2j_2}. \tag{5.31}$$

Since we are now considering in a Cartesian coordinates, (5.31) is written as follows:

$$\begin{aligned}\overset{1}{T}{}^{ij}&=\varepsilon^{ii_1i_24}\varepsilon^{jj_1j_2}_{\cdot\cdot\cdot4}\nabla_{i_1}\nabla_{j_1}\overset{1}{v}_{i_2j_2}\\&=\varepsilon^{ii_1i_2\nu}\varepsilon^{jj_1j_2}_{\cdot\cdot\cdot\nu}\nabla_{i_1}\nabla_{j_1}\overset{1}{v}_{i_2j_2}.\end{aligned} \tag{5.32}$$

Moreover, since $\overset{1}{v}_{ij}$ does not contain x^4, if we put

$$\overset{1}{v}_{i4}=\overset{1}{v}_{4i}=\overset{1}{v}_{44}=0,$$

then we have

$$\overset{1}{T}{}^{ij}=\varepsilon^{i\lambda_1\lambda_2\nu}\varepsilon^{j\mu_1\mu_2}_{\cdots\nu}\nabla_{\lambda_1}\nabla_{\mu_1}\overset{1}{v}_{\lambda_2\mu_2},\quad (\overset{1}{v}_{\lambda_2\mu_2}=\overset{1}{v}_{\mu_2\lambda_2}).$$

Furthermore we have from (5.13),

$$\overset{1}{T}{}^{4\mu}=\varepsilon^{4\lambda_1\lambda_2\nu}\varepsilon^{\mu\mu_1\mu_2}_{\cdots\nu}\nabla_{\lambda_1}\nabla_{\mu_1}\overset{1}{v}_{\lambda_2\mu_2}=0,$$

hence we see that $\overset{1}{v}_{ij}$ must satisfy the condition [1]

$$\Delta\overset{1}{v}{}^i_i-\nabla_i\nabla^j\overset{1}{v}{}^i_j=0$$

Therefore, from (5.24), (5.27), (5.29) and (5.32) we obtain

$$T^{\lambda\mu}=\overset{1}{T}{}^{\lambda\mu}+\overset{0}{T}{}^{\lambda\mu}=\varepsilon^{\lambda\lambda_1\lambda_2\nu}\varepsilon^{\mu\mu_1\mu_2}_{\cdots\nu}\nabla_{\lambda_1}\nabla_{\mu_1}(\overset{0}{v}_{(\lambda_2\mu_2)}+\overset{1}{v}_{\lambda_2\mu_2}),$$

if we put

$$v_{\lambda_2\mu_2}=\overset{0}{v}_{(\lambda_2\mu_2)}+\overset{1}{v}_{\lambda_2\mu_2}$$

then we have

$$T^{\lambda\mu}=\varepsilon^{\lambda\lambda_1\lambda_2\nu}\varepsilon^{\mu\mu_1\mu_2}_{\cdots\nu}\nabla_{\lambda_1}\nabla_{\mu_1}v_{2\lambda\mu_2},\quad (v_{\lambda_2\mu_2}=v_{\lambda_2\mu_2}).\quad q.\ e.\ d.$$

Furthemore we can similarly prove the following theorems in the n dimensional space:

THEOREM 6. *The general solution of*

$$\nabla_b T^{ab}=0 \text{ and } T^{ab}=T^{ba}\quad (a,\ b=1,\ 2,\ \dots,\ n)$$

is given by

$$T^{ab}=\varepsilon^{aa_1a_2\cdots a_{n-1}}\varepsilon^{bb_1b_2\cdots b_{n-1}}\nabla_{a_1}\nabla_{b_1}v_{a_2\cdots a_{n-1}\,b_2\cdots b_{n-1}},\quad (a_i,\ b_i=1,\ \dots,\ n)$$

where

$$v_{a_2\cdots a_{n-1}b_2\cdots b_{n-1}}=v_{b_2\cdots b_{n-1}a_2\cdots a_{n-1}}.$$

