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On the Matrix Space

By

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Let $\mathfrak{S}$ be the space of all the complex matrices of degree n with the usual topology, we shall in this paper define the paths in the matrix space $\mathfrak{S}$ and we shall investigate the properties of the paths in the matrix space $\mathfrak{S}$, the general linear group $\mathfrak{M}$ and the special orthogonal group O^+.

§ 1. The paths in the matrix space $\mathfrak{S}$

We shall first consider $\mathfrak{S}$ as a vector space of dimension n^2 and we shall introduce a sort of parallelism into $\mathfrak{S}$ by saying that the vectors MV at the points M of $\mathfrak{S}$ are parallel to each other. (we can define another parallelism by using of VM in place of MV). By the paths in $\mathfrak{S}$ we shall mean the auto-parallel curve $M=M(t)$ (t is a real parameter) with respect to this parallelism, that is, the curve defined by the differential equation:

$$\frac{dM}{dt}=MA, \quad (A \text{ is a constant matrix}).^{1)} \tag{1.1}$$

Then the path through M_0 is given by

$$M(t)=M_0 \exp tA, \quad (M(0)=M_0). \tag{1.2}$$

Let $\mathfrak{A}(M)$ be the set of matrices S such that $SM=0$, (it will be called a left anihilator of M), and let $\rho(M)$ be the rank of the matrix M, then we shall prove the following lemmas.

LEMMA 1. *There exists a matrix X such that $NX=M$ for the given matrices N and M, if and only if $SN=0$ implies $SM=0$, that is, if and only if $\mathfrak{A}(N)\subset\mathfrak{A}(M)$.*

PROOF. If there exists a matrix X such that $NX=M$, then it is clear that $SN=0$ implies $SM=0$. Conversely, we shall assume that $SN=0$ implies

1) The path defined by $\frac{dM}{dt}=MA$ may be called a right path, on the contrary, the path defined by $\frac{dM}{dt}=AM$ may be called a left path. (See Remark 4, p. 60).

$SM=0$. If $\rho(N)=r$, then there exist the regular matrices P and Q such that

$$N_0 = PNQ = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.3}$$

where E_r is the unit matrix of degree r. And then $SN=0$ is equivalent to $S_0N_0=0$ where $S=S_0P$, and also $SM=0$ is written as $S_0M_0=0$ where $M_0=PM$. Here if we put

$$S_0 = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \ S_{11} \text{ being a matrix of degree } r,$$

then we have from $S_0N_0=0$,

$$S_{11}=0, \quad S_{21}=0, \quad \text{i. e.,} \quad S_0 = \begin{pmatrix} 0 & S_{12} \\ 0 & S_{22} \end{pmatrix}. \tag{1.4}$$

Putting

$$M_0 = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \ M_{11} \text{ being a matrix of degree } r,$$

from the fact that $S_0M_0=0$ for the arbitrary matrices of the form (1.4) it follows that

$$M_{21}=0, \quad M_{22}=0, \quad \text{i. e.,} \quad M_0 = \begin{pmatrix} M_{11} & M_{12} \\ 0 & 0 \end{pmatrix}. \tag{1.5}$$

Therefore, if we put $K=\begin{pmatrix} M_{11} & M_{12} \\ * & * \end{pmatrix}$, then we have $N_0K=M_0$, i. e., $PNQK=PM$; since P is regular, we have $M=NQK=NX$ where $X=QK$. Thus this lemma is proved.

LEMMA 2. *There exists a regular matrix X such that $NX=M$ for the given matrices N and M, if and only if $\mathfrak{A}(N)=\mathfrak{A}(M)$.*

PROOF. If there exists a regular matrix X such that $NX=M$, then also $N=MX^{-1}$, hence by Lemma 1 we have

$$\mathfrak{A}(N) \subset \mathfrak{A}(M) \quad \text{and} \quad \mathfrak{A}(M) \subset \mathfrak{A}(N), \tag{1.6}$$

that is, $\mathfrak{A}(N)=\mathfrak{A}(M)$.

Conversely, we shall assume that $\mathfrak{A}(N)=\mathfrak{A}(M)$. Then by Lemma 1 we see that there exist the matrices X and Y such that

$$M = NX \quad \text{and} \quad N = MY. \tag{1.7}$$

Consequently, we see that $\rho(M)\leqq\rho(N)$ and $\rho(N)\leqq\rho(M)$, that is, $\rho(M)=\rho(N)(=r)$.

So we can take the matrices P and Q such that

(1.8) $$N_0 = PNQ = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}, \text{ (P and Q are regular).}$$

And $NX = M$ is written as

(1.9) $$N_0 X_0 = M_0, \quad \text{where} \quad X_0 = Q^{-1}X \quad \text{and} \quad M_0 = PM.$$

Here if we put

$$X_0 = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \text{ and } M_0 = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where X_{11} and M_{11} are the matrices of degree r, then, from (1.8) and (1.9) we have

(1.10) $$M_{21} = 0, \quad M_{22} = 0, \quad M_{11} = X_{11} \text{ and } M_{12} = X_{12}.$$

And since $\rho(M_0) = r$, the rank of the matrix $(M_{11}\ M_{12})$ is also equal to r; therefore, there exists a regular matrix $\tilde{X}_0 = \begin{pmatrix} M_{11} & M_{12} \\ * & * \end{pmatrix}$, for this regular matrix $\tilde{X}_0$, clearly, we haxe $N_0\tilde{X}_0 = M_0$. If we take $\tilde{X} = Q\tilde{X}_0$, then we have $N\tilde{X} = M$ and $\det \tilde{X} \neq 0$. Thus this lemma is proved.

From Lemma 2 we have

THEOREM 1. *There exist, at least, a countable number*[1] *of paths through N and M, if and only if $\mathfrak{A}(M) = \mathfrak{A}(N)$.*

PROOF. If there exists a path through N and M, then by using a suitable parameter t this path is expressible as

(1.11) $$M(t) = N \exp tA \quad \text{where} \quad M(1) = M,$$

that is, $M = N \exp A$. Since the matrix $\exp A$ is regular, by Lemma 2 we have $\mathfrak{A}(M) = \mathfrak{A}(N)$. Conversely, if $\mathfrak{A}(M) = \mathfrak{A}(N)$, then by Lemma 2, there exists a regular matrix X such that $M = NX$. Since a regular matrix X is always expressible as $X = \exp A$, (there exist a countable number of such A, at least, as seen from the periodicity of the exponential function of matrix,[2]), we have $M = N \exp A$. Therefore, there exist, at least, a countable number of paths: $M(t) = N \exp tA$ through N and M. Thus this theorem is proved.

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1) In the particular case where n=1 and $|N^{-1}M| = 1$, there exists only one path, (regarding as a curve itself).

2) See [2], p. 111. Numbers in brackets refer to the references at the end of the paper.

