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Remarks on Periodic Solutions of van der Pol's Equation

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After having written the paper:

M. Urabe, H. Yanagiwara, and Y. Shinohara, *Periodic solutions of van der Pol's equation with damping coefficients* $\lambda=2\sim10$, J. Sci. Hiroshima Univ., Ser. A. **23** (1960) (325–366),

the writer was informed by L. Cesari of a paper:

A. A. Дородницын, *Асимтотическое решение уравнения ван-дер-Поля*, Прикл. Мат. Мех., **11** (1947), 313–328 (A. A. Dorodnicyn, *Asymptotic solution of van der Pol's equation*, Prikl. Mat. Meh., Amer. Math. Soc. Translation no. 88).

In this interesting paper, A. A. Dorodnicyn gives asymptotic expressions for the amplitude a and the period ω of the periodic solution of van der Pol's equation

$$\frac{d^2x}{dt^2}-\lambda(1-x^2)\frac{dx}{dt}+x=0 \quad (\lambda>0) \tag{1}$$

for large values of the parameter λ. The formulas for a and ω read:

$$a=2+\frac{\alpha}{3}\lambda^{-4/3}-\frac{16}{27}\frac{\log\lambda}{\lambda^2}+\frac{1}{9}(3b_0-1+2\log 2-8\log 3)\frac{1}{\lambda^2} + O(\lambda^{-8/3}), \tag{2}$$

$$\omega=(3-2\log 2)\lambda+3\alpha\lambda^{-1/3}-\frac{22}{9}\frac{\log\lambda}{\lambda} +\left(3\log 2-\log 3-\frac{1}{6}+b_0-2d\right)\frac{1}{\lambda}+O(\lambda^{-4/3}), \tag{3}$$

where α, b_0 and d are the transcendental constants which are determined independently from the above formulas so that

$$\alpha=2.338107, \qquad b_0=0.1723, \qquad d=0.4889.$$

M. Cartwright in her paper of 1952:

Van der Pol's equation for relaxation oscillations, Contributions to the theory of non-linear oscillations, II, Annals Math. Studies 29, Princeton (1952), 3–18, gives also the asymptotic expressions, but her formulas, though published five years later, contain only the first two terms of the expressions above.

In the above paper of Dorodnicyn, as an example, the amplitude a and the period ω of the periodic solution for $\lambda=10$ are computed by the above formulas.

The author observed minor errors in the manipulations leading to the formula (3) for the period. The formula for the amplitude is correct. The corrected formula for the period replacing (3) reads:

$$\omega=(3-2\log 2)\lambda+3\alpha\lambda^{-1/3}-\frac{1}{3}\,\frac{\log\lambda}{\lambda}+\left(3\log 2-\log 3-\frac{3}{2}+b_0-2d\right)\frac{1}{\lambda}+O(\lambda^{-4/3}). \tag{3'}$$

The numerical results which are mentioned below merge well with the ones of the formulas (2) and (3′).

Recently, Yanagiwara has gotten a periodic solution for $\lambda=20$ continuing the computation of the above paper of ours. The details of his results will be published separately.

In addition, recently, there has been published a paper:

W. S. Krogdahl, *Numerical solutions of van der Pol Equation*, ZAMP, **11** (1960), 59–63.

In this paper, the periodic solutions are computed for $\lambda=0(1)10$ using Gill's modification of the Runge-Kutta method in such a way that, starting from each trial point ($x(0)=a_0$, $\dot{x}(0)=0$), the orbit in the (x, $\dot{x}$)-plane meets again the

λ \ by	a: us	a: Krogdahl	a: Dorodnicyn's (2)	ω: us	ω: Krogdahl	ω: Dorodnicyn's (3)	ω: Dorodnicyn's (3′)
0	2.000	2.0000		6.283	6.2832		
1	2.009	2.0086		6.687	6.6633		
2	2.0199	2.0199		7.6310	7.6299		
3	2.0235	2.0233		8.8613	8.8595		
4	2.0231	2.0230		10.2072	10.2037		
5	2.0216	2.0215		11.6055	11.6124		
6	2.0199	2.0198		13.0550	13.0612		
7	——	2.0181		——	14.5380		
8	2.0169	2.0167		16.0740	16.0363		
9	——	2.0154		——	17.5501		
10	2.0145	2.0143	2.0138 (−0.0007)	19.1550	19.0763	18.831 (−0.324)	19.184 (+0.029 …0.15%)
20	2.0077	——	2.0077 (0.0000)	34.7103	——	34.4925 (−0.2178)	34.742 (+0.032 …0.09%)

(The numbers in brackets are the differences from the values computed by us. Though Krogdahl computed the values to five decimal places, they are rounded off to four decimal places for convenience of comparison.)

x-axis in an equal distance from the origin as the trial starting point. The results of this paper coincide well with those of ours in the amplitude, but, in the period, there are some small differences between both results.

Since the details of computation are not mentioned in this paper, the writer cannot decide whether the results of this paper should be preferable to those of ours or not. Much less the method of this paper needs far more numbers of steps of step-by-step numerical integration than that of ours. In this paper, the results of computation are compared with the values computed from Dorodnicyn's formula (2) and (3) (not (3')).

The table of the preceding page shows the values of the amplitude a and the period ω of the periodic solutions computed in various ways.

This table shows that the values computed by (2) and (3') are quite accurate even for $\lambda=10$. Consequently, for values of λ larger than 10, the more accurate values would be gotten by these two asymptotic expressions as is checked for $\lambda=20$. Then it may be unnecessary to continue the computation of periodic solutions for further values of λ over 10 if greater accuracy is not needed.

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