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Lattices of Projections in AW-algebras*

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Introduction.

Kaplansky [1] [1] has introduced the notion of AW*-algebras, as an elegant abstract generalization of weakly closed self-adjoint operator algebras on a Hilbert space (W*-algebras). By an AW*-algebra, it is meant a B*-algebra such that the left annihilator of any subset is a principal left ideal generated by a projection, an idempotent self-adjoint element. Many properties such as complete additivity of equivalence of projections and Decomposition Theorem, which play the fundamental rôles in both W*-algebras and continuous geometries, have been shown in AW*-algebras by Kaplansky [1], [2]. In particular, it has been verified that the projections in a *finite* AW*-algebra form a continuous geometry, and that there exists a unique normalized dimension function in any finite AW*-algebra.

In this paper, we shall investigate the lattice of projections in an *arbitrary* AW*-algebra by making use of the known theory of continuous geometries and AW*-algebras. The set of all projections in an AW*-algebra is a complete orthocomplemented lattice such that any pair of orthogonal projections forms a modular pair [2] (Theorem 1.1), and its subset of all finite projections forms a general continuous geometry (Theorem 1.2). Although it is neither modular nor continuous lattice, it has many analogous properties as a continuous geometry, among others, it is shown that it has a dimension function uniquely determined in some sense (Theorem 5.1 and 5.2), which is an extension of the result of Kaplansky [1] stated above to a general case, and may be considered as an abstract generalization of Segal [1], Theorem 1, and of a part of Dixmier [3], Theorem 1 and 2.

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1) The numbers in square brackets refer to the references at the end of the paper.
2) (b, c) is called a *modular pair* if $a \leq c$ implies $(a \cup b) \cap c = a \cup (b \cap c)$.

§ 1. Projections in AW*-algebras.

According to Kaplansky [3], 853, we shall define an AW*-algebra as follows:

DEFINITION 1.1. By a *B*-algebra*, it is meant a Banach $*$-algebra in which, $\|x^*x\| = \|x\|^2$ for every $x \in A$. An element f in A is called a *projection* if it is self-adjoint and idempotent, i. e., $f=f^*=f^2$. For projections a, b in A, we shall denote by $a \leqq b$ if $a=ab$. We observe that the set of all projections in A is partially ordered by the relation "$\leqq$", and that if a and b commute we have $a \cup b = a+b-ab$ and $a \cap b = ab$.

By an *AW*-algebra*, we mean a B*-algebra A such that the left annihilator $L(S)$ of any subset S of A is a principal left ideal generated by a projection f in A, i. e., $L(S)=Af$.

REMARK 1.1. A B*-algebra is an AW*-algebra if and only if it satisfies the following conditions. Cf. Kaplansky [1], 236, [3], 853.

(A) *In the partially ordered set of projections, any set of orthogonal projections has a least upper bound.*

(B) *Any maximal commutative self-adjoint subalgebra is generated by its projections.*

REMARK 1.2.[3] Any W*-algebra $\mathbf{M}$ is an AW*-algebra. Because $\mathbf{M}$ is a B*-algebra as is well knoun; and if $\{T_\alpha; \alpha \in I\}$ is a subset of $\mathbf{M}$, and $\mathfrak{R}_\alpha$ is the renge of T_α $(\alpha \in I)$, let E be the projection on the orthocomplement of the linear manifold $\vee(\mathfrak{R}_\alpha; \alpha \in I)$, then we can show without difficulty that the left annihilator of $\{T_\alpha; \alpha \in I\}$ is the principal left ideal generated by E.

LEMMA 1.1. *For any subset S of an AW*-algebra A the right annihilator $R(S)$ is a principal right ideal generated by a projection.*

PROOF. Let $S^* = \{s^*; s \in S\}$, then the left annihilator $L(S^*)$ is a principal left ideal generated by a projection, say f. It is easily shown that $R(S)=fA$.

The following three lemmas are immediate consequences of the definiton of AW*-algebras. Cf. Kaplansky [1], 237, Corollary 1, 2 and 3, respectively.

LEMMA 1.2. *In an AW*-algebra, the left annihilator of a left ideal is a principal two-sided ideal generated by a central projection.*

LEMMA 1.3. *An AW*-algebra has the unit element* 1.

LEMMA 1.4. *The projections in an AW*-algebra form a complete lattice.*

DEFINITION 1.2. Let L be a lattice with 0. If there exists a dual-automorphism of period two $a \to a^\perp$ such that $a \cap a^\perp = 0$, then L is called *orthocomplemented*,

3) Cf. Kaplansky [1], 236, Remark.

and $a^{\perp}$ is called the *orthocomplement* of a. If $a \leqq b^{\perp}$, then a is said *orthogonal* to b and is denoted by $a \perp b$.

THEOERM 1.1.[4] *The set A_p of all projections in an AW*-algebra A is a complete orthocomplemented lattice sucht hat $a \perp b$ implies $(a \cup b) \cap b^{\perp} = a$.*

PROOF. Put $a^{\perp} = 1 - a$ for any projection a in A, then $a^{\perp}$ is clearly the orthocomplement of a.

If $a \perp b$, then we have

$$(a \cup b) \cap b^{\perp} = (a + b) b^{\perp} = a.$$

Thus the proof is completed by Lemma 1.4.

Following Maeda [2], 90, we shall define general continuous geometries as follows:

DEFINITION 1.3. A conditionally complete lattice L is called *conditionally continuous*[5] if it satisfies both the following conditons:

For any directed set $\{a_\delta \,;\, \delta \in D\}$ of L,

(α) $a_\delta \uparrow a$ *implies* $a_\delta \cap b \uparrow a \cap b$, *for any* $b \in L$, and

(β) $a_\delta \downarrow a$ *implies* $a_\delta \cup b \downarrow a \cup b$, *for any* $b \in L$.

By a *general continuous geometry*, it is meant a conditionally continuous, relatively complemented, modular lattice with 0.

A complete lattice is called *upper continuous* if it satisfies the condition (α), and is called *continuous* if both conditions (α) and (β) are satisfied.

By a *continuous geometry*, we mean a continuous, complemented, modular lattice.

LEMMA 1.5. *Let L be a lattice with 0. L is a general continuous geometry if and only if $L(0, a) = \{x \in L \,;\, x \leqq a\}$ is a continuous geometry for any $a \in L$.*

PROOF. Let L be a general continuous geometry. For any element a, $L(0, a)$ has the unit element a, whence it is clearly a continuous geometry.

Conversely suppose that $L(0, a)$ is a continuous geometry for any $a \in L$. For any $b, c \in L$, $a \leqq c$ implies $(a \cup b) \cap c = a \cup (b \cap c)$, since $L(0, b \cup c)$ is modular. It follows that L is a modular lattice. By a similar argument, we can easily show that L is a conditionally continuous, relatively complemented lattice.

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4) In an orthocomplemented lattice, any pair of orthogonal projections forms a modular pair (cf. foot-note 2) provided that $a \perp b$ implies $(a \cup b) \cap b^{\perp} = a$. Because, if $a \leqq c \leqq b^{\perp}$, then it holds

$$(a \cup b) \cap c = (a \cup b) \cap b^{\perp} \cap c = a = a \cup (b \cap c).$$

Thus the condition (V) in Sasaki [1] could be replaced by a formally weaker condition: $a \perp b$ *implies* $(a \cup b) \cap b^{\perp} = a$.

5) Cf. Maeda [2], Definition 2.2.

Consequentely L is a general continuous geometry.

DEFINITION 1.4. Let a, b be projections in an AW*-algebra. If there exists an element x such that $xx^* = a$ and $x^*x = b$, then a is called *equivalent* to b and is denoted by $a \sim b$. A projection a is called *finite* if $a \sim b$ and $b \leqq a$ implies $a = b$, and otherwise *infinite*. An AW*-algebra is called *finite* or *infinite* according as 1 is finite or not. We shall write $a \prec b$ if there exists a projection c such that $a \sim c$ and $c < b$. Note that a is infinite if and only if $a \prec a$.

LEMMA 1.6. *Let a be a finite projection in an AW*-algebra A. The sublattice $A_p(0, a)$ of all projections contained in a is a continuous geometry.*

PROOF. By Kaplansky [1], Theorem 2.4, the subalgebra aAa is also an AW*-algebra which is easily shown to be finite,[6] and it follows from Kaplansky [1], Theorem 6.5 that the set of all projections in aAa which is identical with $A_p(0, a)$ forms a continuous geometry.

As an immediate consequence of Lemma 1.5 and 1.6, we obtain the following:

THEOREM 1.2. *The set of all finite projections in an AW*-algebra is a general continuous geometry.*

§ 2. Central projections in AW*-algebras.

In this section, we shall discuss the interrelation of central projections in an AW*-algebra A and central elements of the lattice A_p of projections in A.

DEFINITION 2.1. Let a, b be elements of an orthocomplemented lattice. If it holds $a = (a \cap b) \cup (a \cap b^\perp)$, then a is called *permutable* with b, and is denoted by $a \leftrightarrow b$. Obviously $a \leftrightarrow b$ implies $a \leftrightarrow b^\perp$.

LEMMA 2.1. *Let L be a complete, orthocomplemented lattice such that $a \perp b$ implies $(a \cup b) \cap b^\perp = a$. Then we have*

(i) *$a \leftrightarrow b$ imples $b \leftrightarrow a$,*

(ii) *$a_1 \leftrightarrow b$, $a_2 \leftrightarrow b$ imply $a_1 \cup a_2 \leftrightarrow b$,*

(iii) *$a_\delta \leftrightarrow b$, $a_\delta \uparrow a$ imply $a_\delta \cap b \uparrow a \cap b$, where $\{a_\delta ;\ \delta \in D\}$ is a directed set of L,*

(iv) *$a_\alpha \leftrightarrow b\,(\alpha \in I)$ implies $\vee(a_\alpha ;\ \alpha \in I) \leftrightarrow b$.*

PLOOF. (i) Cf. Sasaki [1], Lemma 5.5.

(ii) Since $a_i = (a_i \cap b) \cup (a_i \cap b^\perp)$ $(i = 1, 2)$, we have

$$a_1 \cup a_2 = \{(a_1 \cap b) \cup (a_2 \cap b)\} \cup \{(a_1 \cap b^\perp) \cup (a_2 \cap b^\perp)\}$$

6) We observe that if b, c are projections in aAa, then b, c are equivalent in aAa if and only if b, c are equivalent in A, since the element x defining the equivalence of b, c is necessarily in bAc.

$$\leqq\{(a_1\cup a_2)\cap b\}\cup\{(a_1\cup a_2)\cap b^\perp\}\leqq a_1\cup a_2.$$

Thus it holds $a_1\cup a_2\leftrightarrow b$.