§ 2. A maximal simply connected domain of the general linear group $\mathfrak{M}$

Let $\mathfrak{M}_0$ be the set of all the regular matrices without the negative characteristic root, and let $\mathfrak{M}_0^c$ be the complement of $\mathfrak{M}_0$ in $\mathfrak{M}$, i.e., the set of all the regular matrices with the negative characteristic root, that is,

$$\mathfrak{M} = \mathfrak{M}_0 \cup \mathfrak{M}_0^c, \quad \mathfrak{M}_0 \cap \mathfrak{M}_0^c = \phi, \tag{2.1}$$

then we can prove

THEOREM 2. *$\mathfrak{M}_0$ is a maximal simply connected domain of $\mathfrak{M}$, and $\mathfrak{M}_0$ is dense in $\mathfrak{M}$. $\mathfrak{M}_0^c$ is the boundary of $\mathfrak{M}_0$.*

PROOF. Let $\mathfrak{A}_0$ be the set of all the matrices satisfying the condition: the imaginary parts of the characteristic roots lie in the open interval $(-\pi, \pi)$, then it is already known [1] that $\mathfrak{M}_0$ is homeomorphic with $\mathfrak{A}_0$ by the exponential mapping. Since it is easily seen that $\mathfrak{A}_0$ is connected and simply connected, $\mathfrak{M}_0$ is also connected and simply connected. (See Remark 1). By the consideration of that the characteristic roots of M are the continuous function of M, we can easily show that $\mathfrak{M}_0$ is open and dense in $\mathfrak{M}$. Now we have only to prove that $\mathfrak{M}_0$ is maximal with respect to the simply connectedness. Suppose that $\mathfrak{M}_0 \subsetneqq \widetilde{\mathfrak{M}}$, then the matrix M belonging to $\widetilde{\mathfrak{M}} - \mathfrak{M}_0$ is written as

$$M = T^{-1}\mathring{M}T, \quad \mathring{M} = \begin{pmatrix} -a_1 & & & & & \\ & -a_2 & & K & & \\ & & \ddots & & & \\ & & & -a_r & & \\ & & & & \alpha_{r+1} & \\ & 0 & & & & \ddots \\ & & & & & & \alpha_n \end{pmatrix}, \quad (r \geqq 1), \tag{2.2}$$

where $a_1, a_2, \cdots, a_r$ are positive, and $\alpha_{r+1}, \alpha_{r+2}, \cdots, \alpha_n$ are not negative. If we put

$$M(\theta) = T^{-1}\mathring{M}(\theta)T, \quad \mathring{M}(\theta) = \begin{pmatrix} -a_1e^{i\theta} & & & & \\ & \ddots & & K & \\ & & -a_re^{i\theta} & & \\ & & & \alpha_{r+1} & \\ & 0 & & & \ddots \\ & & & & & \alpha_n \end{pmatrix}, \quad (i = \sqrt{-1}), \tag{2.3}$$

then $M(\theta)$ $(-\pi < \theta \leqq \pi)$ is a closed curve through the point M; $M(\theta)$ except for M lies in $\mathfrak{M}_0$. If the curve $M(\theta)$ $(-\pi < \theta \leqq \pi)$ is deformable to a point, then the circle $-a_1e^{i\theta}$ $(-\pi < \theta \leqq \pi)$ in the complex plane must be deformable

1) See [2], p. 111, Theorem III.

to a point, that is, the curve, in the way of deformation, must pass through O in the complex plane because the characteristic roots of M are the continuous functions of M. Then, the corresponding matrix becomes singular, this is impossible; therefore, $\widetilde{\mathfrak{M}}$ is not simply connected. That is, $\mathfrak{M}_0$ is a maximal simply connected domain of $\mathfrak{M}$.

REMARK 1. Any closed curve $M(s)$ $(0 \leqq s < 1)$ in $\mathfrak{M}_0$ is expressible as $M(s) = \exp A(s)$; if we put $F(s;t) = \exp tA(s)$, then $F(s;t)$ is a continuous function of s and t, such that $F(s;1) = M(s)$ and $F(s;0) = E$. That is, any closed curve $M(s)$ is deformable to a point E. Since $\mathfrak{M}_0$ is arcwise connected, from this we conclude that $\mathfrak{M}_0$ is simply connected. Moreover, we remark that for a fixed s_0 $(0 \leqq s_0 < 1)$, the curve $F(s_0;t) = \exp tA(s_0)$ is a path through E and $M(s_0)$.

REMARK 2. Since $\mathfrak{M}_0^c$ is the complement of $\mathfrak{M}_0$ in $\mathfrak{M}$, we have $\mathfrak{M}_0^c = \bigcup_{\iota=1}^{n} \mathfrak{M}_\iota$, $\mathfrak{M}_\iota \cap \mathfrak{M}_\kappa = \phi$ $(\iota \neq \kappa)$, where $\mathfrak{M}_\iota$ means the set of all the regular matrices having just ι negative characteristic roots. And we can prove that $\mathfrak{M}_\iota$ is connected: Any matrix M of $\mathfrak{M}_r$ is transformed to $\mathring{M}$ by a regular matrix T:

$$M = T\mathring{M}T^{-1}, \quad \mathring{M} = \begin{pmatrix} -a_1 & \eta_1 & & & & & 0 \\ & -a_2 & \eta_2 & & & & \\ & & \ddots & \ddots & & & \\ & & & -a_r & \ddots & & \\ & & & & \alpha_{r+1} & \ddots & \\ 0 & & & & & \ddots & \eta_{n-1} \\ & & & & & & \alpha_n \end{pmatrix}, \quad (\eta_\kappa = 0,\ 1),$$

where $a_1, a_2, \dots, a_r$ are positive and $\alpha_{r+1}, \alpha_{r+2}, \dots, \alpha_n$ are not negative. If we put

$$M(\theta) = T(\theta)\mathring{M}(\theta)T^{-1}(\theta), \quad T(\theta) = \exp\theta A, \quad (T = \exp A) \quad \text{and}$$

$$\mathring{M}(\theta) = \begin{pmatrix} -a_1(\theta) & \eta_1\theta & & & & & 0 \\ & -a_2(\theta) & \eta_2\theta & & & & \\ & & \ddots & \ddots & & & \\ & & & -a_r(\theta) & \ddots & & \\ & & & & \alpha_{r+1}(\theta) & \ddots & \\ 0 & & & & & \ddots & \eta_{n-1}\theta \\ & & & & & & \alpha_n(\theta) \end{pmatrix}$$

where $a_\iota(\theta) = e^{\theta b_\iota}$, $(a_\iota = e^{b_\iota})$, $(\iota = 1, 2, \dots, r)$ and $\alpha_\iota(\theta) = e^{\theta\beta_\iota}$, $(\iota = r+1, \dots, n)$, β_ι being a complex number such that $\alpha_\iota = e^{\beta_\iota}$ and $|I(\beta_\iota)| < \pi$,[1] then $M(\theta)$ is a continuous function of θ such that $M(1) = M$ and $M(0) = \begin{pmatrix} -1 & & & & & 0 \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}$,

1) $I(\beta)$ means the imaginary part of β.

that is, any element M of $\mathfrak{M}_r$ is connected with $\begin{pmatrix} -1 & & & & & 0 \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}$ (r entries -1) by a continuous curve in $\mathfrak{M}_r$. Hence we see that $\mathfrak{M}_r$ is connected. It is clear that $\mathfrak{M}_{\iota+1}\subset\overline{\mathfrak{M}}_\iota$ $(\iota=0, 1, 2, \dots, n)$, (since our field under the consideration is the complex field). Therefore we know that $\mathfrak{N}_r=\bigcup_{\iota=r}^{n}\mathfrak{M}_\iota$ $(r=0, 1, 2, \dots, n)$ is connected. In particular, $\mathfrak{M}_0^c=\mathfrak{N}_1=\bigcup_{\iota=1}^{n}\mathfrak{M}_\iota$ is also connected. That is, $\mathfrak{M}_0^c$ is the connected boundary of $\mathfrak{M}_c$.

Moreover, the set $P\mathfrak{M}_0-R\equiv\{X\,;\,X=PM-R,\ M\in\mathfrak{M}_0\}$ (P being regular) is homeomorphic with $\mathfrak{M}_0$, and therefore the set $P\mathfrak{M}_0-R$ is also a maximal simply connected domain of $P\mathfrak{M}-R$. And also, $\mathfrak{M}=P\mathfrak{M}_0\cup(P\mathfrak{M}_0)^c$ and $(P\mathfrak{M}_0)^c$ is the connected boundary of $P\mathfrak{M}_0$.

