SOME PROPERTIES OF LORENTZ TRANSFORMATIONS

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(Received May 9, 1952)

§ 1. Property of special Lorentz transformations.

In special relativity and quantum mechanics, physical laws are expressed in the forms which are invariant by special Lorentz transformations. The special Lorentz transformations are given by the equations (Lorentz transformations along x-axis)

$$x'=\frac{x-ut}{\sqrt{1-u^2/c^2}},\quad y'=y,\quad z'=z,\quad t'=\frac{t-ux/c^2}{\sqrt{1-u^2/c^2}},\tag{1.1}$$

and the equations which are obtained by cyclic interchange of x, y and z. Concerning special Lorentz transformations along x-and y-axes, we can prove the following statement. *The physical laws expressed in the forms which are invariant by special Lorentz transformations along x- and y-axes, have necessarily the property which is invariant by rotation in x, y-plane.* In order to prove this statement we shall consider 5 coördinate systems K, K', K'', K''' and $K^{(IV)}$ such that

K' is moving with uniform velocity u in the direction of x-axis of K,
K'' „ v „ y'-axis of K',
K''' „ $\bar{u}$ „ x''-axis of K'',
$K^{(IV)}$ „ $\bar{v}$ „ y'''-axis of K''',

where u and v are arbitrarily chosen and $\bar{u}$ and $\bar{v}$ are determined by the equations

$$\begin{aligned}\bar{u}=-u\sqrt{1-v^2/c^2},\quad \bar{v}&=-v/\sqrt{1-\bar{u}^2/c^2}\\ &=-v/\sqrt{1-u^2/c^2+u^2v^2/c^4}\end{aligned}$$

in terms of u and v. The relations between the coordinates of K and K' are defined by (1.1), and the relations between the coordinates of K' and K'' are given by the equations

$$x''=x',\quad y''=\frac{y'-vt'}{\sqrt{1-v^2/c^2}},\quad z''=z',\quad t''=\frac{t'-vy'/c^2}{\sqrt{1-v^2/c^2}}\tag{1.2}$$

Further, the relations between the coordinates of K'' and K''' or K''' and $K^{(IV)}$ are given in the similar equations. Namely:

$$x''' = \frac{x''-\bar{u}t''}{\sqrt{1-\bar{u}^2/c^2}}, \quad y''' = y'', \quad z''' = z'', \quad t''' = \frac{t''-\bar{u}x''/c^2}{\sqrt{1-\bar{u}^2/c^2}}, \tag{1.3}$$

$$x^{(IV)} = x''', \quad y^{(IV)} = \frac{y'''-\bar{v}t'''}{\sqrt{1-\bar{v}^2/c^2}}, \quad z^{(IV)} = z''', \quad t^{(IV)} = \frac{t'''-\bar{v}y'''/c^2}{\sqrt{1-\bar{v}^2/c^2}}. \tag{1.4}$$

From these equations we can obtain the relations between the coordinates of K and $K^{(IV)}$ by successive substitutions of (1.1), (1.2) and (1.3) into the equations (1.4). The resulting equations are

$$\begin{aligned} x^{(IV)} &= x\cos\theta - y\sin\theta, \quad z^{(IV)} = z, \\ y^{(IV)} &= x\sin\theta + y\cos\theta, \quad t^{(IV)} = t, \end{aligned} \tag{1.5}$$

where θ is defined by

$$\cos\theta = \frac{\sqrt{1-u^2/c^2}}{\sqrt{1-u^2/c^2+u^2v^2/c^4}}, \quad \sin\theta = \frac{uv/c^2}{\sqrt{1-u^2/c^2+u^2v^2/c^4}} \tag{1.6}$$

Namely, the coordinate system $K^{(IV)}$ is obtained directly by the rotation of x-, y-axes of the system K in x, y-plane. Here the time coordinate t is unaltered. On the other hand, by the coordinate system K and $K^{(IV)}$, the physical laws are expressed in the same forms. Therefore, the forms should be invariant by the transformations (1.5). Hence the statement holds good. Also the same statement holds in y, z- and z, x-plane respectively. Therefore, the statement may be generalized in three dimensional form as follows. *The physical laws expressed in the forms which are invariant by special Lorentz transformations along x-, y- and z-axes, have necessarily the property which is invariant by rotation in x, y, z-space (i.e. the property of spherical symmetry.)*

The statement is also predicted from the following consideration. The transformations (1.1), u being regarded as parameter, form the one-parameter continuous transformation group generated by the infinitesimal transformations with the operater

$$X = t\frac{\partial}{\partial x} + \frac{x}{c^2}\frac{\partial}{\partial t}.$$

Similarly, corresponding to the Lorentz transformations along y- or z-axis, there exist the infinitesimal transformations with the following operator Y or Z:

$$Y = t\frac{\partial}{\partial y} + \frac{y}{c^2}\frac{\partial}{\partial t}, \quad Z = t\frac{\partial}{\partial z} + \frac{z}{c^2}\frac{\partial}{\partial t}.$$

Then we can show that, even though these operators X, Y and Z form a one-parameter continuous transformation group respectively, they do not constitute a 3-parameter group. In fact, the commutators of the operators X, Y and Z become the operators generating the group of rotations in x, y, z-space, and do not become the linear combinations of X, Y and Z with constant coefficients. Namely,

$$[XY]=\frac{1}{c^2}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right),\quad [YZ]=\frac{1}{c^2}\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right),\quad \text{etc.}$$

The first equation of the above shows that the invariant of the operators X and Y is necessarily invariant by the operator

$$x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}$$

which generates the group of rotation in x, y-plane. Hence the statement is predicted.