(iii) Since $a_\delta=(a_\delta\cap b)\cup(a_\delta\cap b^\perp)$ $(\delta\in D)$, we obtain

$$\vee(a_\delta\,;\ \delta\in D)=\vee(a_\delta\cap b\,;\ \delta\in D)\cup\vee(a_\delta\cap b^\perp\,;\ \delta\in D).$$

It follows from $\vee(a_\delta\cap b^\perp\,;\ \delta\in D)\perp b$ that

$$b\cap\vee(a_\delta\,;\ \delta\in D)=\{\vee(a_\delta\cap b\,;\ \delta\in D)\cup\vee(a_\delta\cap b^\perp\,;\ \delta\in D)\}\cap b$$
$$=\vee(a_\delta\cap b\,;\ \delta\in D)$$

(iv) Let D be the directd set of all finite subset ν of I, and let $a_\nu=\vee(a_\alpha\,;\ \alpha\in\nu)$, then we have $\vee(a_\nu\,;\ \nu\in D)=\vee(a_\alpha\,;\ \alpha\in I)$, and $a_\nu\leftrightarrow b$ $(\nu\in D)$ by (ii). It follows from (iii) that $b\cap\vee(a_\delta\,;\ \delta\in D)=\vee(a_\delta\cap b\,;\ \delta\in D)$, and $b^\perp\cap\vee(a_\delta\,;\ \delta\in D)=\vee(a_\delta\cap b^\perp\,;\ \delta\in D)$, whence

$$\{b\cap\vee(a_\alpha\,;\ \alpha\in I)\}\cup\{b^\perp\cap\vee(a_\alpha\,;\ \alpha\in I)\}=\vee(a_\alpha\cap b\,;\ \alpha\in I)\cup\vee(a_\alpha\cap b^\perp\,;\ \alpha\in I)$$
$$=\vee((a_\alpha\cap b)\cup(a_\alpha\cap b^\perp)\,;\ \alpha\in I)$$
$$=\vee(a_\alpha\,;\ \alpha\in I),$$

completing the proof.

LEMMA 2.2.[7] *Let L be an orthocomplemented lattice such that $a\perp b$ implies $(a\cup b)\cap b^\perp=a$. Then the following conditions are equivalent to each other:*

(α) *z is a central element of L.*[8]

(β) *z is a neutral element of L.*[9]

(γ) *z has a unique complement.*

(δ) *z is permutable with every element of L.*

(ε) *If z' is a complement of z, then we have $a=(a\cap z)\cup(a\cap z')$ for every a.*

PROOF. $(\alpha)\Rightarrow(\beta)\Rightarrow(\gamma)$ is obvious.

$(\gamma)\Rightarrow(\delta)$. For any $a\in L$, put $v=a\cap(a\cap z)^\perp$, then $v\cap z=0$, whence $z'=v\cup(v\cup z)^\perp$ is a complement of z such that $v\leqq z'$.[10] Since the complement is unique, it follows $v\leqq z^\perp$, accordingly $v\leqq a\cap z^\perp$. Now we have $a=(a\cap z)\cup v\leqq(a\cap z)\cup(a\cap z^\perp)$. Thus it holds $a=(a\cap z)\cup(a\cap z^\perp)$, since the converse inequality

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7) A similar theorem is valid in W*-algebras, and in complemented modular lattices, cf. Maeda [1], Theorem I; and [3], 23, Theorem 3.3, 49, Theorem 2.1, respectively.

8) An element z of a lattice L with 0 and 1 is called a central element if it corresponds to $[1_1, 0_2]$ under an isomorphism of L with $L_1\cdot L_2$ for some lattices L_1, L_2.

9) An element z is called neutral if the triple $\{x, y, z\}$ generates a distributive sublattice of L for every x, $y\in L$.

10) Remark that $z\cap z'=\{v\cup(v\cup z)^\perp\}\cap(v\cup z)\cap z=v\cap z$.

is valid in general.

$(\delta)\Rightarrow(\alpha)$. It holds $a=(a\cap z)\cup(a\cap z^{\perp})$ for every $a\in L$. Therefore if we put $L_1=L(0, z)$, and $L_2=L(0, z^{\perp})$, then we can easily show that L is isomorphic to $L_1\cdot L_2$ and z corresponds to $[1_1, 0_2]$. Thus z is a central element of L.

Thus the conditions (α), (β), (γ), and (δ) are equivalent to each other. It is clear that (β) implies (ε) and (ε) implies (δ), completing the proof.

LEMMA 2.3. *Let a, b be projections in an AW^*-algebra. It holds $ab=ba$ if and only if $a\leftrightarrow b$.*

PROOF. If $ab=ba$, then obviously $ab^{\perp}=b^{\perp}a$, whence $ab=a\cap b$, and $ab^{\perp}=a\cap b^{\perp}$. It follows that $(a\cap b)\cup(a\cap b^{\perp})=ab+ab^{\perp}=a$, whence $a\leftrightarrow b$.

Conversely suppose that $a\leftrightarrow b$, then we have

$$a=(a\cap b)\cup(a\cap b^{\perp}),\quad \text{whence}$$

$$ab=(a\cap b)\,b+(a\cap b^{\perp})b=a\cap b.$$

Similarly we have $ba=a\cap b$, whence $ab=ba$.

THEOREM 2.1.[11] *The center of the lattice A_p of all projections in an AW^*-algebra A consists of central projections in A, and it is a complete Boolean algebra.*

PROOF. Let z be a central projection in A. For any $a\in A_p$, we have $az=za$, and it follows from Lemma 2.2 and 2.3 that z belongs to the center of A_p.

Conversely suppose that z is a central element of A_p. In order to show that z commutes with every $x\in A$, we may assume $x=x^*$, and it is sufficient to verify that z commutes with every projection $a\in A_p$, in view of the condition (B) in Remark 1.1. Thus the assertion is obvious from Lemmas 2.3 and 2.2.

The center of A_p is clearly a Boolean algebra, and the completeness follows immediately from Lemmas 2.1 (iv) and 2.3.[12]

§ 3. Central envelopes of projections. Structure theorems.

DEFINITION 3.1. Let S be a subset of an AW*-algebra A. By the *left projection* of S, it is meant the projection f such that the left annihilator of S is $Af^{\perp}$ (cf. Definition 1.1). The right projection of S is defined similarly. This

11) As to a similar theorem in W*-algebra, cf. Maeda [1], Theorem I.

12) The completeness may be proved by a similar argument as Kaplansky [1], Lemma 2.2, (b).

terminology agrees with that of Rickart [1], 534, if S consists of a single element,[13] in view of the following:

LEMMA 3.1. *Let S be a subset of an AW^*-algebra A, then the following statements are equivalent to each other:*

(α) *f is the left projection of S, i. e., $L(S)=Af^{\perp}$.*

(β) (i) *$fs=s$ for every $s\in S$, and*

(ii) *for any $x\in A$, $xs=0$ ($s\in S$) implies $xf=0$.*

(γ) *For a projection g, it holds $gs=s$ ($s\in S$) if and only if $g\geqq f$.*

PROOF. $(\alpha)\Rightarrow(\beta)$. (i) Since $f^{\perp}\in L(S)$, we have $f^{\perp}s=0$ ($s\in S$), whence $s=fs$ ($s\in S$). (ii) If $xs=0$ ($s\in S$), then $x\in L(S)=Af^{\perp}$, whence $x=xf^{\perp}$, and so $xf=0$.

$(\beta)\Rightarrow(\gamma)$. Let g be a projection with $gs=s$ ($s\in S$), then we have $g^{\perp}s=0$ ($s\in S$), whence $g^{\perp}f=0$ by (β) (ii), and so $f\leqq g$.

Conversely suppose that $f\leqq g$, then $f=gf$, whence it holds by (β) (i) that $s=fs=gfs=gs$ ($s\in S$).

$(\gamma)\Rightarrow(\alpha)$. Since $fs=s$ ($s\in S$), we have $f^{\perp}s=0$ ($s\in S$), whence $f^{\perp}\in L(S)$. Now let g be the left projection of S, then $f^{\perp}\in L(S)=Ag^{\perp}$ implies $f^{\perp}\leqq g^{\perp}$, and $f\geqq g$. On the other hand $gs=s$ ($s\in S$), whence $g\geqq f$ by (γ). Thus $f=g$.

COROLLARY 3.1. *The left and right projections of x are the left projections of xx^* and x^*x, respectively.*

PROOF. It holds $L(x)=L(xx^*)$, because $yx=0$ implies $yxx^*=0$, and conversely $yxx^*=0$ implies $yx(yx)^*=0$, accordingly $yx=0$. It follows that the left projections of x and xx^* are identical. The remaining part of the corollary is obvious, since $R(x)=L(x^*)$.

DEFINITION 3.2. Let S be a subset of an AW*-algebra A, and f and g be its left and right projections, respectively. By the *carrier*[14] of S, we shall mean the projection $f\cup g$. If S consists of a single element x, then the carrier of $\{x\}$ is called the carrier of the element x, and is denoted by $c(x)$.

As an immediate consequence of Lemma 3.1, we have the following:

COROLLARY 3.2. *Let e be the carrier of a subset S of an AW^*-algebra, and a be a projection. It holds $as=sa=s$ ($s\in S$) if and only if $a\geqq e$.*

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13) B_p^*-algebras in the terminology of Rickart [1] migth be defined as B*-algebras with unit elements such that every element has its left projection, while AW*-algebras are B*-algebras such that every subset has its left projection.

14) In W*-algebras, the carrier of S is idencal with "die Haupteinheit von S" in the terminology of J. v. Neumann [1], Definition 4.

DEFINITION. 3.3. Let x be any element of an AW*-algebra A. The left projection of the left ideal Ax is called the *central carrier* of x. In particular, if a is a projection in A, the central carrier of a is called the *central envelope* of a, and is denoted by $e(a)$.[15)]

COROLLARY 3.3. *The central carrier f of x is the least central projection such that $fx=xf=x$. Particularly the central envelope of a projection is the least central projection containing it.*

PROOF. By Lemma 1.2, f is a central projection, whence it follows from Lemma 3.1 that $fx=xf=x$. Now suppose that g is a central projection such that $gx=xg=x$, then we have $gax=ax$ $(a\in A)$, whence $f\leqq g$ by Lemma 3.1. The remainder part is then obvious.

REMARK 3.1. In view of Corollary 3.3, the central carrier of x might be defined as the right projection of xA.

We observe that the cetral carrier of x is the central envelope of the left projection of x. Indeed, let f be the left projection of x, then (i) $fx=x$, and (ii) $yx=0$ implies $yf=0$. It follows clearly $L(Ax)=L(Afx)\supseteqq L(Af)$. Conversely it holds $L(Af)\supseteqq L(Ax)$, for $yax=0$ implies $yaf=0$ by (ii) for every $a\in A$. Thus we have $L(Ax)=L(Af)$, consequently the central carrier of x is equal to $e(f)$.

DEFINITION 3.4. If a, b are equivalent projections in an AW*-algebra, (cf. Definition 1.4), then the element x such that $a=xx^*$ and $b=x^*x$, is called the *partially isometric element* defining the equivalence of a and b. By a *unitary element*, it is meant an element x such that $xx^*=x^*x=1$.

Now we shall give some remarks on the equivalence of projections, which has been discussed in detail by Kaplansky [1].

REMARK 3.2. The relation $\sim$ is clearly an equivalence relation, i. e., (i) $a\sim a$, (ii) $a\sim b$ implies $b\sim a$, and (iii) $a\sim b$, $b\sim c$ imply $a\sim c$. Furthermore it has the complete additivity,[16)] i. e., if $\{a_\alpha\,;\,\alpha\in I\}$ and $\{b_\alpha\,;\,\alpha\in I\}$ are both sets of mutually orthogonal projections such that $a_\alpha\sim b_\alpha$ $(\alpha\in I)$, then it holds $\vee(a_\alpha\,;\,\alpha\in I)\sim\vee(b_\alpha\,;\,\alpha\in I)$

REMARK 3.3. If x is the partially isometric element defining the equivalence of projections a and b, then a and b are the left and right projections of x, respectively, by Corollary 3.1.

15) As to a similar notion in W*-algebras, cf. Maeda [1], Definition 2.1 and Dixmier [1], Definintion 3.1; in continuous geometries, cf. J. v. Neumann [2], Part III, Definition 1.1; in upper-continuous, complemented, modular lattices, cf. Maeda [3], 60, Definition 4.2.

16) Cf. Kaplansky [1], Theorem 5.5.

