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Some Results Deduced from the New Fundamental Group of Transformations in Special Relativity and Quantum Mechanics.

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§ 1. Character of the new fundamental group of transformations.

In the previous paper [1], we have shown that the special Lorentz transformations are available only in the case where the properties under consideration are assumed to be spherically symmetric. In the case where it is not assumed whether the properties considered are spherically symmetric or not, a new fundamental group of transformations should be taken in place of special Lorentz transformations, as representing the relations between the coordinates of two systems one of which is moving with uniform velocity to the other. Such a new fundamental group of transformations has been searched in the previous paper [2]. The form of the equations of the transformations of this group, has been given by the expression (1.2) of the previous paper [3], as follows

$$
\begin{aligned}
x'^i = x^j & \left[\delta^i_j - \frac{d^i - u^i/c}{1-(du)/c}\, d_j - d^i\left\{\frac{u_j/c}{\sqrt{1-(uu)/c^2}} - \frac{d_j\sqrt{1-(uu)/c^2}}{1-(du)/c}\right\}\right] \\
& + t\left[d^i\,\frac{(uu)/c-(du)\{1-\sqrt{1-(uu)/c^2}\}}{\{1-(du)/c\}\sqrt{1-(uu)/c^2}} - \frac{u^i}{1-(du)/c}\right], \qquad (1.1) \\
t' = [t-(ux)/c^2] & \Big/ \sqrt{1-(uu)/c^2} \qquad (i, j=1, 2, 3)
\end{aligned}
$$

the notations which appear in the expression, being explained there.

The essential difference between the transformations (1.1) and the special Lorentz transformations, is that the former transformations (1.1) contains constants d_1, d_2, d_3 which are considered as representing the direction cosines of certain direction because of $(dd)=1$. Thus by the new fundamental group of transformations, certain direction whose direction cosines are d_1, d_2, d_3, is introduced. Concerning such a direction determined by the direction cosines d_1, d_2, d_3, we can show that the following statements hold.

I. When the direction of progress of light coincides with the direction whose direction cosines are d_1, d_2, d_3 in a coordinate system $K(x, y, z, t)$, an aberration of light is not found along the direction perpendicular to the direction of the progress of light.

II. In the case where the properties under consideration are assumed to be axially symmetric, the direction of the axis of axial symmetry coincides with the direction determined by the direction cosines d_1, d_2, d_3.

In the following sections (2,3), we give the proof of the above statements.

§ 2. Aberration of light.

We consider two coordinate systems $K(x, y, z, t)$ and $K'(x', y', z', t)$ one of which K' is moving with uniform velocity to K, the x-, y-, z-components of the velocity being u^1, u^2, u^3 with respect to K. The relations between the coordinates of K and K', are given by the equations (1.1), where x^i $(i=1, 2, 3)$ are used in place of x, y, z. Now we suppose that the direction of the uniform velocity of K' with respect to K, is perpendicular to the direction whose direction cosines are d_1, d_2, d_3, i. e. $(du)=0$. Under this supposition, we can show that the progress of light in the direction whose direction cosines being d_1, d_2, d_3 in the system K, is observed unaltered by the transformations of coordinates from K to K', namely, by the transformations (1.1), the velocity $dx^i/dt=cd^i$ is transformed to $dx'^i/dt'=cd^i$. This result is easily deduced, from (1.1), by calculating dx'^i/dt' under the conditions $(du)=0$ and $dx^i/dt=cd^i$. Hence, in this ease, the aberration of light is not found. Thus the first statement of § **1** is proved.

In other cases, however, we see that the aberration of light is found. For example, to compare the aberration of light introduced by the special Lorentz transformations with the result deduced from (1.1), we consider the case where the direction whose direction cosines are d_1, d_2, d_3, is perpendicular to the direction of the progress of light and the former direction coincides with the direction of the uniform velocity of K', viz. $u^i=d^i u$, u being the magnitude of the uniform velocity of K'. In this case, substituting $u^i=d^i u$ into (1.1), the relations between the coordinates of K and K' are expressed as follows

$$x'^i=x^j\left[\delta^i_j-d^i\,d_j\left\{1-\frac{1}{\sqrt{1-u^2/c^2}}\right\}\right]-\frac{t\,u\,d^i}{\sqrt{1-u^2/c^2}} \tag{2.1}$$

$$t'=[t-(ux)/c^2]\Big/\sqrt{1-u^2/c^2} \qquad (i, j=1, 2, 3)$$

Hence, from the above, we have

$$\frac{d x'^i}{d t'} = \frac{[d x^j/dt][\{\delta^i_j - d^i d_j\}\sqrt{1-u^2/c^2} + d^i d_j] - u d^i}{1 - u d_j [d x^j/d t]} \tag{2.2}$$

From these equations, an aberration of light is introduced similarly as deduced from the special Lorentz transformations.

§ 3. Axis of symmetry in the case of axial symmetry.