THEOREM 7. *The general solution of*

$$\nabla_b T^{ab}=0\quad and\quad T^{ab}=T^{ba}\quad (a,\ b=1,\ \dots,\ n)$$

is given by

$$T^{ab}=\varepsilon^{aa_1a_2c_1\cdots c_{n-3}}\varepsilon^{bb_1b_2}_{\cdots\cdots c_1\cdots c_{n-3}}\nabla_{a_1}\nabla_{b_1}v_{a_2b_2},$$

where $v_{a_2b_2}=v_{b_2a_2}$.

Theorem 7. means that $v_{a_2b_2}g_{a_3b_3}\cdots g_{a_{n-1}b_{n-1}}$ can be taken in place of $v_{a_2\cdots a_{n-1}b_2\cdots b_{n-1}}$ to express the general solution $\nabla_b T^{ab}=0$ and $T^{ab}=T^{ba}$.

Finally we have the following theorem:

1) Here we can easily see that when $T^{4}_{\cdot j}=0$, (5.14) holds for this $\overset{1}{v}_{\lambda\mu}$.

THEOREM 8. *The general solution of*

$$\nabla_b T^{ab}=T^b \quad \textit{and} \quad T^{ab}=T^{ba} \quad (a,\ b=1,\ 2,\ \dots,\ n)$$

is given by

$$T^{ab}=\overset{1}{T}{}^{ab}+\overset{0}{T}{}^{ab},$$

where $\overset{1}{T}{}^{ab}$ *is the general solution of* $\nabla_b T^{ab}=0,\ T^{ab}=T^{ba}$, *and* $\overset{0}{T}{}^{ab}$ *is determined by*

$$\left.\begin{aligned} &\overset{0}{T}{}^{ab}=0 \ \textit{for}\ a\neq b\\ &\nabla_1 \overset{0}{T}{}^{11}=\overset{1}{T}\\ &\quad\vdots\\ &\nabla_n \overset{0}{T}{}^{nn}=\overset{n}{T}\end{aligned}\right\}$$

REMARK 1. In order to obtain the general solution of

$$\nabla_b T^{ab}=T^a \quad \textit{and} \quad T^{ab}=T^{ba}, \tag{5.33}$$

we shall introduce x^{n+1} as a parameter which has no physical meaning and is to put as an arbitrary constant in the solution; where the metric is given by

$$ds^2=g_{ab}dx^a dx^b+(dx^{n+1})^2=g_{\alpha\beta}dx^\alpha dx^\beta,\ (a,\ b=1,\ \dots,\ n;\ \alpha,\ \beta=1,\ \dots,\ n+1).$$

If we put $T^{a,\,n+1}$, $T^{n+1,\,a}$ and $T^{n+1,\,n+1}$ as follows:

$$\nabla_{n+1}T^{a,\,n+1}=T^a,\ T^{a,\,n+1}=T^{n+1,\,a},\ \nabla_{n+1}T^{n+1,\,n+1}=-\nabla_a T^{a,\,n+1},\ (a=1,\ \dots,\ n). \tag{5.34}$$

Then (5.33) is reduced to

$$\nabla_\beta T^{\alpha\beta}=0 \quad \textit{and} \quad T^{\alpha\beta}=T^{\beta\alpha} \quad (\alpha,\ \beta=1,\ 2,\ \dots,\ n+1). \tag{5.35}$$

Since we know the general solution of (5.35) by theorem 6 or theorem 7, considering together the general solution and the additional conditions (5.34), we shall obtain the general solution of (5.33) if we put x^{n+1} as a constant. Thus also, by this method, theorem 8 can be derived.