Generally it holds that the left and right projections of any element are equivalent, and consequently for any projections a, b the projections $a-(a\cap b)$ and $(a\cup b)-b$ are equivalent, since they are left and right projections of $ab^{\perp}$, respectively.[17]

REMARK 3.4. If the projections a and b are perspective in the lattice A_p, i. e., $a\cup c=b\cup c$, $a\cap c=b\cap c$ for some projection c, then it holds $a\sim b$, because $a-(a\cap c)\sim(a\cup c)-c=(b\cup c)-c\sim b-(b\cap c)$ by Remark 3.3. As an immediate consequence of this results, we have that $a\cap b^{\perp}=a^{\perp}\cap b=0$ implies $a\sim b$.[18]

If a and b are finite projections, then $a\sim b$ implies $a\cup c=b\cup c$ and $a\cap c=b\cap c$ for some projection c.[19]

LEMMA 3.2.[20] *Let a, b and $\{a_\alpha;\ \alpha\in I\}$ be projections in an AW^*-algebra A. Then it holds:*

(i) $a\sim b$ *implies* $e(a)=e(b)$,

(ii) $a\lesssim b$ *implies* $e(a)\leqq e(b)$,

(iii) $e(za)=ze(a)$ *for any central projection* z,

(iv) $e(\vee(a_\alpha;\ \alpha\in I))=\vee(e(a_\alpha);\ \alpha\in I)$.

PROOF. (i) Let x be the partially isometric element defining the equivalence of a and b, then we have $L(Axx^*)=Ae(a)^{\perp}$ and $L(Ax^*x)=Ae(b)^{\perp}$ by Definition 3.3. Since $yAxx^*=0$ if and only if $yAx^*x=0$ by Kaplansky [1], Lemma 4.9, it holds $L(Axx^*)=L(Ax^*x)$, whence $e(a)=e(b)$.

(ii) It is obvious from (i).

(iii) Since $za\lesssim ze(a)$, it holds $e(za)\leqq ze(a)$. As $e(za)\leqq z$, we have $e(za)+z^{\perp}\geqq za+z^{\perp}a=a$, whence $e(za)+z^{\perp}\geqq e(a)$. By multiplying both sides of this inequality by z, we obtain $e(za)\geqq ze(a)$. Thus we have $e(za)=ze(a)$.

(iv) Since $\vee(a_\alpha;\ \alpha\in I)\geqq a_\alpha$ $(\alpha\in I)$, we have $e(\vee(a_\alpha;\ \alpha\in I))\geqq e(a_\alpha)$ $(\alpha\in I)$, whence $e(\vee(a_\alpha;\ \alpha\in I))\geqq\vee(e(a_\alpha);\ \alpha\in I)$. On the other hand, it holds $e(a_\alpha)\geqq a_\alpha$ $(\alpha\in I)$, whence $\vee(e(a_\alpha);\ \alpha\in I)\geqq\vee(a_\alpha;\alpha\in I)$, and accordingly $\vee(e(a_\alpha);\alpha\in I)\geqq e(\vee(a_\alpha;\ \alpha\in I))$, completing the proof.

LEMMA 3.3. *Let a, b, a_1 and b_1 be projections in an AW^*-algebra A, the following conditions are equivalent to each other:*

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17) Cf. Kaplansky [1], Theorem 5.2, Lemma 5.3 and Theorem 5.4.

18) Projections a, b such that $a\cap b^{\perp}=a^{\perp}\cap b=0$ are called to be in positions p' in the terminology of Dixmier [1].

19) Cf. Kaplansky [1], Theorem 6.6.

20) As to analogous properties in continuous geometries and in upper continuous, complemented, modular lattices, cf. J. v. Neumann [2], Part III, Theorem 1.3, 1.4 and Maeda [3], 60, Lemma 4.7, respectively.

(α) $aAb = 0$.

(β) $e(a)e(b) = 0$.

(γ) $a \geqq a_1$, $b \geqq b_1$ *and* $a_1 \sim b_1$ *imply* $a_1 = b_1 = 0$.

PROOF. $(\alpha) \Rightarrow (\beta)$. $aAb = 0$ implies $a \in L(Ab) = Ae(b)^{\perp}$, whence $ae(b) = 0$, and it follows from Lemma 3.2 (iii) that $e(a)e(b) = 0$.

$(\beta) \Rightarrow (\gamma)$. Let $a \geqq a_1$, $b \geqq b_1$ and $a_1 \sim b_1$. By Lemma 3.2 (i), (ii), we have $e(a_1) = e(b_1) \leqq e(a) \cap e(b) = 0$, whence $a_1 = b_1 = 0$.

$(\gamma) \Rightarrow (\alpha)$. Suppose that $aAb \neq 0$. Then there exists a non-zero element $x \in aAb$. Let a_1 and b_1 be respectively the left and right projections of x, then it holds $a_1 \sim b_1$ by Remark 3.3. Since $ax = a$, it follows from Lemma 3.1 that $a \geqq a_1$ and similarly $b \geqq b_1$. Thus we have $a_1 = b_1 = 0$, contrary to $x \neq 0$.

REMRAK 3.5. It has been shown that (γ) implies (α) in AW*-algebras and B_p^*-algebra, cf. Kaplansky [1], Lemma 3.3, and Rickart [1], Theorem 5.2. As to an analogous theorem in W*-algebras, in continuous geometries, and in upper continuous complemented modular lattices, cf. Maeda [1], Theorem II, J. v. Neumann, [2], Part I, Theorem 5.7, Part III, Lemma 1.1, and Maeda [3], 61, Theorem 4.5, respectively.

As Kaplansky [1] has shown, the Decomposition Theorem, one of the most important theorems in dimension theory, is now at hand in view of Lemma 3.3 and the complete additivity of the equivalence of projections (Remark 3.2). We shall for convenience cite it here for reference.

DECOMPOSITION THEOREM.[21] *Let* a, b *be projections in an* AW^**-algebra. There existsts a central projection* z *such that* $za \lesssim zb$ *and* $z^{\perp}a \gtrsim z^{\perp}b$.

DEFINITION 3.5. A B^*-algebra is called a *factor* if its center consists of scalar multiples of the unit element.

COROLLARY 3.4. *In an* AW^**-algebra* A, *the following conditions are equivalent to each other:*

(α) *A is a factor.*

(β) *For projections a, b in A, it holds either $a \lesssim b$ or $a \gtrsim b$.*

(γ) *For projections a, b in A, $aAb = 0$ implies either $a = 0$ or $b = 0$.*

PROOF. $(\alpha) \Rightarrow (\beta)$. By applying the Decomposition Theorem to a, b, there

21) Cf. Kaplansky [1], Lemma 3.4 and Theorem 5.6. As to an analogous theorem in W^*-algebra, cf. Maeda [1], Theorem III, and Dixmier [1], Theorem 6. In cases continuous geometries and upper continuous complemented modular lattices, cf. J. v Neumann [2], Part III, Theorem 2.7, and Maeda [3], 77, Thorem 1.2, respectively.

exists a central projection z such that $za \gtrsim zb$ and $z^{\perp}a \lesssim z^{\perp}b$. Since either $z=0$ or $z=1$, it follows that either $a \lesssim b$ or $a \gtrsim b$.

$(\beta) \Rightarrow (\gamma)$. Suppose that $aAb=0$, then it holds from Lemma 3.3 that $a \geqq a_1$, $b \geqq b_1$, $a_1 \sim b_1$ implies $a_1 = b_1 = 0$. Now since either $a \lesssim b$ or $a \gtrsim b$, there exists a projection c such that $a \sim c \leqq b$ or $b \sim c \leqq a$, and it follows that either $a=0$ or $b=0$.

$(\gamma) \Rightarrow (\alpha)$. Let z be any central projection in A, then we have obviously $z^{\perp}Az=0$, whence either $z=1$ or $z=0$. Thus the center of A contains only scalar multiples of 1.

REMARK. 3.6. The equivalence of (α) and (γ) in specified B_p^*-algebras has been shown by Rickart [1], Theorem 5.18. As to the equivalence of (α) and (β) in any W*-algebra, cf. Maeda [1], Theorem IV.

THEOREM 3.1.[22] *Let Z be the center of an AW*-algebra A, and let a be any projection in A. The center of the AW*-algebra aAa is identical with aZ, and it is *-isomorphic to $e(a)Z$.*

PROOF. Let f be any central projection in the AW*-algebra aAa, whose unit element is clearly a. Since $(a-f)Af=(a-f)aAaf=(a-f)faAa=0$, it follows from Lemma 3.3 that $e(a-f)e(f)=0$. Now by Lemma 3.2, we have

$$e(ae(f)-f)=e(ae(f)-fe(f))=e(a-f)e(f)=0,$$

whence $f=ae(f) \in aZ$.

Conversely any projection in aZ is clearly a central projection in aAa, and it follows that central projections in aAa consist of projections in aZ.

Since aZ and the center of aAa are both AW*-algebras, they are generated by the set of projections (Remark 1.1 (B)), and it follows that the center of aAa is aZ.

The correspondence $az \leftrightarrow e(a)z$ is one-to-one between aZ and $e(a)Z$. Because, if $az_1 = az_2$ and $z_1, z_2 \in Z$, then $e(a)z_1 = e(a)z_2$. This mapping is obviously *-isomorphism of aZ and $e(a)Z$, completing the proof.

COROLLARY 3.5.[23] *Let a, b be projections in an AW*-algebra A. The following condition is equivalent to the conditions in* Lemma 3.3 :

(δ) $a \cap b = 0$ *and* (a, b) D, i. e., $(a \cup b) \cap x = (a \cap x) \cup (b \cap x)$ *for any* $x \in A_p$.

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22) As to an analogous theorem in W*-algebras, cf. Dixmier [3], Lemma 2.2, 2.4, and 2.5; in cases continuous geometries and upper continuous complemented modular lattices, cf. J. v. Neumann [2], Part III, Theorem 1.6, and Maeda [3], 78, Theorem 1.4, respectively.

23) As to an analogous theorem in W*-algebras, cf. Maeda [1], Theorem II.

PROOF. We shall show that (δ) is equivalent to (β) in Lemma 3.3.

(β)$\Rightarrow$(δ). Since $ae(b)\leqq e(a)e(b)$, it follows that $a\frown b\leqq ae(b)=0$, and $(a\smile b)e(b)=b$ and similarly $(a\smile b)e(a)=a$, and so a and b are central projections in $(a\smile b)A(a\smile b)$ by Theorem 3.1. Accordingly we have $y=(a\frown y)\smile(b\frown y)$ for every projection $y\leqq a\smile b$. Thus it holds for any $x\in A_p$,

$$(a\smile b)\frown x=\{a\frown(a\smile b)\frown x\}\smile\{b\frown(a\smile b)\frown x\}=(a\frown x)\smile(b\frown x),$$

since $(a\smile b)\frown x\leqq a\smile b$. Consequently (a, b) D is valid.

(δ)$\Rightarrow$(β). Since a and b are complements to each other in $L(0, a\smile b)$, it follows from Lemma 2.2, and Theorem 2.1 that a, b are central projections in $(a\smile b)A(a\smile b)$, whence $a\smile b=a+b$. Now by Theorem 3.1, we have $a=e(a)(a+b)$, whence $e(a)b=0$, accordingly $e(a)e(b)=0$ by Lemma 3.2, (iii).

Now we shall enter into structure theorems of AW*-algebras.

DEFINITION 3.6. Let A be an AW*-algebra, and a, b be its projections. We shall write $a\gg b$ [24] if it holds either $za\succ zb$ or $za=zb=0$ for every central projection z. A non-zero projection a is called *properly infinite* if $a\gg a$ is valid, and *purely infinite* if a contains no finite projection ($\neq 0$). An AW*-algebra is called *purely* [25] (resp. *properly*) *infinite* provided that the unit element 1 is purely (resp. properly) infinite, and it is called *semi-finite* [26] if every non-zero central projection contains a finite non-zero projection.