In this section we shall prove the second statement of § **1.** For this purpose we take the case where the properties under consideration are axially symmetric (not spherically symmetric), and suppose that the direction cosines of the axis of axial symmetry are a^1, a^2, a^3. In this case physical laws should be expressed in the forms which are invariant not only by the transformations (1.1) but also by the rotations about the axis of axial symmetry. Hence the transformations (1.1) and the rotations about the axis of axial symmetry should constitute a fundamental group of transformations. The rotations about the axis determined by the direction cosines a^1, a^2, a^3, are generated by the infinitesimal transformations with the operator

$$A = a^h R_h \tag{3.1}$$

where $R_h = \epsilon_{hij}\, x^i\, \partial/\partial x^j$

$$\epsilon_{hij} = \begin{cases} 1 \text{ if } h,\ i,\ j \text{ is an even permutation of } 1,\ 2,\ 3, \\ -1 \text{ if } h,\ i,\ j \text{ is an odd permutation of } 1,\ 2,\ 3, \\ 0 \text{ in any other cases.} \end{cases}$$

and double suffixes are summed from 1 to 3. On the other hand, the group of transformations (1.1), u^1, u^2, u^3 being regarded as parameters, are generated by the infinitesimal transformations with the operators

$$S_i = \frac{x^i}{c}\frac{\partial}{\partial t} + \left[\{ct - (dx)\}\, \delta^j_i + d_j\, x^i\right] \frac{\partial}{\partial x^j} \tag{3.2}$$

$$= Q_i - \epsilon_{ijk}\, d^k R_j \tag{3.3}$$

where
$$Q_i = \frac{x^i}{c}\frac{\partial}{\partial t} + ct \frac{\partial}{\partial x^i}$$

(The equations (3.2) are obtained from the equations (3.3) of the previous paper [**3**], and (3.3) are obtained from the equations (2.4) of the previous paper [**2**], by putting $h=k=0$ and replacing d^l by $-d^l$). Therefore, the condition that the transformations (1.1) and the rotations about the axis of axial symmetry constitute a group, is equivalent to the condition that the operators (3.1) and (3.3) generate a group, i. e. the commutators of (3.1) and (3.3) are expressed by linear combinations of (3.1) and (3.3) with constant coefficients b^i and b^j_i as follows

$$[A\, S_i]=b_i\, A+b^j_i\, S_j\,. \tag{3.4}$$

The left hand side of the above is calculated as follows

$$\begin{aligned}[A\, S_i]&=a^h\,[R_h\, Q_i]-a^h\,\epsilon_{ijk}\,d^k\,[R_h\, R_j]=a^h\,\epsilon_{ihj}\,Q_j-a^h\,d^k\,\epsilon_{ijk}\,\epsilon_{jhl}\,R_l\\ &=a^h\,\epsilon_{ihj}\,Q_j+a^h\,d^k\,(\delta_{ih}\,\delta_{kl}-\delta_{kh}\,\delta_{il})\,R_l\\ &=a^h\,\epsilon_{ihj}\,Q_j+a^i\,d^k\,R_k-a_h\,d^h\,R_i\,,\end{aligned} \tag{3.5}$$

while the right hand side of (3.4) is expressed as

$$b_i\,a^k\,R_k+b^j_i\,(Q_j-\epsilon_{jkl}\,d^l\,R_k)\,. \tag{3.6}$$

Hence, in order that (3.5) may be equal to (3.6), the coefficients of the operators Q_j and R_k of (3.5) and (3.6), must be equal to each other, i. e.

$$a^h\,\epsilon_{ihj}=b^j_i \tag{3.7}$$

$$a^i\,d^k-a_h\,d^h\,\delta^k_i=b_i\,a^k-b^j_i\,\epsilon_{jkl}\,d^l \tag{3.8}$$

Substituting the left hand member of (3.7) into the place of b^j_i of (3.8), we have

$$a^i\,d^k-a_h\,d^h\,\delta^k_i=b_i\,a^k-a^h\,\epsilon_{ihj}\,\epsilon_{jkl}\,d^l\,. \tag{3.9}$$

The last term of the above is calculated as

$$a^h\,\epsilon_{ihj}\,\epsilon_{jkl}\,d^l=a^h\,(\delta_{ik}\,\delta_{hl}-\delta_{il}\,\delta_{hk})\,d^l=a^h\,d^h\,\delta_{ik}-a^k\,d^i \tag{3.10}$$

Hence (3.9) becomes

$$a^i\,d^k=b_i\,a^k+a^k\,d^i \tag{3.11}$$

Multiplying (3.11) by d^k and summing the resulting equations for $k=1,2,3$, we have

$$a^i=b_i\,(a\,d)+(a\,d)\,d^i \qquad (i=1,2,3) \tag{3.12}$$

From these equations, if follows that $(a\,d)\neq 0$. Because if $(a\,d)=0$, (3.12) become $a^i=0$ which contradict the supposition that the direction cosines of the axis of axial symmetry are denoted by a^1, a^2, a^3. Hence, dividing each term of (3.12) by $(a\,d)$, it follows that

$$b_i=-d^i+a^i/(a\,d)\,. \tag{3.13}$$

Substituting (3.13) into (3.11), we have $d^k=a^k/(a\,d)$, which shows that the direction whose direction cosines are a^1, a^2, a^3, coincides with the direction determined by the direction cosines d^1, d^2, d^3. So we have proved the second statement of § **1**.