Next we shall consider some properties of $\mathfrak{M}_0$.

THEOREM 3. *If $M\mathfrak{M}_0\subset\mathfrak{M}_0$, then $M=kE$ $(k>0)$.*

PROOF. If $M\mathfrak{M}_0\subset\mathfrak{M}_0$, then it is clear that $M\in\mathfrak{M}_0$. Then M is similar to

$$\mathring{M}=\begin{pmatrix} r_1e^{i\theta_1} & & \# \\ & r_2e^{i\theta_2} & \\ 0 & \ddots & \\ & & r_ne^{i\theta_n}\end{pmatrix},\qquad (i=\sqrt{-1},\ r_\iota>0,\ \ -\pi<\theta_\iota<\pi). \tag{2.4}$$

And then $M\mathfrak{M}_0\subset\mathfrak{M}_0$ is equivalent to $M_0\mathfrak{M}_0\subset\mathfrak{M}_0$, (since $T\mathfrak{M}_0T^{-1}=\mathfrak{M}_0$). Here if $\theta_1\neq0$ then $L=\begin{pmatrix} e^{i\{(\mathrm{sign}\,\theta_1)\pi-\theta_1\}} & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & \ddots & \\ & & & 1\end{pmatrix}$ belongs to $\mathfrak{M}_0$; and $\mathring{M}L=\begin{pmatrix} -r_1 & & \# \\ & r_2e^{i\theta_2} & \\ & & \ddots \\ 0 & & & r_ne^{i\theta_n}\end{pmatrix}$ belongs to $\mathfrak{M}_0^c$. That is, $\mathring{M}\mathfrak{M}_0\not\subset\mathfrak{M}_0$; therefore, the characteristic roots of M must be all positive, i. e., $\theta_\iota=0$.

We shall first consider the case where $n=2$. In this case, M is similar to

$$\mathring{M}=\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix},\quad(\lambda>0),\quad\text{or}\quad\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix},\quad(\lambda,\ \mu>0). \tag{2.5}$$

(i) $\mathring{M}=\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$, $(\lambda>0)$: If we put $L=\begin{pmatrix}0 & \frac{1-i}{\lambda^2}\\ i & 0\end{pmatrix}$, then the characteristic roots of L are equal to $\pm\frac{1}{\lambda}\sqrt{1+i}$, that is, these are not negative; therefore

$L \in \mathfrak{M}_0$. On the other hand, the characteristic roots of $\mathring{M}L$ are -1 and $1+i$, hence $\mathring{M}L \in \mathfrak{M}_0^c$. So we see that $\mathring{M}\mathfrak{M}_0 \not\subset \mathfrak{M}_0$, i. e., $M\mathfrak{M} \not\subset \mathfrak{M}$.

(ii) $\mathring{M} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $(\lambda \neq \mu,\ \lambda,\ \mu > 0)$: We may assume that $\lambda > \mu > 0$. If we take $L = \begin{pmatrix} -\dfrac{2}{\mu} & \dfrac{1}{\lambda\mu} \\ 1 - \dfrac{2\lambda}{\mu} & 0 \end{pmatrix}$, then it is easily seen that the characteristic roots of L are $-\dfrac{1}{\mu}\left(1 \pm \sqrt{\left(\dfrac{\mu}{\lambda} - 1\right)}\right)$, and the characteristic roots of $\mathring{M}L$ are -1 and $1 - \dfrac{2\lambda}{\mu}$. That is, $L \in \mathfrak{M}_0$ and $\mathring{M}L \notin \mathfrak{M}_0$; consequently $\mathring{M}\mathfrak{M}_0 \not\subset \mathfrak{M}_0$. Thus, in the case where $n=2$, we see that $M\mathfrak{M}_0 \subset \mathfrak{M}_0$ implies $M = kE$ $(k>0)$. The converse is clear.

Next we shall consider the case where $n>2$. In this case, M is similar to

$$\mathring{M} = \begin{pmatrix} \lambda_1 & \eta_1 & & & 0 \\ & \lambda_2 & \eta_2 & & \\ & & \lambda_3 & \ddots & \\ & & & \ddots & \eta_{n-1} \\ 0 & & & & \lambda_n \end{pmatrix}, \tag{2.6}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n > 0$, and $\eta_1, \eta_2, \dots, \eta_{n-1} = 0$ or 1. If we put

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & E_{n-2} \end{pmatrix}, \tag{2.7}$$

where L_1 is an arbitrary matrix of degree two, then we have

$$\mathring{M}L = \left(\begin{array}{c|ccccc} \mathring{M}_1 L_1 & \begin{matrix} 0 \\ \eta_2 \end{matrix} & & 0 & & \\ \hline & \lambda_3 & \eta_3 & & & \\ 0 & & \lambda_4 & \ddots & & \\ & & & \ddots & \eta_{n-1} & \\ & & & & \lambda_n & \end{array}\right).$$

Therefore, $\mathring{M}\mathfrak{M}_0 \subset \mathfrak{M}_0$ implies $\mathring{M}_1\mathfrak{M}_0 \subset \mathfrak{M}_0$ for the case where $n=2$; hence, from the above consideration, we have $M_1 = kE_2$, i. e., $\lambda_1 = \lambda_2 = k$ and $\eta_1 = 0$. Thus, repeating this procedure, we obtain $\mathring{M} = kE$, i. e., $M = kE$, $(k>0)$.

REMARK 3. If $MN = NM$ for all $N \in \mathfrak{M}_0$, then $M = \kappa E$. For, if we take $N = \begin{pmatrix} \alpha_1 & & 0 \\ & \alpha_2 & \\ 0 & & \ddots \\ & & & \alpha_n \end{pmatrix} \in \mathfrak{M}_0$, where α_ι are distinct non-negative comlpex numbers, then from $MN=NM$ it follows that $M=\begin{pmatrix}\kappa_1 & & & 0\\ & \kappa_2 & & \\ & & \ddots & \\ 0 & & & \kappa_n\end{pmatrix}$. And next if we take $N=\begin{pmatrix}1 & 1 & & 0\\ & 1 & \ddots & \\ & & \ddots & 1\\ 0 & & & 1\end{pmatrix}$, then $N\in\mathfrak{M}_0$; from $MN=NM$ it follows that $\kappa_1=\kappa_2=\cdots=\kappa_n$, consequently $M=\kappa E$.

THEOREM 4. $\mathfrak{M}=M_1\mathfrak{M}_0\cup M_2\mathfrak{M}_0\cup\cdots\cup M_r\mathfrak{M}_0$, *where* $M_\iota=e^{i\theta_\iota}E$, $(-\pi<\theta_\iota<\pi$, θ_ι being distinct, $\iota=1,2,\ldots,r$, $r\geqq n+1)$. *And* $M_\iota\mathfrak{M}_0$ $(\iota=1,2,\ldots,r)$ *are maximal simply connected domains of* $\mathfrak{M}$.