REMRAK 3.7. Since the least upper bound of finite (resp. purely infinite) central projections is also finite (resp. purely infinite), it follows from Zorn's lemma that there exists a unique maximal finite (resp. purely infinite) central projection, say z^f (resp. z^{pi}). Put $z^i=1-z^f-z^{pi}$, then the algebras z^fA, $z^{pi}A$, and z^iA are finite, purely infinite and semi-finite together with properly infinite, respectively.

Thus it holds the following theorem. Cf. Kaplansky [1], Theorem 4.2.

THEOREM 3.2. *Any AW*-algebra is a direct sum of a finite algebra, a purely infinite algebra and a properly infinite and semi-finite algebra.*

REMARK 3.8. In view of Theorems 3.1. and 3.2, for any projection a, there exists a central projection z such that az is finite and $az^{\perp}$ is properly infinite.

24) A similar notation is used in continuous geometries by J. v. Neumann [2], Part III, Definition 2.3.

25) A purely infinite algebra in this sense is called "of type III" by Kaplansky [2], Definitions 1.3 and 2.1.

26) Cf. Griffin [1], Definition 1.4.

LEMMA 3.4.[27] *If a, b are equivalent projections in an AW^*-algebra, there exist unitary elements w_1, w_2 such that $a = a_1 + a_2$, $b = b_1 + b_2$ and $w_i^* a_i w_i = b_i$ $(i = 1, 2)$, where $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are orthogonal pairs of projections.*

PROOF. In view of Remark 3.7, we may assume without loss of generality that a and b are both finite or both properly infinite. In case a, b are finite, it follows from Remark 3.4, that a is perspective to b in A_p accordingly $a^{\perp}$ is also perspective to $b^{\perp}$ in A_p, whence $a^{\perp} \sim b^{\perp}$. Let x and y be the partially isometric elements defining $a \sim b$ and $a^{\perp} \sim b^{\perp}$ respectively, then $w_1 = x + y$ and $w_2 = 1$ are desired elements. In case a, b are properly infinite, there exists projections a_1, a_2; b_1, b_2 such that $a = a_1 + a_2$, $b = b_1 + b_2$ and $a \sim a_1 \sim a_2$, $b \sim b_1 \sim b_2$,[28] whence $a_1{}^{\perp} \sim b_1{}^{\perp}$.[29] By a similar argument as above, there exists a unitary element w_1 such that $w_1^* a_1 w_1 = b_1$, and similarly w_2 such that $w_2^* a_2 w_2 = b_2$. These elements w_1, w_2 are the asserted.

COROLLARY 3.6.[30] *Let a be a projection in an AW^*-algebra, and let $\{a_\alpha ; \alpha \in I\}$ be the set of all projections equivalent to a. Then it holds*

$$e(a) = \vee (a_\alpha ; \alpha \in I),$$

PROOF. Put $b = \vee (a_\alpha ; \alpha \in I)$, then we have $e(a) e(b^{\perp}) = 0$. For, if otherwise, there exist, by Lemma 3.3, non-zero projections a', b' such that $a' \leqq a$, $b' \leqq b^{\perp}$, and $a' \sim b'$, and it follows from Lemma 3.4 that there exist pairs of orthogonal projections $\{a_1', a_2'\}$, $\{b_1', b_2'\}$ and unitary elements w_1, w_2 such that $a' = a_1' + a_2'$, $b' = b_1' + b_2'$ and $w_i a_i' w_i^* = b_i'$ $(i = 1, 2)$. Let $a_i = w_i a w_i^*$ $(i = 1, 2)$, then $b_i' \leqq a_i \sim a$ $(i = 1, 2)$, whence $b_i' \leqq b$ $(i = 1, 2)$ and also $b_i' \leqq b^{\perp}$ $(i = 1, 2)$, and it follows $b_i' = 0$ $(i = 1, 2)$, contrary to $b' \neq 0$. Thus it holds $e(a) e(b^{\perp}) = 0$. Now since we have $e(a) - b = e(a) b^{\perp} \leqq e(a) e(b^{\perp})$, it follows $e(a) = b$.

Now we shall refine the decomposition of AW*-algebras in Theorem 3.2. We need the following lemmas.

LEMMA 3.5. *Let A be a properly infinite AW^*-algebra.*

(i) *If a, b are finite projections in A, then it holds $a \lesssim b^{\perp}$.*

(ii) *If a is a finite projection in A, then there exists a sequence of orthogonal projections $\{a_n\}$, each a_n being equivalent to a.*

PROOF. (i). By applying the Decomposition Theorem to a and $b^{\perp}$, there exists a central projection z such that $az \lesssim b^{\perp} z$ and $az^{\perp} \gtrsim b^{\perp} z^{\perp}$. Since $az^{\perp}$ is

27) As to a similar lemma in W*-algebra, cf Dixmier [2], Lemma 1.7.

28) Apply Kaplansky [1], Lemma 4.5 to aAa and to bAb.

29) Proof: $1 - a_1 = (1 - a) + a_2 \sim (1 - a) + a = 1$ and similarly $b_1{}^{\perp} \sim 1$, whence $a_1{}^{\perp} \sim b_1{}^{\perp}$,

30) As to an analogous property in W*-algebras, cf. Dixmier [1], Lemma 3.1.

finite, it follows that $b^{\perp}z^{\perp}$ is also finite; accordingly $b^{\perp}z^{\perp}+bz^{\perp}=z^{\perp}$ is finite, too. Since A is properly infinite, it follnws hat $z^{\perp}=0$, i. e., $z=1$, consequently $a\lesssim b^{\perp}$.

(ii). Let $\{a_1, a_2, \ldots, a_n\}$ be a set of orthogonal projections, each a_i being equivalent to a. Then $b=\bigvee_{i=1}^{n} a_i$ is also finite and it follows from (i) that there exists a projection a_{n+1} such that $a\sim a_{n+1}\lesssim b^{\perp}$. Then $\{a_1, a_2, \ldots, a_{n+1}\}$ are mutually orthogonal and each a_i is equivalent to a. Thus we obtain the result by induction.

LEMMA 3.6.[31] *In a semi-finite AW*-algebra, there exists a finite projection u_0 such that $e(u_0)=1$.*

PROOF. By Zorn's lemma, there exists a maximal set of finite projections $\{u_\alpha;\ \alpha\in I\}$ such that $e(u_\alpha)e(u_\beta)=0$ $(\alpha\neq\beta)$. The projection $u_0=\bigvee(u_\alpha;\ \alpha\in I)$ is an asserted one.

DEFINITION 3.7.[32] By a *subunit projection* in a semi-finite AW*-algebra, it is meant a finite projection u_0 such that $e(u_0)=1$.

LEMMA 3.7.[33] *Let A be a semi-finite and properly infinite AW*-algebra, and let u_0 be any subunit projection in A. There exists a non-zero central projection z which is equal to the least upper bound of mutually orthogonal projections each of which is equivalent to zu_0.*

PROOF. Let $\{u_\alpha;\ \alpha\in I\}$ be a maximal set of mutually orthogonal projections each of which is equivalent to u_0 (Zorn's lemma), and put $u=\bigvee(u_\alpha;\ \alpha\in I)$. It is not valid $u_0\lesssim u^{\perp}$, since otherwise we could enlarge $\{u_\alpha;\ \alpha\in I\}$. It follows from the Decomposition Theorem that there exists a non-zero central projection z such that $zu^{\perp}\prec zu_0$. Then we have $z=\bigvee(zu_\alpha;\ \alpha\in I)+zu^{\perp}$. Assume that the index set I is well ordered, then $zu_\gamma\sim zu_{\gamma+1}$ $(\gamma\geqq 1)$, and $\bigvee(zu_\gamma;\ \gamma\geqq 1)\sim\bigvee(zu_\gamma;\ \gamma>1)$, since I is an infinite set by Lemma 3.5. (ii). Accordingly we have $\bigvee(zu_\gamma;\ \gamma\geqq 1)\gtrsim\bigvee(zu_\gamma;\ \gamma>1)+zu^{\perp}\sim z$. It is obvious that $z\gtrsim\bigvee(zu_\gamma;\ \gamma\geqq 1)$, and it follows that $z\sim\bigvee(zu_\gamma;\ \gamma\geqq 1)$. Now let x be the partially isometric element defining the equivalence of z and $\bigvee(zu_\gamma;\ \gamma\geqq 1)$, and put $v_\gamma=xzu_\gamma x^*$ $(\gamma\geqq 1)$. Then we have $v_\gamma\sim zu_\gamma\sim zu_0$ $(\gamma\geqq 1)$ and $z=\bigvee(v_\gamma;\ \gamma\geqq 1)$.

DEFINITION 3.8.[34] Let u_0 be a subunit projection in a semi-finite AW*-algebra. If a projection a is the least upper bound of an infinite set of orthogo-

31) As to a similar lemma in W*-algebras, cf. Dixmier [2], Lemma 2.4.
32) Cf. Maeda [2], 91.
33) As to the case u_0 is an abelian projection, cf. Kaplansky [1], Lemma 4.8.
34) Cf. Griffin [1], Definition 1.5.

nal projections $\{v_\alpha ; \alpha \in I\}$, each of which is equivalent to $e(a)u_0$, and if the cardinal of the index set I is $\mathfrak{n}$, then a is said to be *of type* $S_\mathfrak{n}$ relative to u_0.

LEMMA 3.8. *Let A be a semi-finite and properly infinite AW*-algebra, u_0 be its subunit projection, and let π be the set of infinite cardinals not greater than that of A. There exists a set of central projections $\{z_\mathfrak{n} ; \mathfrak{n} \in \pi\}$ such that:*

(i) $\vee (z_\mathfrak{n} ; \mathfrak{n} \in \pi) = 1$,

(ii) *Either* $z_\mathfrak{n} = 0$ or $z_\mathfrak{n}$ *is a projection of type* $S_\mathfrak{n}$ *relative to* u_0,

(iii) *If z is a central projection of type $S_\mathfrak{n}$ relative to u_0, then it holds $z \leqq z_\mathfrak{n}$.*

PROOF. If there exists no central projection of type $S_\mathfrak{n}$ relative to u_0, then put $z_\mathfrak{n} = 0$. Otherwise, there exists (by Zorn's lemma) a maximal set of orthogonal central projections $\{z_\alpha ; \alpha \in I\}$ such that each z_α is of type $S_\mathfrak{n}$ relative to u_0. Put $z_\mathfrak{n} = \vee (z_\alpha ; \alpha \in I)$, then $z_\mathfrak{n}$ is clearly of type $S_\mathfrak{n}$ relative to u_0.

It is an immediate consequence of Lemma 3.7 that $\vee (z_\mathfrak{n} ; \mathfrak{n} \in \pi) = 1$.

Suppose that z is a central projection of type $S_\mathfrak{n}$ relative to u_0 and $z \not\leqq z_\mathfrak{n}$. Then we have $z - z_\mathfrak{n} z \neq 0$, which is of type $S_\mathfrak{n}$ relative to u_0 and orthogonal to $z_\mathfrak{n}$, contrary to the maximality of $\{z_\alpha ; \alpha \in I\}$. Thus we have $z \leqq z_\mathfrak{n}$. This completes the proof.

Now let us consider AW*-algebras satisfying the following condition, which is suggested by Griffin [1]:

(C) *If a projection is of types $S_\mathfrak{m}$ and $S_\mathfrak{n}$ relative to subunit projections u_1 and u_2 respectively, then it holds $\mathfrak{m} = \mathfrak{n}$.*

Remark that any W*-algebra satisfies the condition (C) by Griffin [1], Theorem 2.

