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Non-degenerate Divisors on an Algebraic Surface

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1. Introduction. The notion of non-degenerate divisors on an abelian variety was introduced first by Morikawa [2][1], then after Weil proved that if X is a non-digenerate divisor on an abelian variety, then the complete linear system $|mX|$ is ample[2] for a sufficiently large m [6]. The latter property of divisors can be transferred to any variety and we shall call, in this paper, a divisor X on an algebraic variety "*non-degenerate*", if $|mX|$ is ample and has no fixed component for a sufficiently large m. Now we ask how one can distinguish the class of non-degenerate divisors among the set of divisors. For this purpose we shall introduce the notion of (p)-*cycles* on a non-singular variety V^n. Let X be an r-dimensional cycle on V. If the Kronecker index $(X.Y)$ is positive for all positive cycles Y of codimension r, we shall call X a (p)-cycle. The main theorem in this paper asserts that *if V is a non-singular surface, any (p)-divisor is non-degenerate and conversely.* On the other hand a non-degenerate divisor is a (p)-divisor and it is seen easily that any positive (p)-divisor is non-degenerate on an abelian variety[3]. It would be interesting to investigate the relation between these two notions in general situation.

2. Summary of known results. **(A)** Let F be a complete non-singular surface and let X be a divisor on F. Let $\mathfrak{L}(X)$ be the sheaf of germs of rational functions f on F such that $(f)+X>O$[4]. Then the cohomology groups with coefficients in the sheaf $\mathfrak{L}(X)$ have respectively the following interpretations. The O-dimensional cohomology group $H^0(F, \mathfrak{L}(X))$ is the module $L(X)$ of functions f such that $(f)+X>O$ on F. Its dimension will be denoted by $l(X)$ as usual. The dimension of the linear system $|X|$ is defined by $\dim|X|=l(X)-1$. The dimension of 1-cohomology group $H^1(F, \mathfrak{L}(X))$ will be denoted in the following by $\omega(X)$ and will be called the *superabundance* of the linear system $|X|$. The linear system whose superabundance is zero will be called a *regular system*. The 2-nd cohomology group $H^2(F, \mathfrak{L}(X))$ is, by Serre's duality, canonically isomorphic to $H^0(F, \mathfrak{L}(K\text{-}X))$, where K is the canonical divisor on F. Hence its

1) The numbers in the bracket refer to the biliography at the end of the paper.
2) A linear system N is ample if the rational map associated with N gives a biregular embedding into a projective space.
3) Let X be a positive degenerate divisor on an abelian variety, then there exists an abelian subvariety B such that $X_t=X$ for $t\in B$. Then any curve C contained in B has the property that $(X.C)=0$. Hence X cannot be a (p)-divisor.
4) On the results of sheaf theory the readers are expected to refer to the papers [4] and [8].

dimension, denoted usually by $i(X)$, is equal to $l(K\text{-}X)$, and it is called the *speciality index* of X. Let $\mathcal{O}$ be the sheaf of local rings on F. The Euler characteristic of F with coefficients in the sheaf $\mathcal{O}$ will be denoted by $\chi(F)$, i. e. $\chi(F)=\sum_{q=0}^{2}(-1)^q \dim H^q(F, \mathcal{O}))$. The *arithmetic genus* $p(F)$ of F is related to $\chi(F)$ by the equation $1+p(F)=\chi(F)$. Let X be a divisor on F, then we shall put $\chi_F(X)=\chi(F)-\chi(F, \mathfrak{L}(-X))$. The arithmetic genus of X, which we shall denote by $\pi(X)$, is defined by $1-\pi(X)=\chi_F(X)$. If X is an irreducible curve on F, $\pi(X)$ coincides with the genus of X in the sense of Rosenlicht[5]. Now we can give the Theorem of Riemann-Roch in the following form.

$$l(X)=1+p(F)+1/2[(X^2)+(K.X)]+\omega(X)-i(X)$$

where $(X.Y)$ stands for the Kronecker index of X and Y.

(**B**) The following two theorems play essential roles in our investigations. For the proof the readers are asked to refer to the original ones.