PROOF. As used in the proof of Theorem 2, $\mathfrak{M}_0=\exp\mathfrak{A}_0$, where $\exp\mathfrak{A}_0$ means the set of $\exp A$ such that $A\in\mathfrak{A}_0$. And then it is clear that

$$e^{i\theta}\mathfrak{M}_0=e^{i\theta}\exp\mathfrak{A}_0=\exp(\mathfrak{A}_0+i\theta E)=\exp\mathfrak{A}_\theta, \tag{2.8}$$

where $\mathfrak{A}_\theta$ means the set of all the matrices satisfying the condition: the characteristic roots lie in the open interval $(-\pi+\theta,\ \pi+\theta)$. Let Θ be a set of $\theta_1,\theta_2,\ldots,\theta_r$ such that $-\pi<\theta_\iota<\pi$, $\iota=1,2,\cdots,r$ and $r\geqq n+1$, then it is easily seen that for any element M of $\mathfrak{M}$ there exists such a θ_{j_0} that $M=\exp A$, $A\in\mathfrak{A}_{\theta_{j_0}}$, (since the number of the distinct characteristic roots of M is equal to n at most), that is, we see that any element M of $\mathfrak{M}$ belongs to a set $M_{j_0}\mathfrak{M}_0$, where $M_{j_0}=e^{i\theta_{j_0}}E$. Thus, we obtain that

$$\mathfrak{M}=M_1\mathfrak{M}_0\cup M_2\mathfrak{M}_0\cup\cdots\cup M_r\mathfrak{M}_0,$$

where $M_\iota=e^{i\theta_\iota}E$, $(-\pi<\theta_\iota<\pi,\ \iota=1,2,\ldots,r,\ r\geqq n+1)$. And it is clear that $M_\iota\mathfrak{M}_0$ is a maximal simply connected domain of $\mathfrak{M}$.

§ 3. The paths in the general linear group $\mathfrak{M}$

In this section we shall consider the paths in the general linear group $\mathfrak{M}$. If $M, N\in\mathfrak{M}$, then it is clear that $\mathfrak{A}(M)=\mathfrak{A}(N)=\{0\}$, and hence, by Theorem 1, we see that there exist, at least, a countable number of paths through the given two points N and M of $\mathfrak{M}$. Since any right path is translated to a path through the unit element E of $\mathfrak{M}$ by a left translation: $X'=N^{-1}X$, we shall restrict ourselves to the paths through the unit element E of $\mathfrak{M}$. Let $\mathfrak{F}_M$ be the set of all the paths through E and M, and let $\boldsymbol{P}_M$ [1] be the set

1) $\mathfrak{F}_M$ contains a countable number of paths at least, and P_M contains a countable number of matrices at least.

of all the matrices A such that $\exp A = M$, we shall prove the following theorems.

THEOREM 5. *For an element M of $\mathfrak{M}_0$, there exists one and only one path through E and M which is entirely contained in $\mathfrak{M}_0$. And for an element M of $\mathfrak{M}_0^c$, there exist, at least, two paths from E to M which are contained in $\mathfrak{M}_0$ except for M.*

PROOF. The paths from E to M are given by

$$M(t) = \exp tA, \ (0 \leqq t \leqq 1), \text{ where } A \in \boldsymbol{P}_M. \tag{3.1}$$

If $M(t) \subset \mathfrak{M}_0$ for all t such that $0 \leqq t \leqq 1$, then $A \in \mathfrak{A}_0$. For, if $A \notin \mathfrak{A}_0$, then, for some characteristic root μ of A, the imaginary part $I(\mu)$ satisfies $|I(\mu)| \geqq \pi$, consequently, $\exp\, t_0 A \in \mathfrak{M}_0^c$ for $t_0 = \dfrac{\pi}{|I(\mu)|}$; this is a contradiction. It is already known [1] by us that there exists one and only one A such that $\exp A = M$ and $A \in \mathfrak{A}_0$. Therefore, the path asserted above is given by $M(t) = \exp tA$, $A \in \mathfrak{A}_0 \cap \boldsymbol{P}_M$.

If $M \in \mathfrak{M}$, then M is expressible as $M = \exp A$, $A \in \tilde{\mathfrak{A}}$,[1] $\tilde{\mathfrak{A}}$ being the set of all the matrices satisfying the condition: the imaginary parts of the characteristic roots lie in the half-closed interval $(-\pi, \pi]$. And if $M \in \mathfrak{M}_0^c$, then A has the characteristic root $a + i\pi$ (a is real). Let A_1 be the matrix obtained by taking $a - i\pi$ in place of the characteristic root $a + i\pi$ in Jordan's canonical form of A, then both $\exp tA$ $(0 \leqq t \leqq 1)$ and $\exp tA_1$ $(0 \leqq t \leqq 1)$ are the paths from E to M which are contained in $\mathfrak{M}_0$ except for M. Thus this theorem is proved.

THEOREM 6. *Any path from E to M intersects $\mathfrak{M}_0^c$ at most, in a finite number of points.*

PROOF. Any path from E to M is expressible as

$$M(t) = \exp tA, \quad (0 \leqq t \leqq 1), \ A \in \boldsymbol{P}_M.$$

Let $a_\iota + ib_\iota$ $(\iota = 1, 2, \ldots, n)$ be the characteristic roots of A, then, $M(t) \in \mathfrak{M}_0^c$ if and only if $tb_\iota = (2m_\iota + 1)\pi$ for some integer m_ι. Since $0 \leqq t \leqq 1$, we have $0 < \dfrac{(2m_\iota + 1)}{b_\iota}\pi \leqq 1$; consequently, the path intersects $\mathfrak{M}_0^c$ at most, in $\sum_{\iota=1}^{n} p(b_\iota)$ points, where

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1) See [2], p. 111, Theorem II.

$$(3.2)\qquad p(b)=\begin{cases}\left[\frac{1}{2}\left(\frac{b}{\pi}-1\right)\right]+1 & \text{for } b\geqq\pi,\\ 0 & \text{for } \pi>b\geqq 0,\\ \left[\frac{1}{2}\left|\frac{b}{\pi}-1\right|\right] & \text{for } b<0.\end{cases}$$

More precisely, the path intersects $\mathfrak{M}_0^c$ in just $\sum_{\iota=1}^{n}p(b_\iota)$ points, unless $\frac{b_\iota}{b_\kappa}=\frac{2m_\iota+1}{2m_\kappa+1}$ $(\iota\neq\kappa=1,2,\dots,n)$, m_ι being any integer. Thus this theorem is proved.

Let $\boldsymbol{C}(M)$ be the set of all the matrices which are commutative with M, and let $\mathfrak{C}_M$ be the set of all the closed paths through E which are contained in $\boldsymbol{C}(M)$, and here we shall mean by the principal path through E and M the path: $M(t)=\exp tA_0$, $A_0\in\tilde{\mathfrak{A}}\frown\boldsymbol{P}_M$. Then we have

THEOREM 7. *$\mathfrak{F}_M$ is decomposed into $\mathfrak{C}_M$ and the principal path through E and M.*

PROOF. If $M(t)\in\mathfrak{F}_M$, then $M(t)=\exp tA$, where $A=A_0+\mathfrak{p}$, $A_0\in\tilde{\mathfrak{A}}$ and $\mathfrak{p}\in\boldsymbol{P}_E\frown\boldsymbol{C}_{(M)}$,[1] and vice versa. Since $A_0\mathfrak{p}=\mathfrak{p}A_0$, we have $\exp tA=\exp t(A_0+\mathfrak{p})$ $=\exp t\mathfrak{p}\exp tA_0$, where $\exp tA_0$ is the principal path through E and M, and $\exp t\mathfrak{p}\in\mathfrak{C}_M$. Thus, this theorem is proved.

REMARK 4. A right path through M_1 and M_2 is also regarded as a left path through M_1 and M_2, and vice versa. For, any right path through M_1 and M_2 is expressible as $M(t)=M_1\exp tA$, $(M_2=M_1\exp A)$; and it is clear that $M(t)=(\exp tB)M_1$, $(B=M_1AM_1^{-1})$; hence $M(t)$ is regarded as a left path through M_1 and M_2.

Moreover, a right path from M_1 to M_2 is regarded as a right path from M_2 to M_1 in the opposite direction, as easily seen from the fact that $M(t)=M_1\exp tA=M_2M_2^{-1}M_1\exp tA=M_2(\exp(-A))\exp tA=M_2\exp(t-1)A$, i. e., $M(t)=M_2\exp s(-A)$, $(s=1-t)$.