DEFINITION 3.9. An AW*-algebra satisfying the condition (C) is called *of type* $S_\mathfrak{n}$ if its unit element is of type $S_\mathfrak{n}$ relative to any (or every by virtue of the condition (C)) subunit projection.

THEOREM 3.3.[35] *Let A be a semi-finite and properly infinite AW*-algebra, then A is a direct sum of AW*-algebra of type $S_\mathfrak{n}$ ($\mathfrak{n} \in \pi$), π being a set of infinite cardinals not grater than that of A.*

PROOF. In view of Lemma 3.8, it is sufficient to prove that the central projections $\{z_\mathfrak{n} ; \mathfrak{n} \in \pi\}$ in the lemma are mutually orthogonal.

Suppose that $z_\mathfrak{m} z_\mathfrak{n} \neq 0$ for some $\mathfrak{m}$, $\mathfrak{n}$ with $\mathfrak{m} \neq \mathfrak{n}$, then $z_\mathfrak{m} z_\mathfrak{n}$ is of types $S_\mathfrak{m}$ and $S_\mathfrak{n}$, which contradits the condition (C). This completes the proof.

We obtain the following immediately from Theorems 3.2 and 3.3:

35) As to a similar theorem in W*-algebras, cf. Griffin [1], Theorem 3.

COROLLARY 3.7. *Let A be an AW^*-algebra satisfying the condition* (C). *Then A is a direct sum of a finite algebra, a purely infinite algebra and algebras of type $S_{\mathfrak{n}}$ $(\mathfrak{n}\in\pi)$, π being a set as in Theorem* 3.3.

§ 4. Ideals and neutral ideals in AW*-algebras.[36)]

In this section, we shall investigate the interrelation of ideals in an AW*-algebra and the sets of projections contained in the ideals.

DEFINITION 4.1. *Left, right,* and *closed ideals* in an AW*-algebra are defined as usual. By a *self-adjoint ideal,* we mean an ideal I such that $I=I^*$, i. e., $x\in I$ implies $x^*\in I$.

LEMMA 4.1. *A linear subset I of an AW^*-algebra is a left ideal if and only if $x\in I$ impies $wx\in I$ for every unitary element w.*

PROOF. This is an immediate consequence of the fact that every element is a linear combination of unitary elements, which is easily shown by a similar argument as in Dixmier [1], Theorem 10.

LEMMA 4.2.[37)] *Let x be any element of an AW^*-algebra, then there exists a unique decomposition of x snch that $x=ua$, where a is a positive element and u is a partially isometric element such that u^*u is the right projection of a.*

PROOF. (i) Put $(x^*x)^{\frac{1}{2}}=a$, then a is a positve element. Let e be the right projection of a, then e is also the right projection of x by Corollay 3.1. By a similar argument as in Kaplansky [1], 244, we can see that there exists a sequence $\{e_n\}$ of orthogonal projections commutative with a such that $e=\bigvee_{n=1}^{\infty}e_n$ and that if we put $a_n=ae_n(n=1, 2, \ldots)$, then there exists a positive element a'_n with properties that

$$a\,a'_n=a'_n\,a=e_n \;\cdots\cdots\; (1),\text{ and}$$

$$e_n\,a'_n=a'_n\,e_n=a'_n \;\cdots\cdots\; (2),$$

for every $n=1, 2, \ldots$.

(ii) Put
$$u_n=x\,a'_n \;\cdots\cdots\; (3),$$

then it follows from (1) that $u_n^*u_n=e_n$ $(n=1, 2, \ldots)$. If we put $u_nu_n^*=f_n$, then f_n is a projection by (2), and we can show that $f_nf_m=0$ $(n\neq m)$, indeed, we have

36) Cf. Wright [1]. An analogous investigation in W*-algebras as in this section has been performed by Dixmier [3], 16-23. *Added in proof*: During the preparation of the paper, the following was not available to the author: Wright, A reduction for algebras of finite type, Ann. of Math. **60** (1954), 560-570.

37) It is well known that a similar lemma is valid in W*-algebras, and is called the *canonical decomposition*, cf. Murray and J. v. Neumann [1], Definition 4.4.1 and Lemma 4.4.1.

$f_n f_m = u_n u_n^* u_m u_m^* = x a'_n a'_n x^* x a'_m a'_m x^*$ by (3) and $a'_n x^* x a'_m = a'_n a^2 a'_m = e_n e_m = 0$, by (1).

Let $f = \bigvee_{n=1}^{\infty} f_n$, then it follows from Kaplansky [2], Lemma 20 that there exists a partially isometric element u such that $u^* u = e$, $u u^* = f$, $u^* f_n = u_n^*$, and

$$u e_n = u_n \quad \cdots\cdots \ (4).$$

(iii) Now we shall prove $x = ua$. We can see $uae_n = xe_n$ for every $n = 1, 2, \ldots$, because the left member of the equation is $u_n a$ by (4), the right member being equal to $x a'_n a = u_n a$ by (1) and (3). It follows $uae = xe$ and accordingly $ua = x$.[38]

(iv) Now suppose that $x = vb$, $b \geqq 0$, and $v^* v$ is the right projection of b. Then we have $x^* x = b^2$, and accordingly $b = a$.

Since $x = va$, it follows $x a'_n = v a a'_n$, the left hand member of which is equal to $u_n = u e_n$ by (3) and (4), the right hand member being $v e_n$ by (2). It follows that $v e_n = u e_n$ $(n = 1, 2, \ldots)$, whence $ve = ue$ and accordingly $v = u$.[38] This completes the proof.

As an immediate consequence of Lemma 4.2, we obtain the following :

LEMMA 4.3.[39] *A left (right) ideal in an AW*-algebra is two-sided if and only if it is self-adjoint.*

DEFINITION 4.2. A subset J of projections in an AW*-algebra A is called a *neutral ideal* provided that

(i) $a \in J$ and $b \in J$ imply $a \cup b \in J$,

(ii) $b \in J$, $a \in A_p$, and $a \leqq b$ imply $a \in J$, and

(iii) $b \in J$ and $a \sim b$ imply $a \in J$.

REMARK 4.1. A subset of a relatively complemented lattice with 0 is called a neutral ideal provided that (i) $a, b \in J$ imply $a \cup b \in J$, (ii) $a \leqq b$ and $b \in J$ imply $a \in J$, and (iii) if a is perspective to b and $b \in J$ then $a \in J$.[40]

It follows from Remark 3.4 that any neutral ideal in an AW*-algebra A in Definition 4.2 is a neutral ideal in the lattice A_p in the preceding sense, but the converse is not always true except for a finite AW*-algebra A. However, by a neutral ideal in an AW*-algebra, we shall mean a subset J in Definition 4.2 throughout the remainder of this paper, unless otherwise stated.

It follows immediately from Lemma 3.4.

38) Note that elements a, x, u, v have one and the same right projection e.

39) Cf. Dixmier [3], Lemma 3.3.

40) Cf. Birkhoff [1], 125, Theorem 11, and Maeda [3], 32, Definition 4.5, and Remark 4.4.

LEMMA 4.4.[41] *A subset J of projections in an AW^*-algebra satisfying the conditions* (i) *and* (ii) *in Definition* 4.2 *is a neutral ideal if and only if it satisfies the following condition:*

(iii′) $a \in J$ *implies* $w^*aw \in J$ *for every unitary element* w.

LEMMA 4.5.[42] *Let I be any two-sided ideal in an AW^*-algebra, then the set $J(I)$ of all projections in I is a neutral ideal.*

PROOF. If a is a projection such that $a \leqq b \in J(I)$, then a is clearly in $J(I)$, because $a = ab$. It is obvious that $a \in J(I)$ implies $w^*aw \in J(I)$ for every unitary element w. Suppose that $a, b \in J(I)$ and put $c = (a \cup b) - a$, then it follows from Remark 3.3 that $c \sim b - (a \cap b) \leqq b$, accordingly $c \in J(I)$ and $a \cup b = a + c \in J(I)$. Thus $J(I)$ is a neutral ideal.

LEMMA 4.6.[43] *Let J be any neutral ideal in an AW^*-algebra. Then the set $I(J)$ of all elements whose carriers belong to J is the two-sided ideal generated by J.*

PROOF. It is sufficient to prove that $I(J)$ is a two-sided ideal. Suppose that $x, y \in I(J)$, then their carriers $c(x)$, $c(y)$ belong to J, and accordingly $c(x) \cup c(y) \in J$. It follows from Corollary 3.2 that

$$\{c(x) \cup c(y)\}(\lambda x + \mu y)\{c(x) \cup c(y)\} = \lambda x + \mu y,$$

for any complex numbers λ, μ, whence $c(\lambda x + \mu y) \leqq c(x) \cup c(y)$. Thus $\lambda x + \mu y \in I(J)$. Next let w be any unitary element, then it holds

$$\{c(x) \cup wc(x)w^*\}\, wx\, \{c(x) \cup wc(x)w^*\} = wx,$$

whence $wx \in I(J)$, and similarly $xw \in I(J)$. It follows from Lemma 4.1 that $I(J)$ is a two-sided ideal.

LEMMA 4.7.[44] *Let J be any neutral ideal in an AW^*-algebra A. Then it holds $J = J(I(J)) = J(\overline{I(J)})$, $\overline{I(J)}$ being the closure of $I(J)$.*

PROOF. Since it is obvious that $J \subseteqq J(I(J)) \subseteqq J(\overline{I(J)})$, we shall show that $J(\overline{I(J)}) \subseteqq J$. Suppose that a is any projection in $\overline{I(J)}$. Then there exists a sequence $\{x_n\}$ such that $x_n \in I(J)$ $(n = 1, 2, \dots)$ and $x_n \to a$ $(n \to \infty)$. Since $ax_na \to a$ $(n \to \infty)$, we may assume without loss of generality that $ax_n = x_na = x_n$, and $\|a - x_n\| < 1$ for every n. Put $y = a + \sum_{n=1}^{\infty}(a - x_1)^n$, then we can easily show that $y \in A$ and $x_1 y = a$.[45] It follows from $x_1 \in I(J)$ that $a \in I(J)$, and accordingly

41) Cf. Dixmier [3], Lemma 3.1.
42) Cf. loc. cit., Lemma 3.7, (a).
43) Cf. loc. cit., Lemma 3.7, (b).
44) Cf. Dixmier [3], Lemma 3.8, (b).
45) Cf. Gelfand [1], Hilfssatz 1.

$a=c(a)\in J$, completing the proof.

DEFINITION 4.3[46] Let I be any two-sided ideal in an AW*-algebra. The ideal generated by its projections, i. e., $I(J(I))$ is called *restricted ideal of* I. An ideal is called *restricted* if it is identical with its restricted ideal.

LEMMA 4.8.[47] *Let I be any two-sided ideal in an AW^*-algebra, then it holds* $\overline{I(J(I))}\supseteqq I\supseteqq I(J(I))$, *where* $\overline{I(J(I))}$ *is the closure of* $I(J(I))$.

In particular, if I is closed, then $I=\overline{I(J(I))}$; *and if I is restricted, then* $I=I(J(I))$.

PROOF. Since it is obvious that $I\supseteqq I(J(I))$, we shall prove $\overline{I(J(I))}\supseteqq I$.