THEOREM 1. *Let $|D|$ be an irreducible linear system without fixed component on a normal surface. Then the linear system $|iD|$ cannot have a simple base point for a sufficiently large i.*

This is proved by Zariski in [7].

THEOREM 2. *Let U be a complete normal variety, and let D be a positive divisor on U. Assume that the linear system $|D|$ has neither base points nor a fundamental curve on U, then for all n sufficiently large the linear system $|nD|$ is ample.*

The proof can be found, e.g., in Lang [1].

3. Proof of the theorem.

PROPOSITION 1. *Let F be a non-singular surface and let X be a divisor on F. Let D be an irreducible curve on F with the arithmetic genus π, and let $\mathfrak{F}$ be the quotient sheaf $\mathfrak{L}(X)/\mathfrak{L}(X-D)$. Then if $(X.D)>2\pi-2$, $H^1(F,\mathfrak{F})=0$.*

PROOF. Let Y be a divisor linearly equivalent to X such that Y does not contain any singular point of D. Let f be a function such that $(f)=Y-X$. Then the multiplication of f gives an isomorphism of $\mathfrak{L}(X)$ onto $\mathfrak{L}(Y)$ sending $\mathfrak{L}(X-D)$ onto $\mathfrak{L}(Y-D)$. Hence it is sufficient to prove the Proposition under the assumption that X does not contain any singular point of D. In this case the sheaf $\mathfrak{F}$ is canonically isomorphic to the sheaf $\mathfrak{L}(X.D)$ on D such that

$$\mathfrak{L}(X.D)_x=\begin{cases}\text{the set of functions } \bar{f} \text{ on } D \text{ such that } (\bar{f})+X.D>0, \text{ if } x \text{ is a simple point of } D.\\ \text{the set of functions on } D \text{ regular at } x, \text{ if } x \text{ is a singular point of } D.\end{cases}$$

5) For the proof, see [5].

The Theorem of R-R on a curve (with the singularities) asserts that the speciality index of the divisors $X.D$ on D vanishes[6], i.e.dim $H^1(D,\mathfrak{L}(X.D))=0$, if $(X.D)>2\pi-2$. Which proves the Proposition.

PROPOSITION 2. *Let F be a non-singular surface and let X be a positive divisor on F. Let $D_i(i=1,2,\cdots,s)$ be the distinct components of X. Then there exists an integer N satisfying the following condition: Let Y be any divisor such that $(Y.D_i)>N$ for $i=1,\cdots,s$. Then $H^1(F,\mathfrak{L}(Y)/\mathfrak{L}(Y-X))=0$.*

PROOF. Let $X=\sum_{i=1}^{s} n_i D_i$. We shall use the induction on $n=\sum_{i=1}^{s} n_i$. The case $n=1$ is proved in Prop. 1. Let $X_1=X-D_1$. Then we have an exact sequence of sheaves

$$0\rightarrow\mathfrak{L}(Y-D_1)/\mathfrak{L}(Y-X)\rightarrow\mathfrak{L}(Y)/\mathfrak{L}(Y-X)\rightarrow\mathfrak{L}(Y)/\mathfrak{L}(Y-D_1)\rightarrow 0.$$

Since $Y-X=(Y-D_1)-X_1$, we can use the induction assumption. Hence there exists an integer N_1 such that if $(Y-D_1).\ D_i>N_1$ for $i=1,\cdots,s$, then $H^1(F,\mathfrak{L}(Y-D_1)/\mathfrak{L}(Y-X))=0$. Hence if we put $N=\mathrm{Max}\ [N_1+(D_1.D_i),\ 2\pi(D_1)-2]$, the exact sequence

$$H^1(F,\mathfrak{L}(Y-D_1)/\mathfrak{L}(Y-X))\rightarrow H^1(F,\ \mathfrak{L}(Y)/\mathfrak{L}(Y-X))\rightarrow H^1(F,\mathfrak{L}(Y)/\mathfrak{L}(Y-D_1))$$

asserts

$$H^1(F,\mathfrak{L}(Y)/\mathfrak{L}(Y-X))=0.$$

DEFINITION 1. *Let V^n be a non-siugular variety, and let X be an r-dimensional cycle on V. If the Kronecker index $(X.\ Y)$ is positive for any positive cycle Y of dimension n-r, we shall say that X is an r-dimensional (p)-cycle.*