REMARK 5. The necessary and sufficient condition that there exist a closed path through M_1 and M_2 is that $M_1^{-1}M_2\in(\mathfrak{C})$, where $(\mathfrak{C})=\cup(E^t;\ 0\leqq t<1)$ and $E^t=\exp t\mathfrak{p}$, $(\exp\mathfrak{p}=E)$, E^t may be considered as the t-th power of E. This is clear, as we can easily see by considering the closed path obtained by a left translation: $X'=M_1^{-1}X$.

Next let us suppose that there exists a closed path through M_1 and M_2, then we shall consider the necessary and sufficient condition that a path through M_1 and M_2 be a closed path. Since there exists a closed path through

1) See [2], p. 111, Theorem I.

M_1 and M_2, by the above resnlt we have $M_1^{-1}M_2 = \exp t_0 \mathfrak{p}$ where $\exp \mathfrak{p} = E$ and $\exp t\mathfrak{p} \neq E$, $(0 < t < 1)$; on the other hand, a path through[1] M_1 and M_2 is expressible as $M(t) = M_1 \exp tA$, where $M_1^{-1}M_2 = \exp A$, and therefore we have $\exp A = \exp t_0 \mathfrak{p}$. If t_0 is rational, i. e., $t_0 = \frac{q}{p}$, $(p, q) = 1$, then $\exp pA = \exp p\left(\frac{q}{p}\right)\mathfrak{p} = \exp q\mathfrak{p} = E$; therefore in this case any path through M_1 and M_2 is a closed path. Next if t_0 is irrational, then the set $\{\exp mt_0\mathfrak{p};\ m = 1, 2, \dots\}$ is dense in the closed path: $\exp t\mathfrak{p}$ $(0 \leq t < 1)$; therefore, since $\exp mA = \exp mt_0\mathfrak{p}$, the set $\{M_1 \exp mA;\ m = 1, 2, \dots\}$ is dense in the closed path: $M_1 \exp t\mathfrak{p}$ $(0 \leq t < 1)$. In particular, the path $M(t) = M_1 \exp tA$ approaches the point M_1, periodically, and closely step by step. Now we shall determine all the closed paths through M_1 and M_2. Since $\exp \mathfrak{p} = E$, $\mathfrak{p}$ is written as

$$\mathfrak{p} = 2\pi i S^{-1} \begin{pmatrix} f_1 & & & 0 \\ & f_2 & & \\ & & \ddots & \\ 0 & & & f_n \end{pmatrix} S,$$

where $f_1, f_2, \dots, f_n$ are integers and S is a regular matrix, then the matrix satisfying $\exp A = \exp t_0\mathfrak{p}$ is given by

$$A = 2\pi i S^{-1} \begin{pmatrix} t_0 f_1 + m_1 & & & 0 \\ & t_0 f_2 + m_2 & & \\ & & \ddots & \\ 0 & & & t_0 f_n + m_n \end{pmatrix} S,$$

where $m_1, m_2, \dots, m_n$ are integrs. If the path $M(t) = M_1 \exp tA$ through M_1 and M_2 passes through again the point M_1, then there exists a real number $k \neq 0$ such that $\exp kA = E$, i. e., $k(t_0 f_\iota + m_\iota) = l_\iota$ $(\iota = 1, 2, \dots, n)$, l_ι being integers. (If t_0 is rational, this condition is always satisfied). Since now t_0 is irrational, from this we have $m_\iota = rf_\iota$ $(\iota = 1, 2, \dots, n)$, where r is rational. Therefore, we have $A = (t_0 + r)\mathfrak{p}$; by considering the condition: $\exp A = \exp t_0\mathfrak{p}$ and $\exp t\mathfrak{p} \neq E$ $(0 < t < 1)$, we see that r must be an integer. Conversely, it is clear that the path $M(t) = M_1 \exp tA$ $(A = (t_0 + r)\mathfrak{p}$, r is an integer) passes through again M_1 for $t = \frac{1}{t_0 + r}$, $(t_0 + r$ being not zero). Thus, if t_0 is irrational, then all the closed paths through M_1 and M_2 are given by $M(t) = M_1 \exp t(t_0 + r)\mathfrak{p}$, r being integers. Furthermore, we shall consider any path $M(t) = M_1 \exp tA$ through M_1 and M_2 $(M(1) = M_2)$. Let $\mathring{A}$ be a Jordan's canonical form of A and let $\mathring{A} = \mathring{A}_{(r)} + \mathring{A}_{(i)}$:

1) The term "through" means the path $M(t) = M_1 \exp tA(-\infty < t < \infty)$.

$$A=T^{-1}\mathring{A}T,\quad \mathring{A}=\begin{pmatrix} a_1+ib_1 & \eta_1 & & & 0 \\ & a_2+ib_2 & \eta_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \eta_{n-1} \\ 0 & & & & a_n+ib_n \end{pmatrix},\quad \mathring{A}_{(r)}=\begin{pmatrix} a_1 & \eta_1 & & & 0 \\ & a_2 & \eta_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \eta_{n-1} \\ 0 & & & & a_n \end{pmatrix}$$

$$\text{and}\quad \mathring{A}_{(i)}=i\begin{pmatrix} b_1 & & 0 \\ & b_2 & \\ & & \ddots \\ 0 & & & b_n \end{pmatrix},$$

where a_ι and b_ι are real, and $\eta_1, \cdots, \eta_{n-1}=0$ or 1, then we see that $A=A_{(r)}+A_{(i)}$ and $A_{(r)}A_{(i)}=A_{(i)}A_{(r)}$, where $A_{(r)}=T^{-1}\mathring{A}_{(r)}T$ and $A_{(i)}=T^{-1}\mathring{A}_{(i)}T$. This decomposition of A is independent of the manner taking its canonical form. For, let us suppose that there are such two decompositions:

$$A=T\mathring{A}T^{-1}=T(\mathring{A}_{(r)}+\mathring{A}_{(i)})T^{-1}=A_{(r)}+A_{(i)}$$

and

$$A=S\mathring{A}'S^{-1}=S(\mathring{A}'_{(r)}+\mathring{A}'_{(i)})S^{-1}=A'_{(r)}+A'_{(i)},$$

where $\mathring{A}'=A'_{(r)}+\mathring{A}'_{(i)}$, $A'_{(r)}=S\mathring{A}'_{(r)}S^{-1}$ and $A'_{(i)}=S\mathring{A}'_{(i)}S^{-1}$. Since both $\mathring{A}$ and $\mathring{A}'$ are Jordan's canonical form of A, $\mathring{A}'$ is obtained from A by permuting the order of blocks: $\mathring{A}'=UAU^{-1}$ (U being taken as a real regular matrix). Since U is a real matrix, it follows from $\mathring{A}'=U\mathring{A}U^{-1}$ that $\mathring{A}'_{(i)}=U\mathring{A}_{(r)}U^{-1}$. And since $A=T\mathring{A}T^{-1}=S\mathring{A}'S^{-1}$ and $\mathring{A}'=U\mathring{A}U^{-1}$, we have $V\mathring{A}=\mathring{A}V$, $(V=U^{-1}S^{-1}T)$; consequently, from the forms of $\mathring{A}$ and $\mathring{A}_{(r)}$ we have $V\mathring{A}_{(r)}=\mathring{A}_{(r)}V$, from which it follows that $A'_{(r)}=S\mathring{A}'_{(r)}S^{-1}=SU\mathring{A}_{(r)}U^{-1}S^{-1}=T\mathring{A}_{(r)}T^{-1}=A_{(r)}$. Thus we have $A'_{(r)}=A_{(r)}$ and consequently $A'_{(i)}=A_{(i)}$. By using this decomposition of A we shall decompose the path $M(t)=M_1\exp tA$ as follows: $M(t)=M_1\exp t(A_{(r)}+A_{(i)})$ $=M_1\exp tA_{(r)}\exp tA_{(i)}$. Here the path: $M_1\exp tA_{(r)}$ never approach again the point M_1; but, on the contrary, the path: $\exp tA_{(i)}$ either passes through the point E, or approaches the point E periodically, and closely step by step, because there exist a real number t and n integers l_ι $(\iota=1, 2, \dots, n)$ satisfying the system of inequalities $|tb_\iota-2\pi l_\iota|<\varepsilon$ $(\iota=1, 2, \dots, n)$ for any positive number ε and a system of real numbers b_ι $(\iota=1, 2, \dots, n)$.[1] Conversely, let us suppose that a path: $\exp tB$ approaches the point E periodically, and closely step by step, where the matrix B has the form: $B=T^{-1}\mathring{B}T$, $\mathring{B}=\begin{pmatrix} c_1+id_1 & \zeta_1 & & & 0 \\ & c_2+id_2 & \zeta_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \zeta_{n-1} \\ 0 & & & & c_n+id_n \end{pmatrix}$,