Suppose that $x\in I$. In view of Lemma 4.3, we may assume without loss of generality that x is self-adjoint. By Kaplansky [6], Lemma 2.1, there exists a sequence $\{e_n\}$ of projections such that $\|x-xe_n\|<\frac{1}{n}$, $xe_n=e_nx$, and $xx'_n=e_n$ for some x'_n $(n=1, 2, \ldots)$. Then $x\in I$ implies $e_n\in J(I)$, $(n=1, 2, \ldots)$. Since $e_nxe_n=xe_ne_n=xe_n$, it follows $xe_n\in I(J(I))$ for every n. Consequently we have $x\in\overline{I(J(I))}$, because $xe_n\rightarrow x$ $(n\rightarrow\infty)$. Thus $I\subseteqq\overline{I(J(I))}$.

If I is closed, it follows immediately from the preceding result that $I=\overline{I(J(I))}$. The remaining part of the lemma is trivial from Definition 4.3.

As an immediate consequence of Lemmas 4.7 and 4.8, we obtain the following

THEOREM 4.1. *A two-sided ideal I in an AW^*-algebra contains its restricted ideal and it is contained in the closure of its restricted ideal. These three ideals have the same neutral ideal in common.*

THEOREM 4.2.[48] *Both the sets of all closed and of all restricted two-sided ideals in an AW^*-algebra A are upper continuous lattices, ordered by set inclusion, isomorphic to the lattice of all neutral ideals.*

PROOF. First we shall show that the lattice of neutral ideals is upper continuous. The set of all projections in A is clearly a neutral ideal. Let $\{J_\alpha\ ;\ \alpha\in I\}$ be a set of neutral ideals, then the intersection $\Pi(J_\alpha\ ;\ \alpha\in I)$ is also a neutral ideal. If $\{J_\delta\ ;\ \delta\in D\}$ is a directed monotone increasing system of neutral ideals, then the union $\sum(J_\delta\ ;\ \delta\in D)$ is a neutral ideal, too. Consequently the set of all neutral ideals forms an upper continuous lattice.[49]

46) Cf. Dixmier [3], Definition 3.3.

47) Cf. loc. cit., Lemma 3.8, (a), where the half of Lemma 4.8 above i. e., $I\supseteqq I(J(I))$ has been shown in W*-algebras.

48) Cf. Dixmier [3], Lemma 3.9, (a), where the existence of the one-to-one correspondence between the set of neutral ideals and the of all restricted ideals has been shown in W*-algebras. Cf. also Wright [1].

49) Cf. Maeda [3], 12, Lemma 1.14.

Now let J be any neutral ideal, then $I(J)$ is a restricted ideal, then $\overline{I(J)}$ is a closed ideal, by Lemma 4.6. Conversely I is a closed or restricted ideal, then it follows from Lemm 4.5 that $J(I)$ is a neutral ideal. The mapping $I\rightarrow J(I)$ is a one-to-one correspondence of the set of all closed (or restricted) ideals onto the set of all neutral ideals, by Lemmas 4.7 and 4.8, and clearly preserves the order relation. This completes the proof.

COROLLARY 4.1. *The upper continuous lattice of all two-sided ideals in an AW^*-algebra, ordered by set inclusion, is homomorphic to the lattice of all neutral ideals.*

PROOF. The set of all ideals forms clearly an upper continuous lattice. Let I_1, I_2 be any two-sided ideals. Then $J(I_1)$ and $J(I_2)$ are neutral ideals and obviously $J(I_1\frown I_2)=J(I_1)\frown J(I_2)$, and $J(I_1\smile I_2)\geqq J(I_1)\smile J(I_2)$. It follows from Theorem 4.2 Lemma 4.7 that

$$\begin{aligned} J(I_1\smile I_2) &\leqq J(\overline{I(J(I_1))}\smile \overline{I(J(I_2))}) \\ &= J(\overline{I(J(I_1))})\smile J(\overline{I(J(I_2))}) \\ &= J(I_1)\smile J(I_2). \end{aligned}$$

Thus it holds $J(I_1\smile I_2)=J(I_1)\smile J(I_2)$. Consequently $I\rightarrow J(I)$ is a homomorphic mapping of the lattice of all two-sided ideals on the lattice of all neutral ideals.

COROLLARY 4.2. *Closed (or restricted) two-sided ideals in an AW^*-factor form a chain.*

PROOF. In view of Theorem 4.2, it is sufficient to prove that the set of all neutral ideals forms a chain, i. e., it holds either $J_1\leqq J_2$ or $J_2\leqq J_1$ for any neutral ideals J_1, J_2. Suppose that J_2 is not a part of J_1, then there exists a projection a_2 such that $a_2\in J_2$ and $a_2\notin J_1$. It follows from Corollary 3.4, that we have either $a_1\lesssim a_2$ or $a_2\lesssim a_1$ for any $a_1\in J_1$. Assume $a_2\lesssim a_1$, then it holds $a_2\in J_1$, which is a contradiction. Thus we have $a_1\lesssim a_2$, for any $a_1\in J_1$, consequently $J_1\leqq J_2$, completing the proof.

REMARK 4.2. The following simple examples may clarify the situation of Theorem 4.1 and Corollary 4.2.

Let $\mathfrak{A}$ be the AW*-factor of all bounded operators in a separable Hilbert space $\mathfrak{H}$. The set $J(\aleph_0)$ of all finite dimensional projections forms the unique proper neutral ideal. It follows that proper closed (or restricted) two-sided ideal in $\mathfrak{A}$ is also unique. Indeed, they are respectively the ideal $\mathcal{T}$ of all completely continuous operators and the ideal $\mathcal{F}$ of all finite dimensional operators; and any proper two-sided ideal (for example, the ideal $\mathcal{J}$ of all operators of Hilbert-Schmidt types) contains $\mathscr{F}$ and is contained in $\mathscr{T}$, all of these having $J(\aleph_0)$ in common.[50]

If generally the dimension of $\mathfrak{H}$ is $\aleph_\alpha$, α being an ordinal number; any proper neutral ideal is the set $J(\aleph)$ of all projections whose ranges have dimensions less than $\aleph$, an infinite cardinal less than $\aleph_\alpha$. It follows that the set of all closed (resp. restricted) two-sided ideals in $\mathfrak{A}$ forms a well ordered set, and any non-zero two-sided ideal I contains a restricted ideal $I(J(\aleph))$, and is contained in the closed ideal $\overline{I(J(\aleph))}$, for some infinite cardinal $\aleph$ not greater than $\aleph_\alpha$, uniquely determined by I.

COROLLARY 4.3. *Let A be an AW*-algebra with center Z. There exists a one-to-one correspondence between all maximal two-sided ideals and all maximal neutral ideals. Especially, maximal two-sided ideals in Z correspond one-to-one to maximal ideals in the Boolean algebra Z_p.*

PROOF. First we shall show that any maximal two-sided ideal I in A is necessarily closed. We have $J(I) < A_p$, since $J(I) = A_p$ would imply $I(J(I)) = A$, and accordingly $I = A$ by Theorem 4.1, contrary to $I < A$. Then it follows from Theorems 4.1 and 4.2 that $I \leqq \overline{I(J(I))} < \overline{I(A_p)} = A$, consequently $I = \overline{I(J(I))}$ from the maximality of I.

Then the first half of the corollary is obvious from Theorem 4.2.

Since the center Z is a finite AW*-algebra, it follows from Remark 4.1 that any neutral ideal in Z is a lattice theoretic neutral ideal, which is trivially an ideal in the lattice Z_p, since Z_p is a Boolean algebra. Thus the remainder part of the corollary is obvious from the preceding result.

REMARK. 4.3. Let Z be the center of an AW*-algebra. Then it follows from Corollary 4.3 that the representative Boolean space of the Boolean algebra Z_p is homeomorphic to the representative Gelfand space of the center Z.

REMARK 4.4. It follows from Theorem 4.2 that any finite AW*-algebra is strongly semi-simple in the sense that the intersection of all maximal two-sided ideals is the null ideal; indeed, it corresponds to the intersection of all maximal neutral ideals in the continuous geometry A_p, which is known to be the null ideal.[51]

It follows from the preceding result that any finite AW*-algebra A is isomorphic to a subdirect product of a simple B*-algebra A/I ($I \in \Omega$), where Ω is the set of all maximal two-sided ideals in A.

50) Cf. Calkin [1], Theorem 1.4 and 1.7.

51) Cf. Maeda [3], 109, Remark 3.1,

An infinite AW*-algebra A is not always strongly semi-simple, because, for example, $\mathcal{I}$ in Remark 4.2 is the unique maximal two-sided ideal in $\mathfrak{A}$ and is not a null ideal. However, A is also isomorphic to a subdirect product of B*-algebras. Indeed, let Ω be the set of all maximal ideals, say $\mathfrak{p}$, in the Boolean algebra Z_p, and let $I(\mathfrak{p})$ be the least closed two-sided ideal in A containing $\mathfrak{p}$, then A is isomorphic to a subdirect product of B*-algebras $A/I(\mathfrak{p})$, $(\mathfrak{p}\in\Omega)$.

§ 5. Dimension functions of AW*-algebras.

Kaplansky [1] has shown that there exists a unique normalized dimension function in any *finite* AW*-algebra. In this section, we shall extend this result to an *arbitrary* AW*-algebra.[52]

DEFINITION 5.1. Let A be an AW*-algebra with center Z, and Ω be the representative Boolean space for the Boolean algebra Z_p,[53] and let $\mathbf{Z}$ be the set of all non-negative, finite or infinite valued continuous functions on Ω.[54] To every central projection z there corresponds an open compact set $E(z)$ in Ω, the characteristic function of which will be denoted by z itself in the sequel.

By a *dimension function* of A, it is meant a mapping $d(a)$ of A_p into $\mathbf{Z}$ satisfying the following conditions:[55]

(1°) *If a is a finite projection, then $d(a)<\infty$ except on a non-dense set of Ω.*

(2°) *$d(a)=0$ implies $a=0$.*

(3°) *$a\sim b$ implies $d(a)=d(b)$.*

(4°) *If z is a central projeciton, then $d(za)=zd(a)$.*

(5°) *If a, b are orthogonal projections, then $d(a+b)=d(a)+d(b)$.*

REMARK. 5.1. Any continuous geometry has a dimension function $d(a)$ satisfying the conditions (1°)–(5°) above, uniquely determined provided $d(1)=1$.

52) As to the dimension functions in W*-algebras, cf. Dixmier [3], Theorems 1 and 2, and Segal [1], Theorem 1. Dimension functions of semi-finite W*-algebras are identical with the restrictions to projections of "pseudo-application ♮ normal fidlèle et essentielle" in the terminology of Dixmier [3], to which we are deeply indebted in writing this section.

53) In view of Remark 4.3, Ω may be considered as the representative Gelfand space for the center Z.

54) $\mathbf{Z}$ is a complete lattice, cf. Stone [1], Theorem 18, Dixmier [4], 25. If $f_\alpha\in\mathbf{Z}$, $(\alpha\in I)$, we shall denote by $\sum(f_\alpha\,;\,\alpha\in I)$ the L. U. B. of the directed set of sums of finite subsets of $\{f_\alpha\,;\,\alpha\in I\}$.

55) The complete additivity of a dimension function is a consequence of these conditions, cf. Corollary 5.3, below. As to this fact, I was suggested by Mr. S. Maeda.