From the above statement we have the following inference. *In the case where the properties considered are not spherically symmetric, it is meaningless to express the physical laws in the forms which are invariant by special Lorentz transformations along x-, y- and z-axes.* In this case *new fundamental group of transformations* should be considered in place of special Lorentz transformations. And the physical laws should be expressed in the forms which are invariant by this fundamental group of transformations. In the next section we shall consider the problem to determine such a new fundamental group of transformations.

§ 2. Establishment of new fundamental group of transformations.

We consider the general case where the properties considered are not assumed to be spherically symmetric or not, and suppose that a coordinate system K' is moving with uniform velocity to the other system K, the x-, y-, z- components of the velocity of the origin of K' being u^1, u^2, u^3 with respect to K. In order to determine the relations between the coordinates of K and K', we shall consider the conditions which should be satisfied by the transformations of the coordinates of K and K'. First, it must be postulated that the velocity of light is constant ($=c$) for K and K'. We represent this postulate by the condition that the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{c^2\partial t^2}\right)\varphi=0$$

is invariant by the transformations of the coordinates of K and K'. Next, we put the second postulate that the relations between the coordinates of K and K' should be determined uniquely by the components u^1, u^2, u^3 of the uniform velocity. This postulate means that K' does not undergo an arbitrary rotation while K' is moving with uniform velocity to K. According to this postulate let us denote the relations between the coordinates of K and K' by

$$\begin{aligned} x'^i &= f^i(x^1,\ x^2,\ x^3,\ x^4\ ;\ u^1,\ u^2,\ u^3) \\ &\equiv f^i(x\ ;\ u) \end{aligned} \qquad (i=1, 2, 3) \tag{2.1}$$

where x^1, x^2, x^3, x^4 are used in place of x, y, z, ct, and $f^i(i=1, \ldots, 4)$ are certain functions of x's and u's. Here we assume that u^1, u^2, u^3 are implied as parameters and the functional forms of f^i are independent of u's. Lastly, in order to consider the third postulate, we take a coordinate system K'' which is moving with uniform velocity to K', the x'-, y'-, z'-components of the velocity of the origin of K'' being v^1, v^2, v^3 with respect to K'. Then the coordinates x''^i of K'' are expressed by the equations $x''^i=f^i(x'\ ;\ v)$ in the same functional forms as (2.1). Since K' is moving with uniform velocity with respect to K, physical laws are expressed in the same forms by K and K' as well as by K' and K'', accordingly by K and K''. Hence, it will be natural to assume that K'' is also moving with certain uniform velocity (say w) with respect to K, viz. x''^i are expressed in the forms $x''^i=f^i(x\ ;\ w)$ by the same functions as (2.1), replacing u^1, u^2, u^3 by w^1, w^2, w^3, where w^1, w^2, w^3 are the x-, y-, z-components of the uniform velocity w with respect to K. This means that the transformations (2.1), u^1, u^2, u^3 being regarded as parameters, form a 3-parameter group. From this consideration, we put the third postulate as follows. The transformations of (2.1), u^1, u^2, u^3 being regarded as parameters, must form a 3-parameter continuous transformation group.

In the previous paper(1), from the postulates above mentioned, we have obtained the transformations of the coordinates of K and K'. These transformations constitute a sub-group of the general Lorentz transformation group. This sub-group of transformations are considered as representing the relations between the coordinates of two systems, one of which is moving with uniform velocity to the other. Therefore, the physical laws should be expressed in the forms which are invariant by this group of transformations (sub-group of general Lorentz transformation group).

In order to bring out the basic part of this group of transformations, we consider the operators of the infinitesimal transformations which generate this group. The operators are given by the equations (2.4) in the previous paper[1]. Namely,

$$\begin{aligned} P_i &= x^i\frac{\partial}{\partial x^4}+x^4\frac{\partial}{\partial x^i}+d_p x^p\frac{\partial}{\partial x^i}-x^i d^p\frac{\partial}{\partial x^p} \qquad (i,\ j,\ p,\ q=1,\ 2,\ 3)\\ &\quad + d_i\left(hd^j \in_{jpq} x^p\frac{\partial}{\partial x^q}+kx^\lambda\frac{\partial}{\partial x^\lambda}\right) \qquad (\lambda=1,\ \dots,\ 4) \end{aligned} \tag{2.2}$$