1) See [4], p. 157.

c_ι and d_ι $(\iota=1, 2, \dots, n)$ being real, and $\zeta_\kappa=0$ or 1 $(\kappa=1, 2, \dots, n-1)$, then it is easily seen that $c_\iota=0$ $(\iota=1, 2, \dots, n)$ and $\zeta_\kappa=0$ $(\kappa=1, 2, \dots, n-1)$. That is, B must have the form:

$$B=T^{-1}\mathring{B}T,\ \mathring{B}=i\begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & & d_n \end{pmatrix},\ \text{and } d_\iota\ (\iota=1, 2, \dots, n) \text{ being real.}$$

From the above consideration we can say that the path $M(t)=M_1 \exp tA$ approaches the path $M(t)=M_1 \exp tA_{(r)}$ periodically, and closely step by step.

Finally we remark that the closed path through M_1 and M_2 considered above is a simple closed path. For, as mentioned above, we have $M_1^{-1}M_2=\exp t_0\mathfrak{p}$, where $\exp\mathfrak{p}=E$ and $\exp t\mathfrak{p}\neq E$ for $0<t<1$; if $M(t_1)=M(t_2)$ for $0<t_1<t_2<1$, then we have $\exp t_1\mathfrak{p}=t_2\mathfrak{p}$, consequently, $\exp(t_2-t_1)\mathfrak{p}=E$, $(0<t_2-t_1<1)$, this contradicts the above assumption that $\exp t\mathfrak{p}\neq E$ for $0<t<1$. Therefore, the closed path through M_1 and M_2 is a simple closed path.

REMARK 6. If we define $N^t=\{\exp tA$ for any A such that $\exp A=N\}$, then we have $MN^tM^{-1}=(MNM^{-1})^t$ for the branches corresponded suitably to each other. From the above definition we have $N^t=\exp t(L(N)+\mathfrak{p}_N)$, where $L(N)$ means the matrix $A_0\in\tilde{\mathfrak{A}}$ such that $\exp A_0=N$, and $\mathfrak{p}_N$ means the period of N (that is, $\exp\mathfrak{p}_N=E$ and $\mathfrak{p}_N\in \boldsymbol{C}(N)$[1]). Then, we have $(MNM^{-1})^t=\exp t\{L(MNM^{-1})+\mathfrak{p}_{MNM^{-1}}\}$, and $L(MNM^{-1})=ML(N)M^{-1}$ (since $L(N)$ is a polynomial of N), so, if we correspond the periods to each other as $M\mathfrak{p}_NM^{-1}=\mathfrak{p}_{MNM^{-1}}$, then we have, clearly, $MN^tM^{-1}=(MNM^{-1})^t$.

Now we shall consider some algebraic properties of paths in $\mathfrak{M}$. On any path $M(t)$ through E and M it holds that $M(t_1)M(t_2)=M(t_2)M(t_1)$, and hence $\mathfrak{F}_M\subset\boldsymbol{C}(M)$. Concerning the commutativity of elements on the union of all the paths through E and M, we have.

THEOREM 8. *All the points on the union of all the paths through E and M are commutative if and only if the minimal polynomial of M is of degree n.*

PROOF. If the minimal polynomial of M is of degree n, then the matrices of $\boldsymbol{P}_M$ are the polynomials of M,[2] and therefore, all the points on the union of all the paths through E and M are commutative. If the minimal polynomial of M is of degree less than n, then Jordan's canonical form of M contains the following blocks:

1) See [2], p. 111, Theorem I.
2) See [2], p. 112, Thorem IV.

$$\hat{M} = \overbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}^{r_1} \dotplus \overbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}^{r_2}. \tag{3.3}$$

We shall show that $\boldsymbol{P}_M$ is not commutative; to do this we have only to show that $\boldsymbol{P}_E \cap \boldsymbol{C}(M)$ is not commutative, because the matrices A of $\boldsymbol{P}_M$ are expressible as $A = A_0 + \mathfrak{p}$ where $A_0 \in \tilde{\mathfrak{A}}$, $\mathfrak{p} \in \boldsymbol{P}_E \cap \boldsymbol{C}(M)$ and $A_0 \mathfrak{p} = \mathfrak{p} A_0$. We shall consider the periods for the block $\hat{M}$. Now we can take as the two periods of $\hat{M}$,[1)]

$$\hat{\mathfrak{p}}_1 = O_{r_1} \dotplus 2f\pi i E_{r_2} \text{ and } \hat{\mathfrak{p}}_2 = S\hat{\mathfrak{p}}_1 S^{-1}, \tag{3.4}$$

where O_{r_1} is the zero matrix of degree r_1, f is a non-zero integer, $S \in \boldsymbol{C}(\hat{M})$ and $S = \begin{pmatrix} E_{r_1} & K \\ 0 & E_{r_2} \end{pmatrix}$, $(K \neq 0)$, then easily we see that $\hat{\mathfrak{p}}_1 \hat{\mathfrak{p}}_2 \neq \hat{\mathfrak{p}}_2 \hat{\mathfrak{p}}_1$. Thus, the theorem is proved.

COROLLARY. In the space of complex quaternions, if α is a regular complex quaternion such that $\alpha \neq k1$ (k is a complex number), then all the points on the union of all the paths through 1 and α are commutative.

PROOF. For the complex quaternion $\alpha \neq k1$ (k is a complex number), the minimal polynomial is degree two; hence this corollary follows from Theorem 8.

REMARK 7. If the minimal polynomial of M is of degree n then the set $\boldsymbol{P}_E \cap \boldsymbol{C}(M)$ — it may be called the set of periods of M — is additive. For, if $\mathfrak{p}_1, \mathfrak{p}_2 \in \boldsymbol{P}_E \cap \boldsymbol{C}(M)$, then, by Theorem 8, $\mathfrak{p}_1 \mathfrak{p}_2 = \mathfrak{p}_2 \mathfrak{p}_1$, and hence $\exp(\mathfrak{p}_1 + \mathfrak{p}_2) = \exp \mathfrak{p}_1 \exp \mathfrak{p}_2 = E$. And clearly, $\mathfrak{p}_1 + \mathfrak{p}_2 \in \boldsymbol{C}(M)$, therefore, we have $\mathfrak{p}_1 + \mathfrak{p}_2 \in \boldsymbol{P}_E \cap \boldsymbol{C}(M)$. That is, the set of periods of M is additive,

Finally we shall consider the group generated by the set $\mathfrak{E}(t_0) \equiv \{M(t_0);\ M(t) \in \mathfrak{F}_E\}$, t_0 being a fixed number such that $0 < t_0 < 1$. As easily seen, $\mathfrak{E}(t_0) = E^{t_0} = \{X;\ X^{\frac{1}{t_0}} = E\}$, where X^t means $\exp(t \log X)$, i. e., it may be considered as the t-power of X. Here by saying that a group G is generated by a set S we shall mean that G is a topological closure of the union of the products of elements of S; and we shall indicate this fact by $G = [S]$.