REMARK 2. In the theorem 6, let $\overset{0}{v}$ be a system of stress-function corresponding to a given stress tensor T^{ab}, then the general system v of stress-functions corresponding to the same stress tensor T^{ab} is given by

$$\boldsymbol{v}=\overset{0}{\boldsymbol{v}}+\overset{1}{\boldsymbol{v}}$$

where

$$X\overset{1}{\boldsymbol{v}}\equiv\varepsilon^{aa_1\cdots a_{n-1}}\varepsilon^{bb_1\cdots b_{n-1}}\nabla_{a_1}\nabla_{b_1}\overset{1}{v}_{a_2\cdots a_{n-1}b_2\cdots b_{n-1}}=0. \tag{5.36}$$

Similarly, in the theorem 7, let $\overset{0}{\tilde{v}}$ be a system of stress-functions corresponding to a given stress tensor T^{ab}, then the general system $\tilde{v}$ of stress-functions corresponding to the same stress tensor T^{ab} is given by

$$\tilde{v} = \overset{0}{\tilde{v}} + \overset{1}{\tilde{v}}$$

where
$$\tilde{X}\overset{1}{\tilde{v}} = \varepsilon^{aa_1a_2c_1\cdots c_{n-3}}\varepsilon^{bb_1b_2}{}_{\cdots c_1\cdots c_{n-3}}\nabla_{a_1}\nabla_{b_1}\overset{1}{\tilde{v}}_{a_2b_2}=0. \tag{5.37}$$

§ 6. Integrability conditions of $\nabla_{(i}u_{j)}=c_{ijkl}T^{kl}$.

Next we shall consider the integrability conditions of

$$\nabla_{(i}u_{j)}=e_{ij}=c_{ijkl}T^{kl}, \quad \textit{(generalized Hooke's law)} \tag{6.1}$$

(6.1) is equivalent to

$$\nabla_i u_j = e_{ij} + \omega_{ij}, \ (e_{[ij]}=0, \ \omega_{(ij)}=0). \tag{6.2}$$

And the integrability conditions of (6.2) are

$$\nabla_{[k}e_{i]j} + \nabla_{[k}\omega_{i]j}=0, \ (e_{[ij]}=0, \ \omega_{(ij)}=0). \tag{6.3}$$

Since $e_{(ij)}=0$, from (6.3) we have

$$\nabla_{[k}\omega_{ij]}=0. \tag{6.4}$$

By using $\omega_{(ij)}=0$, from (6.4) we have $2\nabla_{[k}\omega_{i]j}=\nabla_j\omega_{ik}$.
Hence (6.3) is reduced to

$$2\nabla_{[k}e_{i]j}=\nabla_j\omega_{ki}, \quad (e_{[ij]}=0, \omega_{(ij)}=0). \tag{6.5}$$

Conversely from (6.5) we have (6.4), accordingly (6.3).
Moreover the integrability conditions of (6.5) are

$$\nabla_{[h}\nabla_{[k}e_{i]j]}=0 \quad \textit{i. e.,} \quad \varepsilon^{phi}\varepsilon^{qkj}\nabla_h\nabla_k e_{ij}=0\,; \tag{6.6}$$

or making use of the operator X, we may write this

$$Xe=0. \tag{6.7}$$

Hence, from (6.1) and (6.7) we obtain

$$XCt=0, \ (t=(T^{ij})). \tag{6.8}$$

So we have from Theorem 8 $\quad XCXv+XC\overset{0}{t}=0, \qquad (6.9)$

If we put $XC=Y$, $C^{-1}v=\bar{v}$, it becomes

$$Y^2\bar{v}+Y\overset{0}{t}=0. \tag{6.10}$$

Thus we have the theorem:

Theorem 9. *The fundamental equvtions in the theory of elastisity are reduced to*

$$Y^2\bar{v}+Y\overset{0}{t}=0.$$

§ 7. General solution of the fundamental equations of electromagnetic field in the n-dimensional space. Conclusion.