Note also that $d(a)$ *is completely additive* and *if* $d(a) \leqq d(b)$ *then* a *is perspective to a part of* b.[56]

LEMMA 5.1. *Let* $d(a)$ *be a dimension function of an AW*-algebra, then it holds* $d(a) = \infty \cdot e(a)$ *if and only if* a *is properly infinite.*

PROOF. Let a be a properly infinite projection, then there exist orthogonal projections a_1, a_2 such that $a = a_1 + a_2$, and $a \sim a_1 \sim a_2$. It follows from (3°) and (5°) in Definition 5.1 that $d(a) = 2d(a)$, and accordingly $d(a, \mathfrak{p}) = \infty$[57] or 0 for any point $\mathfrak{p} \in \Omega$. Since $d(a) = e(a)d(a)$ by (4°), we have $d(a, \mathfrak{p}) = 0$ for any $\mathfrak{p} \notin E(e(a))$.[58] It follows from (2°) and the continuity of $d(a)$ that $d(a, \mathfrak{p}) = \infty$ for every $\mathfrak{p} \in E(e(a))$, for otherwise there exists a non-zero central projection $z \leqq e(a)$ such that $zd(a) = 0$, whence it holds $za = 0$ by (2°) and (4°), accordingly $0 = e(za) = ze(a) = z$, contrary to $z \neq 0$.

Conversely suppose that $d(a) = \infty \cdot e(a)$. It follows from Remark 3.8 that there exists a central projection z such that az is finite and $az^{\perp}$ is properly infinite. Assume $az \neq 0$, then $e(az) \neq 0$ and we have $e(az)d(a) = \infty \cdot e(az)$ contrary to (1°). Thus $a = az^{\perp}$ and consequently a is properly infinite.

COROLLARY 5.1. *Any purely infinite AW*-algebra has a unique dimension function* $d(a) = \infty \cdot e(a)$ *for every projection* a.

PROOF. It follows from Lemma 3.2 that $d(a) = \infty \cdot e(a)$ is a dimension function.

Conversely suppose that $d(a)$ is a dimension function, then we have $d(a) = \infty \cdot e(a)$ by Lemma 5.1, since every non-zero projection is properly infinite. This completes the proof.

It follows from Theorem 3.2, Corollary 5.1 and the result of Kaplansky [1] stated above, that in order to show the existence of a dimension function in an arbitrary AW*-algebra, it is sufficient to prove in case it is semi-finite and properly infinite.

We need the following lemmas.

LEMMA 5.2.[59] *Let* u_0 *be a projection with* $e(u_0) = 1$ *in an AW*-algebra, then for any projection* a, *there exists a set of orthogonal projections* $\{a_\alpha ; \alpha \in I)$ *such that* $a =$

56) The condition (1°) may be read "$d(a) < \infty$ for every element a", and "$a \sim b$" in (3°) must be read "a is perspective to b". Cf. Iwamura [1], and Maeda [3], 98, Theorem 1.4 and 1.5. As to the dimension funtion in a general continuous geometry, cf. Maeda [2], 90–92,

57) By $d(a, \mathfrak{p})$, we denote the value of $d(a)$ at the point $\mathfrak{p}$ of Ω.

58) By $E(e(a))$, we denote the open compact set in Ω corresponding to $e(a)$.

59) Cf. Dixmier [3], Lemma 3.10.

$\vee(a_\alpha\,;\,\alpha\in I)$ *and* $a_\alpha\precsim u_0$ $(\alpha\in I)$.

PROOF. There exists, by Zorn's lemma, a maximal set of orthogonal projections $\{a_\alpha\,;\,\alpha\in I\}$ such that $a_\alpha\leqq a$ and $a_\alpha\precsim u_0$ $(\alpha\in I)$. Clearly we have $a\geqq\vee(a_\alpha\,;\,\alpha\in I)$. By applying Lemma 3.3 to u_0 and $b=a-\vee(a_\alpha\,;\,\alpha\in I)$, it holds $e(u_0)e(b)=0$, since otherwise $\{a_\alpha\,;\,\alpha\in I\}$ would be enlarged. Since $e(u_0)=1$. it follows $e(b)=0$, and accordingly $b=0$. This completes the proof.

LEMMA 5.3. *Let A be a properly infinite AW^*-algebra.*

(i) *If b is a finite projection, and $c\precsim b\precsim d$, and $c\leqq d$, then there exists a projection b_1 such that $c\leqq b_1\leqq d$ and $b_1\sim b$.*

(ii) *If a, b are finite projections, then there exists a projection b_1 such that $b_1\perp a$, $b_1\sim b$ and $a\cup b\leqq a+b_1$.*

PROOF. (i) Since $c\precsim b\precsim d$, there exist projections b', c' such that $d\geqq b'\geqq c'\sim c$ and $b'\sim b$. As c is a finite projection, there exists a unitary element w in dAd such that $ww^*=w^*w=d$ and $w^*c'w=c$.[60] Put $b_1=w^*b'w$, then b is easily shown to be the asserted projection.

(ii) Put $c=(a\cup b)-a$, then we have $c\leqq a^\perp$ and $c\precsim b$, since $(a\cap b)-a\sim b-(a\cap b)$ by Remark 3.3. It follows from Lemma 3.5 (i) that $b\precsim a^\perp$. Hence there exists a projection b_1 such that $c\leqq b_1\leqq a^\perp$ and $b_1\sim b$. The projection b_1 is the asserted one.

LEMMA 5.4.[61] *Let a be a finite projection in a properly infinite AW^*-algebra. The set J of all projections which are contained in some projection $a_1+a_2+\cdots+a_n$ $(n=1,2,\ldots)$, where each a_i is equivalent to a and mutually orthogonal, is the neutral ideal generated by a, which will be denoted by $J(a)$ in the sequel.*

PROOF. It is sufficient to show that J is a neutral ideal. Suppose $b_1, b_2\in J$, then there exist sets of mutually orthogonal projections $\{a_1, a_2, \ldots, a_n\}$, $\{a_1', a_2', \ldots, a_m'\}$, each of which is equivalent to a and $b_1\leqq a_1+a_2+\cdots+a_n$, and $b_2\leqq a_1'+a_2'+\cdots+a_m'$. By Lemma 5.3, there exist projections $c_1, c_2, \ldots, c_m$, each of which is equivalent to a and

$$b_1\cup b_2\leqq(a_1+a_2+\cdots+a_n)\cup(a_1'+a_2'+\cdots+a_m')$$
$$\leqq a_1+a_2+\cdots+a_n+c_1+c_2+\cdots+c_m,$$

where $\{a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_m\}$ are mutually orthogonal. Consequently we have $b_1\cup b_2\in J$. The conditions (ii) in Definition 4.2 and (iii$'$) in Lemma 4.5 are obvious. This completes the proof.

60) Cf. the proof of Lemma 3.4.
61) Cf. Dixmier [3], Lemma 3.9 (b).

LEMMA 5.5. *Let a be a properly infinite projection, and u_0 be a finite projection in an AW*-algebra. For every $n=1, 2, \dots,$ there exist orthogonal projections $a_1, a_2, \dots, a_n$ such that $a_i \sim e(a)u_0$ $(i=1, 2, \dots, n)$ and $a \geqq a_1+a_2+\cdots+a_n$.*

PROOF. From Lemma 3.5, (ii), it is sufficient to prove that there exists a projection a_0 such that $a_0 \sim u_0 e(a)$ and $a_0 \in aAa$.

Apply the Decomposition Theorem to a and $u_0 e(a)$, then there exists a central projection z such that

$$a z \gtrsim u_0\, e(a)\, z \quad \ldots\ldots (1), \text{ and}$$

$$a z^{\perp} \lesssim u_0\, e(a)\, z^{\perp} \quad \ldots\ldots (2).$$

Since a is properly infinite, and $u_0 e(a) z^{\perp}$ is finite, it follows from (2) that $az^{\perp}=0$, i. e., $a \leqq z$, and accordingly $e(a) \leqq z$. From (1), we have $a \geqq az \sim u_0 e(a)$, whence there exists a projection a_0 such that $a \geqq a_0 \sim u_0 e(a)$. This completes the proof.

LEMMA 5.6. *Let A be a properly infinite AW*-algebra. If a, u_0 are finite non-zero projections with $a \leqq e(u_0)$, then there exists a central non-zero projection z such that $0 \neq az \in J(u_0)$.*

PROOF Let $\{u_1, u_2, \dots, u_n, \dots\}$ be a sequence of orthogonal projections equivalent to u_0 (Lemma 3.5 (ii)), and let $u^n = u_1+u_2+\cdots+u_n$. Apply the Decomposition Theorem to a and u^n, and there exists a central projection z_n such that $az_n^{\perp} \lesssim u^n z_n^{\perp}$ and

$$a\, z_n \gtrsim u^n\, z_n \quad \ldots\ldots (1).$$

Then we can show $az_n^{\perp} \neq 0$ for some n. For otherwise we have $a \leqq z_n$ $(n= 1, 2, \dots)$, accordingly $e(a) \leqq z_n$ $(n=1, 2, \dots)$. It follows from (1) that $e(a)az_n \gtrsim u_n z_n e(a)$, whence $a \gtrsim u_n e(a)$ for every n, contrary to the finiteness of a.[62] It follows that $0 \neq az_n^{\perp} \lesssim u^n z_n^{\perp} \leqq u^n \in J(u_0)$ for some n. This completes the proof.

LEMMA 5.7.[63] *Let u_0 be a fixed subunit projection in a semi-finite and properly infinite AW*-algebra A. For every projection a in $J(u_0)$, the neutral ideal generated by u_0, there exists a continuous function $\varphi(a)$ in* **Z** *satisfying the conditions in Definition* 5.1.

PROOF. (i) Let u be any subunit projection in A, then it follows from Lemma 1.6 that the set of projections contained in u forms a continuous geometry. Hence in view of Remark 5.1, there exists for every projection a $(\leqq u)$,

62) By a similar argument as in Kaplansky [1], Lemma 6.4, there exists a sequence of orthogonal projections $\{v_n\}$ such that $a \geqq v_1+v_2+\cdots+v_n+\cdots$, where each v_n is equivalent to $u_0 e(a)$, contrary to the finiteness of a. Cf. also Maeda [3], 86, Theorem 4.3.

63) Cf. Dixmma [3], Lemma 4.12.

a finite valued continuous function $\delta_u(a)$, which may be assumed to be defined on the Boolean space Ω in Definition 5.1, from Theorems 2.1 and 3.1. Consequently we do assume $\delta_u(a)\in \mathbf{Z}$ and $\delta_u(u)=1$ for every subunit projection u and any projection a $(\lesssim u)$ in the sequel.

(ii) For every $a\in J(u_0)$, there exist from Lemma 5.4, orthogonal projections $u_1, u_2, \dots, u_n$ such that $a\lesssim u_1+u_2+\cdots+u_n$ and $u_i\sim u_0$ $(i=1, 2, \dots, n)$. Put $u=u_1+u_2+\cdots+u_n$, and $\varphi(a)=n\delta_u(a)$. We shall prove that $\varphi(a)$ is independent of u and n.

Suppose $a\lesssim v_1+v_2+\cdots+v_m=v$, where $v_1, v_2, \dots, v_m$ are orthogonal projections equivalent to u_0. It follows from Lemma 3.5 (ii) that there exist projections $v_{m+1}, v_{m+2}, \dots, v_{mn}$, equivalent to u_0 such that $v_1, v_2, \dots, v_{mn}$ are mutually orthogonal. Similarly there exist projections $u_{n+1}, u_{n+2}, \cdots, u_{mn}$, equivalent to u_0 such that $u_1, u_2, \dots, u_{mn}$ are mutually orthogonal. Put $\bar{u}=u_1+u_2+\cdots+u_{mn}$, and $\bar{v}=v_1+v_2+\cdots+v_{mn}$. Then it holds $\bar{u}\sim\bar{v}$ in the finite AW*-algebra $(\bar{u}\cup\bar{v})A(\bar{u}\cup\bar{v})$, and accordingly

$$\delta_{\bar{u}\cup\bar{v}}(\bar{u}) = \delta_{\bar{u}\cup\bar{v}}(\bar{v}) \quad \dots\dots \quad (1).$$

On the other hand, we have [64]

$$\delta_u(a) = \frac{\delta_{\bar{u}}(a)}{\delta_{\bar{u}}(u)} = m\,\delta_{\bar{u}}(a) = m\,\frac{\delta_{\bar{u}\cup\bar{v}}(a)}{\delta_{\bar{u}\cup\bar{v}}(\bar{u})},$$

and similarly

$$\delta_v(a) = n\,\frac{\delta_{\bar{u}\cup\bar{v}}(a)}{\delta_{\bar{u}\cup\bar{u}}(\bar{v})},$$

whence $n\delta_u(a)=m\delta_v(a)$. Thus $\varphi(a)$ is independent of u and n.