In the following section, we will consider some results which will be deduced from the transformations of the group generated by the operators (3.1) and (3.2).

§ 4. Momentum and mass in the case of axial symmetry.

In the previous paper [**3**], we have defined momentum and mass as an invariant vector of the group of transformations (1.1). Namely, the momentum M^i and mass M of a particle moving with the velocity whose x-, y-, z-components being v^1, v^2, v^3, have been defined by the equations (2.2) of the previous paper [**3**], as follows

$$\begin{aligned} M^i &= \frac{m}{\sqrt{1-(vv)/c^2}}\,v^i + \frac{n\sqrt{1-(vv)/c^2}+(lv)/c}{1-(dv)/c}\,c\,d^i + c\,l^i \\ M &= \frac{m}{\sqrt{1-(vv)/c^2}} + \frac{n\sqrt{1-(vv)/c^2}+(lv)/c}{1-(dv)/c} \qquad (i=1,2,3) \end{aligned} \tag{4.1}$$

where m and n are arbitrary constants and $l^i\,(i=1,2,3)$ are any constants subject to the condition $(d\,l)=0$.

In this section, we will determine M^i and M as an invariant vector not only of the group of transformations (1.1) but also of the group of rotations about the axis of axial symmetry. Such an invariant vector may be regarded as giving a definition of momentum and mass in the case where the properties under consideration are axially symmetric.

In the previous section **3**, we have proved that the direction of the axis of axial symmetry coincides with the direction determined by the direction cosines d^1, d^2, d^3. Hence the group of rotations about the axis of axial symmetry, is generated by the infinitesimal transformations with the operator

$$A=d^i\,R_i=d^i\,\epsilon_{ijk}\,x^j\,\partial/\partial x^k\,. \tag{4.2}$$

The infinitesimal transformations corresponding to the operator (4.2), are expressed as follows

$$\left\{\begin{aligned} x'^k &= x^k+d^i\,\epsilon_{ijk}\,x^j\,\delta\tau \\ t' &= t \qquad\qquad (k=1,2,3) \end{aligned}\right. \tag{4.3}$$

$\delta\tau$ being an infinitesimal quantity. The condition that M^i and M defined by (4.1) are invariant by the rotations about the axis whose direction cosines are d^1, d^2, d^3, is that (4.1) is invariant by the infinitesimal transformations (4.3). By the transformations (4.3), velocity v^k and momentum M^k are transformed to

$$v'^k=\frac{\partial x'^k}{\partial x^j}v^j=v^k+d^i\epsilon_{ijk}v^j\delta\tau, \tag{4.4}$$

$$M'^k=\frac{\partial x'^k}{\partial x^j}M^j=M^k+d^i\epsilon_{ijk}M^j\delta\tau, \tag{4.5}$$

respectively, and momentum M is unaltered, i. e.

$$M'=M. \tag{4.6}$$

Hence the condition that M^i and M defined by (4.1) are invariant by (4.3), is that M'^k and M' determined by (4.5) and (4.6) are equal to quantities which are obtained by replacing v^i in the expression (4.1) by v'^i. Namely

$$M'^i=\frac{m}{\sqrt{1-(v'v')/c^2}}v'^i+\frac{n\sqrt{1-(v'v')/c^2}+(lv')/c}{1-(dv')/c}cd^i+cl^i \tag{4.7}$$

$$M'=\frac{m}{\sqrt{1-(v'v')/c^2}}+\frac{n\sqrt{1-(v'v')/c^2}+(lv')/c}{1-(dv')/c} \tag{4.8}$$

But we see that $(v'v')=(vv)$ and $(dv')=(dv)$ by the relations (4.4). Hence (4.8) becomes

$$(lv')=(lv). \tag{4.9}$$

Therefore, substituting (4.9) into (4.7) and using the relations (4.4) and (4.5), we have

$$d^h\epsilon_{hji}l^j\delta\tau=0 \qquad (h,i,j=1,2,3).$$

These equations are rewritten as

$$d^h l^j-d^j l^h=0.$$

Combining these equations with the condition $(dl)=0$, we have

$$l^j=0. \qquad (j=1,2,3) \tag{4.10}$$

The equations (4.10) are the condition that M^i and M defined by (4.1), are invariant by the infinitesimal transformations (4.3). Under this condition, (4.1) becomes

$$M^i=\frac{m}{\sqrt{1-(vv)/c^2}}v^i+\frac{n\sqrt{1-(vv)/c^2}}{1-(dv)/c}cd^i \tag{4.11}$$

$$M=\frac{m}{\sqrt{1-(vv)/c^2}}+\frac{n\sqrt{1-(vv)/c^2}}{1-(dv)/c} \tag{4.12}$$

where m and n are arbitrary constants. The expressions (4.11) and (4.12) give a definition of momentum and mass in the case of axial symmetry.

References.

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