THEOREM 9. *The group generated by* $\mathfrak{E}(\frac{q}{p})$, *being* $(p, q) = 1$, *is a direct product*

1) See [2], p. 111, Theorem I.

of the special linear group $SL(n, C)$ *and the cyclic group* $Z=\{1, \omega, \dots, \omega^{p-1}\}$ *of order* p, *where* ω *is a p-th primitive root of* 1.

PROOF. Any element of $\mathfrak{E}(\frac{q}{p})$ is written as $\exp \frac{q}{p}\mathfrak{p}$, $\mathfrak{p}\in \boldsymbol{P}_E$. Since $(p, q)=1$, there exists an integer m such that $m(\frac{q}{p})\equiv\frac{1}{p}(\text{mod. } 1)$, it is clear that $\left[\mathfrak{E}(\frac{q}{p})\right]=\left[\mathfrak{E}(\frac{1}{p})\right]$. And $\mathfrak{E}(\frac{1}{p})$ is considered as the set of all the elements U such that $U^p=E$, and then we see that $U^{-1}=U^{p-1}$. Let M be any element of $\left[\mathfrak{E}(\frac{1}{p})\right]$, since $\det U=\omega^r$, $(r=0, 1, \dots, \text{or } p-1)$ for all $U\in\mathfrak{E}(\frac{1}{p})$, then it is easily seen that $\det M=\omega^l$, $(l=0, 1, \dots, \text{ or } p-1)$, ω being p-th primitive root of 1. If we put $G_0=\left[\mathfrak{E}(\frac{1}{p})\right]\cap SL(n, C)$, then easily we see that $\left[\mathfrak{E}(\frac{1}{p})\right]=G_0\times Z$, where $Z=\{1, \omega, \dots, \omega^{p-1}\}$. Since G_0 is a closed subgroup of $SL(n, C)$, by Cartan's Theorem,[1] G_0 is a Lie subgroup of $SL(n, C)$, Since $\mathfrak{E}(\frac{q}{p})$ is invariant by any transformation, i. e., $T^{-1}\mathfrak{E}(\frac{q}{p})T\subset\mathfrak{E}(\frac{q}{p})$ for all $T\in\mathfrak{M}$, G_0 is a non-discrete invariant subgroup of $SL(n, C)$. Since $SL(n, C)$ is a simple Lie group, we see that $G_0=SL(n, C)$. Thus, we obtain that $\left[\mathfrak{E}(\frac{q}{p})\right]=\left[\mathfrak{E}(\frac{1}{p})\right]=SL(n, C)\times Z$.

THEOREM 10. *The group generated by* $\mathfrak{E}(a)$, a *being an irrational number, is a direct product of the special linear group* $SL(n, C)$ *and the group* $T=\{e^{i\theta}\,;\ -\pi<\theta\leqq\pi\}$.

PROOF. Any element of $\mathfrak{E}(a)$ is written as $\exp a\mathfrak{p}$, $\mathfrak{p}\in\boldsymbol{P}_E$. Since a is an irrational number, it is clear that $[(\exp a\mathfrak{p})^m\,;\ m=1, 2, \dots]=\{\exp t\mathfrak{p}\,;\ 0\leqq t<1\}$, $\mathfrak{p}$ being fixed; consequently, $[\mathfrak{E}(a)]\supset\{\exp t\mathfrak{p}\,;\ 0\leqq t<1\}$, for all $\mathfrak{p}\in\boldsymbol{P}_E$, and hence $[\mathfrak{E}(a)]=[\exp t\mathfrak{p}, (0\leqq t<1)\,;\ \mathfrak{p}\in\boldsymbol{P}_E]$. Let M be any element of $[\mathfrak{E}(a)]$, then it is easily seen that $\det M=e^{i\theta}$, θ being real. If we put $G_0=[\mathfrak{E}(a)]\cap SL(n, C)$, then we see easily that $[\mathfrak{E}(a)]=G_0\times T$, where $T=\{e^{i\theta}\,;\ -\pi<\theta\leqq\pi\}$. By the same reason as in the proof of Theorem 9, we can conclude that $G_0=SL(n, c)$. Thus, we see that $[\mathfrak{E}(a)]=SL(n, C)\times T$.

§ 4. The paths in the special orthogonal group O^+

Let O_0^+ be the set $O^+\cap\mathfrak{M}_0$, and as in the previous paper,[2] let O_1^+ be the

1) See, for example, [1], p. 135, corollary.
2) See [3], pp, 316–319.

set of the matrices M of O^+ connected with E by an orthogonal path, i.e., a path contained in O^+, and moreover let O^+_{II} be the set $O^+ - O^+_{\mathrm{I}}$, then we have shown in the previous paper[1] that $O^+_0 \subsetneqq O^+_{\mathrm{I}}$ and that O^+_{II} is not empty and it is the set of the matrices M of O^+ connected with E by a curve which is a product of two orthogonal paths: $M(t) = M_1(t)M_2(t)$, $(0 \leqq t \leqq 1)$, where $M_1(t)$ and $M_2(t)$ are the orthogonal paths. Let $O^+_{(1)}$ be the set of the non-exceptional orthogonal matrices M (i.e., det $(M+E) \neq 0$), and let $O^+_{(2)}$ be the set of the exceptional orthogonal matrices M (i.e., det $(M+E)=0$), then, from consideration in the previous paper,[1] it follows that $O^+_0 \subsetneqq O^+_{(1)} \subsetneqq O^+_{\mathrm{I}}$ and $O^+_{\mathrm{II}} \subsetneqq O^+_{(2)}$.

In this section we shall first consider the subset O^+_0 of O^+.

THEOREM 11. *O^+_0 is a maximal simply connected domain of O^+; and O^+_0 is dense in O^+.*

PROOF. Since $O^+_0 = O^+ \cap \mathfrak{M}_0$, it is clear that O^+_0 is open in O^+; and O^+_0 is homeomorphic with the set $\mathfrak{M}^s \cap \mathfrak{A}_0$ by the exponential mapping, where $\mathfrak{M}^s$ is the set of all the skew-symmetric matrices. Here, it is easily seen, that $\mathfrak{M}^s \cap \mathfrak{A}_0$ is connected and simply connected, therefore, O^+_0 is also similarly as in Remark 1, connected and simply connected. We have considered the orthogonal canonical form of an orthogonal matrix by the coordinate transformation; by the results obtained there, the canonical matrix $\mathring{M}$ of $M \in (O^+ - O^+_0)$ contains, at least, one of the following blocks:[2]

(i) $$\hat{M}: \overbrace{\begin{pmatrix} a & & * \\ & a & \\ 0 & & \ddots \\ & & & a \end{pmatrix}}^{r} \dotplus \overbrace{\begin{pmatrix} a^{-1} & & * \\ & a^{-1} & \\ 0 & & \ddots \\ & & & a^{-1} \end{pmatrix}}^{r}, \quad (a<0), \quad \hat{g}: \mathring{g}(2r),$$

(ii) $$\hat{M}: \overbrace{\begin{pmatrix} -1 & & * \\ & -1 & \\ 0 & & \ddots \\ & & & -1 \end{pmatrix}}^{2s}, \qquad \hat{g}: \mathring{g}(2s),$$