(iii) Then $\varphi(a)$ is clearly an element of $\mathbf{Z}$, and it is not difficult to show that $\varphi(a)$ satisfies the conditions (1°)–(5°) in Definition 5.1, in view of Remark 5.1. Remark that $\varphi(a)$ is finite for every $a\in J(u_0)$, and $a\lesssim b$ if $\varphi(a)\leqq\varphi(b)$, and also that φ is completely additive.

LEMMA 5.8.[65] *Let A be a semi-finite and properly infinite AW*-algebra. For any projection a in A, put*

$$d(a) = \sup\{\varphi(b)\,;\, a\geqq b\in J(u_0)\},$$

where $\varphi(b)$ is the same as in Lemma 5.7. Then $d(a)$ is a dimension function of A.

PROOF. (i) We can easily show that $d(a)$ satisfies the conditions (2°)–(4°)

64) Cf. Maeda [3], 114, Remark 3.4.

65) Cf. Dixmier [3], Lemma 4.6.

in Definition 5.1. Note that $d(a)=\varphi(a)$ for every $a\in J(u_0)$.

(ii) We shall show the condition (1°): *For any finite projection* a, $d(a)<\infty$ *except on a non-dense set of* Ω.

If otherwise, there exists an open set E of Ω, on which $d(a)=\infty$. The closure of E is an open compact set of Ω, to which corresponds a central projection z_0, and $d(az_0)=\infty\cdot z_0$, accordingly it holds $d(az)=\infty\cdot z$ for every central projection $z\leqq z_0$. Now since az_0 is finite, it follows from Lemma 5.6 that thereexists a central projection z_1 such that $0\neq az_0z_1\in J(u_0)$, whence $d(az_0z_1)=\varphi(az_0z_1)$is finite, which is a contradiction.

(iii) Here we shall remark that $d(a)=\infty\cdot e(a)$ provided that a is properly infinite. Indeed, if a is properly infinite, it follows from Lemma 5.5 that $d(a)\geqq ne(a)$ for every $n=1, 2, \dots$, and accordingly $d(a)=\infty\cdot e(a)$.

(iv) Now it will be convenient to verify the following proposition:

If $\{a_\alpha\,;\,\alpha\in I\}$ *is a set of mutually orthogonal projections such that* $a=\vee(a_\alpha\,;\,\alpha\in I)$ *and* $a_\alpha\precsim u_0$ $(\alpha\in I)$, *then it holds* $d(a)=\sum(\varphi(a_\alpha)\,;\,\alpha\in I)$.

PROOF. Let D be the directed set of all finite subsets ν of I, and let $a^{(\nu)}=\vee(a_\alpha\,;\,\alpha\in\nu)$, then clearly $a^{(\nu)}\uparrow a$, and $\sum(\varphi(a_\alpha)\,;\,\alpha\in I)=\sup(\varphi(a^{(\nu)})\,;\,\nu\in D)$, Now it is sufficient to consider the following two cases from Remark 3.8.

Case 1. *a is finite.*

Since $a^{(\nu)}\leqq a$ and $a^{(\nu)}\in J(u_0)$, it follows $\sum(\varphi(a_\alpha)\,;\,\alpha\in I)\leqq d(a)$, whence it suffices to prove the converse inequality. Suppose $b\leqq a$ and $b\in J(u_0)$. Since a is finite, it follows from Lemma 1.6 that $a^{(\nu)}\uparrow a$ implies $a^{(\nu)}\cap b\uparrow a\cap b=b$, accordingly $\varphi(b)=\sup(\varphi(a^{(\nu)}\cap b)\,;\,\nu\in D)$. As it holds $\varphi(a^{(\nu)}\cap b)\leqq\varphi(a^{(\nu)})$, we have $\varphi(a)\leqq\sum(\varphi(a_\alpha)\,;\,\alpha\in I)$. Thus it is valid $d(a)\leqq\sum(\varphi(a_\alpha)\,;\,\alpha\in I)$, and consequently $d(a)=\sum(\varphi(a_\alpha)\,;\,\alpha\in I)$.

Case 2. *a is properly infinite.*

From (iii), it is sufficient to show that $\sup(\varphi(a^{(\nu)})\,;\,\nu\in D)=\infty\cdot e(a)$. Suppose the contrary, then it follows from the continuity of $\sup(\varphi(a^{(\nu)})\,;\,\nu\in D)$ on Ω that there exists a non-zero central projection $z\leqq e(a)$ such that

$$z\sup(\varphi(a^{(\nu)})\,;\,\nu\in D)\leqq nz$$

for some positive integer n. Accordingly it holds $\varphi(za^{(\nu)})\leqq\varphi(u^n z)$ for every $\nu\in D$, where $u^n=u_1+u_2+\cdots+u_n$; $u_1, u_2, \cdots, u_n$ being mutually orthogonal and equivalent to u_0. Thus we have [66] $za^{(\nu)}\precsim u^n z$ for every $\nu\in D$, and accordingly

66) Cf. the remark at the end of the proof of Lemma 5.7.

it holds $za \precsim u^n z$.[67] Since $u^n z$ is finite, it follows that za is finite, contrary to the assumption.

(v) Now we are in a position to prove the condition (5°), but we shall show more generally the complete additivity of $d(a)$ for later use.

Let $\{a_\alpha ; \alpha \in I\}$ be a set of mutually orthogonal projections, and put $a = \vee(a_\alpha ; \alpha \in I)$. From Lemma 5.2, there exist mutually orthogonal projections $\{a_\alpha^\kappa ; \kappa \in K_\alpha\}$ $(\alpha \in I)$ such that $a_\alpha = \vee(a_\alpha^\kappa ; \kappa \in K_\alpha)$ and each $a_\alpha^\kappa \in J(u_0)$. Since $a = \vee(a_\alpha^\kappa ; \kappa \in K_\alpha, \alpha \in I)$, it follows from (iv) that $d(a) = \sum(\varphi(a_\alpha^\kappa) ; \kappa \in K_\alpha, \alpha \in I) = \sum(d(a_\alpha) ; \alpha \in I)$. This completes the proof.

From the remark following Corollary 5.1, we obtain

THEOREM 5.1, *Any AW^*-algebra has a dimension function.*

Next we shal discuss the uniqueness of dimension functions of an AW*-algebra.

LEMMA 5.9.[68] *Let $d(a)$ be a dimension function of a semi-finite AW^*-algebra A, and let u_0 be a subunit projection in A. Then $d(u_0) > 0$ except on a non-dense set of Ω.*

PROOF. Suppose the contrary, then from the continuity of $d(a)$ we have $d(u_0) = 0$ on an open compact set, to which a non-zero central projection z corresponds and it holds $zd(u_0) = d(zu_0) = 0$, accordingly $zu_0 = 0$, whence $e(u_0) \leqq z^\perp$ contrary to $e(u_0) = 1$.

THEOREM 5.2. *Let both $d_1(a)$ and $d_2(a)$ be dimension functions of an AW^*-algebra A, and let u_0 be a subunit projection in the semi-finite part of A. If $d_1(u_0) = d_2(u_0)$, then $d_1(a) = d_2(a)$ for every projection a in A.*

PROOF. Put $\delta_i(b) = \dfrac{d_i(b)}{d_i(u_0)}$ $(i = 1, 2)$ for any projection $b \leqq u_0$, then both $\delta_1(b)$ and $\delta_2(b)$ are the normalized dimension functions of the continuous geometry $A_p(0, u_0)$, which consists of all projections in A contained in u_0 (Lemma 1.6). It follows the uniqueness of the normalized dimension function of a continuous geometry (Remark 5.1) that we have $\delta_1(b) = \delta_2(b)$, and accordingly $d_1(b) = d_2(b)$ for every $b \leqq u_0$. In view of the condition (3°) in Definition 5.1, we have $d_1(c) = d_2(c)$ for every $c \precsim u_0$.

Let a be any projection in A, then by Remark 3.8 there exists a central projection z such that az is finite and $az^\perp$ is properly infinite. In view of Lemma 5.1, we have $d_1(az^\perp) = d_2(az^\perp) = \infty \cdot e(az^\perp)$. Hence we may assume without loss of generality that a is a finite projection. It follows from Lemma

67) Cf. Kaplansky [1], Lemma 6.4, which is valid in any AW*-algebra provided the projection f in the lemma is finite.

68) Cf. Dixmier [3], Lemma 4.16 (a).

5.2 that there exist orthogonal projections $\{a_\alpha\,;\,\alpha\in I\}$ such that $a=\vee(a_\alpha\,;\,\alpha\in I)$ and $a_\alpha\lesssim u_0$ $(\alpha\in I)$. By a similar argument as above, the restrictions of d_1, d_2 to $A_p(0,a)$ are dimension functions of the continuous geometry $A_p(0,a)$. And it follows from Remark 5.1. that $d_1(a)=\sum(d_1(a_\alpha)\,;\,\alpha\in I)$ and $d_2(a)=\sum(d_2(a_\alpha)\,;\,\alpha\in I)$. Since $a_\alpha\lesssim u_0(\alpha\in I)$, it follows $d_1(a_\alpha)=d_2(a_\alpha)$ $(\alpha\in I)$, and consequently $d_1(a)=d_2(a)$. This completes the proof.

COROLLARY 5.2.[69] *Let $d_0(a)$ be any fixed dimension function of a semi-finite AW*-algebra A, and let $\mathfrak{F}$ be the set of all functions in* **Z** *such that $0<f(\mathfrak{p})<\infty$ except on a non-dense set of Ω. Then $d(a)$ is a dimension function of A if and only if $d(a)=fd_0(a)$ for some $f\in\mathfrak{F}$.*

PROOF. The "if" part of the corollary is obvious. So we assume that $d(a)$ is any dimension function of A, and u_0 is a subunit projection in A, then it follows from Lemma 5.9 and the condition (1°) in Definition 5.1 that $0<d(u_0)<\infty$ and $0<d_0(u_0)<\infty$ except on a non-dense set. Put $f=\dfrac{d(u_0)}{d_0(u_0)}$, then we have $f\in\mathfrak{F}$ and accordingly $d'(a)=fd_0(a)$ is a dimension function of A. Since $d'(u_0)=d(u_0)$, it follows from Theorem 5.2 that $d(a)=fd_0(a)$, completing the proof.

COROLLARY 5.3 *Let $d(a)$ be a dimension function of an AW*-algebra A. If $\{a_\alpha\,;\,\alpha\in I\}$ is a set of mutually orthogonal projections, then ue have*

$$d(\vee(a_\alpha\,;\,\alpha\in I))=\sum(d(a_\alpha)\,;\,\alpha\in I).$$

PROOF. If A is finite or purely infinite, the result is obvious. Thus we may assume that A is semi-finite and properly infinite. Let d_0 be the dimension function in Lemma 5.8, then we have shown in (v) of the proof of the lemma that $d_0(\vee(a_\alpha\,;\,\alpha\in I))=\sum(d_0(a_\alpha)\,;\,\alpha\in I)$, and consequently the result is obvious from Corollary 5.2.

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