We shall finally add the consideration of the equations

$$\nabla_b F^{ab}=J^a \quad \textit{and} \quad F^{ab}=-F^{ba} \quad (a,\ b=1,\ \ldots,\ n), \tag{7.1}$$

which are, in the case where $n=4$, the fundamental equations of electro-magnetic field.

If $J^{a}\neq 0$, (7.1) is reduced to

$$\nabla_{\beta}F^{\alpha\beta}=0 \quad and \quad F^{\alpha\beta}=-F^{\beta\alpha} \quad (\alpha,\ \beta=1,\ 2,\ \dots,\ n+1), \qquad (7.2)$$

similarly as in the remarks in § 6. Hence we shall merely consider the case where $J^{a}=0$: i.e.,

$$\nabla_{b}F^{ab}=0 \quad and \quad F^{ab}=-F^{ba} \quad (a,\ b=1,\ 2,\ \dots,\ n+1). \qquad (7.3)$$

Also, in this consideration, we obtain the following results:

RESULTS: *The general solution of*

$$\nabla_{b}F^{ab}=0 \quad and \quad F^{ab}=-F^{ba} \quad (a,\ b=1,\ 2,\ \dots,\ n)$$

is given by

$$F^{ab}=\varepsilon^{aa_1\cdots a_{n-1}}\varepsilon^{bb_1\cdots b_{n-1}}\nabla_{a_1}\nabla_{b_1}f_{a_2\cdots a_{n-1}b_2\cdots b_{n-1}}$$

where

$$f_{a_2\cdots a_{n-1}b_2\cdots b_{n-1}}=-f_{b_2\cdots b_{n-1}a_2\cdots a_{n-1}};$$

or, by

$$F^{ab}=\varepsilon^{aa_1a_2c_1\cdots c_{n-3}}\varepsilon^{bb_1b_2}{}_{c_1\cdots c_{n-3}}\nabla_{a_1}\nabla_{b_1}f_{a_2b_2}$$
$$=(n-3)!\{2\nabla^{[a}\nabla_{c}f^{b]c}+\Box f^{ab}\}$$

where

$$f_{ab}=-f_{ba}, \quad and \quad \Box \ denotes \ \nabla_{a}\nabla^{a}.$$

Moreover the integrability conditions of $\nabla_{[a}\varphi_{b]}=F_{ab}$, which has the physical meaning in the case where $n=4$, is expressed by

$$\Box\nabla_{[a}f_{bc]}=0.$$

Now we shall conclude that, by applying the operators $\tilde{X}$ and X on the tensors $\tilde{v}_{ab}$ and $v_{a_2\cdots a_{n-1}b_2\cdots b_{n-1}}$ respectively, where

$$(\tilde{X}\tilde{\boldsymbol{v}})^{ab}=\varepsilon^{aa_1a_2c_1\cdots c_{n-3}}\varepsilon^{bb_1b_2}{}_{c_1\cdots c_{n-3}}\nabla_{a_1}\nabla_{b_1}\tilde{v}_{a_2b_2},$$
$$(X\boldsymbol{v})^{ab}=\varepsilon^{aa_1a_2\cdots a_{n-1}}\varepsilon^{bb_1\cdots b_{n-1}}\nabla_{a_1}\nabla_{b_1}v_{a_2\cdots a_{n-1}b_2\cdots b_{n-1}},$$

we can obtain the stress tensor T^{ab} from the symmetric part of $\tilde{\boldsymbol{v}}$ or $\boldsymbol{v}$, and the electro-magnetic force F^{ab} from the antisymmetric part of $\tilde{\boldsymbol{v}}$ or $\boldsymbol{v}$. Therefore $\tilde{\boldsymbol{v}}$ and $\boldsymbol{v}$ seem to be the intrinsic tensors of the field in the unified standpoint.

Concerning the boundary problems, it seems, too, to be more essential to consider it by $\tilde{\boldsymbol{v}}$ or $\boldsymbol{v}$ than to consider it by T^{ab} or F^{ab}.

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