where h and k are arbitrary constants and d_i are any constants satisfying the condition $d_i d_i=1$. The last term $x^\lambda\partial/\partial x^\lambda$ (the coefficient of k) of the above, represents the operator of the infinitesimal transformations of the dilatation group $x'^\lambda=ax^\lambda$. Hence, excluding the dilatation, we choose an arbitrary constant k equal to zero. Next, the first term $d^j \in_{jpq} x^p\partial/\partial x^q$ (the coefficient of an arbitrary constant h) in the last bracket of the above equation, represents the operator of the infinitesimal transformations of the one-parameter rotation group, the direction cosine of the axis of rotation being d_1, d_2, d_3. Now, excluding this term, we put $h=0$. This condition that $h=0$, is also derived from the following consideration. Any one-parameter sub-group of the group of transformations defined by (2.2), is generated by the infinitesimal transformations with the operator of the form e^iP_i. Here, we put the condition that any two one-parameter sub-groups of the group defined by (2.2), should constitute a two-parameter sub-group. This condition is equivalent to the condition that any two operators e^iP_i and f^jP_j, e^i and $f^j (i, j=1, 2, 3)$ being arbitrary constants, should generate a two-parameter group, i.e. the commutator of e^iP_i and f^jP_j should be expressed by linear combination of e^iP_i and f^jP_j with constant coefficients:

$$[e^iP_i,\ f^jP_j]=\text{const.}\times e^iP_i+\text{const.}\times f^jP_j\,.$$

In order that this equation may hold for arbitrary constants e^i and f^j, it must be that the commutators of P_i and P_j for all i and j of $i, j=1, 2, 3$, are expressed by linear combinations of P_i and P_j with constant coefficients in the forms

$$[P_iP_j]=\text{const.}\times P_i+\text{const.}\times P_j \tag{2.3}$$

However, on the other hand, we have the relations $[P_iP_j]=d_{ijk}P_k$ where d_{ijk} are given by the equations (2.3) of the previous paper[1], i.e.

$$d_{ij}\ = hd^l(d_j \in_{lik}-d_i \in_{ljk})+d_i\delta_{jk}-d_j\delta_{ik}. \tag{2.4}$$

Comparing (2.3) with (2.4), we have $h=0$.

If we put arbitrary constants h and k equal to zero, the form of P_i given by (2.2), becomes a simpler form, which we denote by S_i as follows:

$$S_i = x^i \frac{\partial}{\partial x^4} + x^4 \frac{\partial}{\partial x^i} + d_p x^p \frac{\partial}{\partial x^i} - x^i d_p \frac{\partial}{\partial x^p} \quad (i,\ p=1,\ 2,\ 3) \tag{2.5}$$

The equations of the transformations of the group generated by the infinitesimal transformations with the operators S_i above defined, are expressed in the finite form as follows:

$$\begin{aligned} x'^i &= x^j\left[\delta^i_j - \frac{d^i + u^i/c}{1+(du)/c} d_j + d^i \left\{ \frac{u_j/c}{\sqrt{1-(uu)/c^2}} + \frac{d_j\sqrt{1-(uu)/c^2}}{1+(du)/c} \right\}\right] \\ &\quad - t\left[d^i \frac{(uu)/c + (du)\{1-\sqrt{1-(uu)/c^2}\}}{\{1+(du)/c\}\sqrt{1-(uu)/c^2}} + \frac{u^i}{1+(du)/c}\right], \\ t' &= [t-(ux)/c^2]/\sqrt{1-(uu)/c^2}, \qquad (i,\ j=1,\ 2,\ 3) \end{aligned} \tag{2.6}$$

which are obtained by putting $h=k=0$ in the equations (3.7) of the previous paper(1). These equations are considered as representing *the relations between the coordinates of the two systems K and K', one of which K' is moving with uniform velocity with respect to the other K, the x-, y-, z-components of the velocity being u^1, u^2, u^3.* Corresponding to these equations we have the formula for *sum* (say w^i) of velocities u^i (with respect to K) and v^i (with respect to K') as follows:

$$w^i = \frac{\begin{bmatrix} u^i\{1+(dv)/c\}\{1+(du)/c\}/\sqrt{1-(uu)/c^2} + d^i(dv)\sqrt{1-(uu)/c^2} \\ +v^i\{1+(du)/c\} - d^i\{(dv)+(uv)/c\} \end{bmatrix}}{\begin{bmatrix} [\{(du)/c+(uu)/c^2\}(dv)/c + \{1+(du)/c\}]/\sqrt{1-(uu)/c^2} \\ +\{(uv)-(dv)(du)\}/c^2 \end{bmatrix}} \tag{2.7}$$

which is obtained by putting $h=k=0$ in the equations (4.2) of the previous paper(1).

The next problem is to express physical laws in the forms which are invariant by the group of transformations given by (2.6). Thus, by considering invariants of the group of transformations (2.6), physical quantities e.g. momentum, mass, energy etc. will be defined (see the next paper), and wave equations in quantum mechanics may also be considered.

Reference

(1) T. Shibata: Fundamental group of transformations in special relativity and quantum mechanics, this journal Vol. **16**, No. 1 (1952), 61—66.