(iii) $$\hat{M}': \begin{pmatrix} -1 & & * \\ & -1 & \\ 0 & & \ddots \\ & & & -1 \end{pmatrix} \dotplus \begin{pmatrix} -1 & & * \\ & -1 & \\ 0 & & \ddots \\ & & & -1 \end{pmatrix}, \quad \hat{g}': \mathring{g}(r_1) \dotplus \mathring{g}(r_2),$$
(r_1 and r_2 being any odd numbers),

1) See [3], pp. 316-319.
2) See [3], pp. 310-311.

where $\mathring{g}(m)=\overbrace{\begin{pmatrix} 0 & & 1 \\ & 1 & \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}}^{m}$. Furthermore, this last form is transformed to [1]

$$\hat{M}:\ \overbrace{\begin{pmatrix} -1 & & & * \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{pmatrix}}^{r_1+r_2},\quad \hat{g}:\ \mathring{g}(r_1+r_2).$$

Let $\mathring{M}(\theta)$ be the matrix obtained from $\mathring{M}$ by taking $\Theta(\theta)\hat{M}$ in place of $\hat{M}$, where $\Theta(\theta)=e^{i(\pi+\theta)}E_p \dotplus e^{-i(\pi+\theta)}E_p$, $2p$ being the degree of $\hat{M}$, then we have $\mathring{M}(\theta)\in O_0^+(\mathring{g})$, where $O^+(\mathring{g})$ means the special orthogonal group with respect to the metric tensor $\mathring{g}$; therefore, in the original coordinate system, we have $M(\theta)\in O_0^+$ for all θ: $-\pi<\theta<\pi$. If $\theta\rightarrow\pi$, then $M(\theta)\rightarrow M$, that is, O_0^+ is dense in O^+. Finally, we shall show that O_0^+ is a maximal simply connected domain. If $\tilde{O}$ is a set such that $O_0^+\subsetneqq\tilde{O}\subset O^+$, then there exists a matrix M such that $M\in\tilde{O}-O_0^+$. For this M, we shall consider $M(\theta)$ mentioned avove. $M(\theta)$ $(-\pi<\theta\leqq\pi)$ is a closed curve in $\tilde{O}$ which contained in O_0^+ except for M. If this closed curve is deformable to a point, then the circle $ae^{i\theta}$ $(-\pi<\theta\leqq\pi)$ in the complex plane must shrink into a point, (since the characteristic roots of $M(\theta)$ are the continuous functions of $M(\theta)$), consequently, the closed curve, in the way of deformation, must pass through O in the complex plane. Then, the corresponing matrix becomes singular, this is a contradiction. That is, the set $\tilde{O}$ is not simply connected.

Therefore, O_0^+ is a maximal simply connected domain. Thus the theorem is completely proved.

Let $\mathfrak{C}_*$ be the set of all the closed orthogonal paths $M(t)$ through E: $M(t)=\exp t\mathfrak{p}$, $\mathfrak{p}\in\boldsymbol{P}_E\subset\mathfrak{M}^s$, and let $\mathfrak{C}_*(t_0)\equiv\{M(t_0)\,;M(t)\in\mathfrak{C}_*\}$, then we have

THEOREM 12. *$O^+(n)$ $(n\geqq 3)$ is generated by $\mathfrak{C}_*(t_0)$, where t_0 is any real number such that* $0<t_0<1$.

PROOF. It is easily seen that

$$\exp t2\pi\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\dotplus O_{n-2}\right\}=\begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix}\dotplus E_{n-2},$$

where O_{n-2} means the zero matrix of degree $n-2$, and that $\mathfrak{C}_*(t_0)$ contains the following elements:

1) See [3] p. 318.

$$\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix} \dotplus E_{n-3}\right\}\left\{\begin{pmatrix}\cos 2\pi t_0 & \sin 2\pi t_0 & 0\\ -\sin 2\pi t_0 & \cos 2\pi t_0 & 0\\ 0 & 0 & 1\end{pmatrix} \dotplus E_{n-3}\right\}$$
$$\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix} \dotplus E_{n-3}\right\}^{-1}$$

$$=\begin{pmatrix}\cos 2\pi t_0 & \sin 2\pi t_0 \cos\theta & -\sin 2\pi t_0 \sin\theta\\ -\sin 2\pi t_0\cdot\cos\theta & \cos 2\pi t_0\cdot\cos^2\theta+\sin^2\theta & (1-\cos 2\pi t_0)\sin\theta\cos\theta\\ \sin 2\pi t_0\cdot\sin\theta & (1-\mathrm{con}\, 2\pi t_0)\sin\theta\cos\theta & \cos 2\pi t_0\sin^2\theta+\cos^2\theta\end{pmatrix} \dotplus E_{n-3},$$

where θ is any complex number. That is, $[\mathfrak{C}_*(t_0)]$ is not discrete. Since it is clear that $[\mathfrak{C}_*(t_0)]\subset O^+$ and that $[\mathfrak{C}_*(t_0)]$ is a closed subgroup of O^+, by Cartan's Theorem, we see that $[\mathfrak{C}_*(t_0)]$ is a Lie subgroup of O^+. And also the set $\mathfrak{C}_*(t_0)$ is invariant under the orthogonal transformation, therefore, $[\mathfrak{C}_*(t_0)]$ is a non-discrete invariant Lie subgroup of O^+. For the case where $n\neq 4$, O^+ is a simple Lie group, and hence we have that $[\mathfrak{C}_*(t_0)]=O^+$. For the case where $n=4$, it is well known that $O^+(4)=G_1\times G_2$ (direct product), where G_1 and G_2 are the invariant Lie subgroup of $O^+(4)$, both being simple and isomorphic to $O^+(3)$. G_1 an G_2 are generated by the following infinitesimal operators respectively:

$$G_1:\ R_1=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix},\quad R_2=\begin{pmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix},\quad R_3=\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix};$$

$$G_2:\ S_1=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix},\quad S_2=\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix},\quad S_3=\begin{pmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}.$$

And moreover, easily we see that $T^{-1}R_kT=S_k$ $(k=1, 2, 3)$, where $T=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$ $(T\in O(4)$ but $T\notin O^+(4))$, that is, we have $T^{-1}G_1T=G_2$. Therefore, since $[\mathfrak{C}_*(t_0)]$ is orthogonal invariant, if $[\mathfrak{C}_*(t_0)]$ contains G_1 then it also contains G_2, and vice versa. Thus, by considering that $[\mathfrak{C}_*(t_0)]$ is an invariant Lie subgroup of $O^+(4)$, we have that $[\mathfrak{C}_*(t_0)]=O^+(4)$. The theorem is completely proved.

REMERK 8. In the case where $n=2$, if t_0 is an irrational number, then we have $[\mathfrak{C}(t_0)]=[\exp t_0\mathfrak{p}]=\{\exp t\mathfrak{p}\ ;\ 0\leqq t\leqq 1\}$, $\mathfrak{p}=2\pi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, that is, $[\mathfrak{C}_*(t_0)]$ is a one-parameter real Lie group $\mathfrak{C}_*$, therefore we have $[\mathfrak{C}_*(t_0)]=\mathfrak{C}_*\subsetneqq O^+(2)$. If t_0 is a rational number, i. e., $t_0=\dfrac{q}{p}$, $(p, q)=1$, then we have

$$\left[\mathfrak{C}_*\left(\frac{q}{p}\right)\right]=\left[\mathfrak{C}_*\left(\frac{1}{p}\right)\right]=\left\{\exp\frac{k}{p}\mathfrak{p}\ ;\ k=0, 1, 2, \dots, p-1\right\},\ \mathfrak{p}=2\pi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

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