PROPOSITION 3. *Let X be an positive (p)-divisor on a non-singular surface F. Then the superabundance $\omega(tX)$ is a non-increasing sequence for a sufficiently large t.*

PROOF. Let us consider the exact sequence of sheaves

$$0\rightarrow\mathfrak{L}(tX)\rightarrow\mathfrak{L}((t+1)X)\rightarrow\mathfrak{L}((t+1)X)/\mathfrak{L}(tX)\rightarrow 0$$

From this we get the exact sequence of cohomology groups

$$H^1(F,\mathfrak{L}(tX))\rightarrow H^1(F,\mathfrak{L}(t+1)X))\rightarrow H^1(F,\mathfrak{L}((t+1)X)/\mathfrak{L}(tX))$$

Let C be any component of X, then $(t+1)\ (X.C)$ can be greater than any preassigned number for a large t, since X is a (p)-divisor. Then by Prop. 2 the third term vanishes if t is large. This proves that $\omega(tX)\geqq\omega((t+1)X)$ for large t.

COROLLARY 1. *$\omega(tX)$ is bounded for $t\geqq 0$ if X is a positive (p)-divisor.*

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6) Cf. Serre [5] and Rosenlicht [3].

COROLLARY 2. *Let X be a positive (p)-divisor and let D be a component of X, then $\omega(tX-D)$ is bounded for $t \geqq 0$.*

PROOF. Let $X-D=X_1$, then X_1 is a positive divisor. Since $(t+1)X-D=tX+X_1$, the similar arqument as in Prop. 3 shows that $\omega(tX) \geqq \omega(tX+X_1)=\omega((t+1)X-D)$ for large t. Then Cor. 2 follows from Cor. 1.

COROLLARY 3. *Let X be an irreducible curve on F such that $(X^2)>0$. Then $\omega(tX)$ is also bounded for $t \geqq 0$.*

THEOREM 3. *Let X be a (p)-divisor, then the linear system $|tX|$ exists for large t and has no fixed component for t large enough.*

PROOF. First we shall remark that $|tX|$ is non-special for large t. In fact assume that tX is special, then $l(K-tX) \geqq 1$, and there exist a positive divisor D linearly equivalent to $K-tX$. Hence $(X.K)-t(X^2)=(X.D)>0$. From this we have $t<(X.K)/(X^2)$. The first assertion follows immediately from R-R Theorem

$$l(tX) \geqq 1+p(F)+1/2[t^2(X.X)-t(K.X)]$$

and the assumption $(X.X)>0$.

To prove the second part it is sufficient to assume that X is a positive divisor. In this case tX is itself a member of the linear system $|tX|$. Hence if there exist a fixed component D it must be a component of the divisor X. Moreover we must have $l(tX)=l(tX-D)$. Applying the R-R theorem we have

$$l(tX)=1+p(F)+(1/2)[t^2(X.X)-t(K.X)]+\omega(tX)$$
$$l(tX-D)=1+p(F)+(1/2)[(tX-D)^2-(K.(tX-D))]+\omega(tX-D).$$

Hence we get

$$t(X.D)=\omega(tX-D)-\omega(tX)-\chi_F(D).$$

But this is impossible if t is large enough, because the right hand side is bounded by Prop. 3 for $t \geqq 0$ and the left hand side is unlimited since $(X.D)>0$.

Using Cor. 3 instead of Prop. 3, we can prove in a similar way

COROLLARY 1. *Let X be an irreducible curve such that $(X^2)>0$, then $|tX|$ has no fixed component for large t.*

COROLLARY 2. *Let X be a positive divisor and let C be a component of X such that $(C^2)>0$, then C cannot be a fixed component of $|tX|$ for a sufficiently large t.*

REMARK. If a divisor X is not a (p)-divisor, $|tX|$ may have a fixed component for any t even though $(X^2)>0$. In fact let Γ be a non-singular curve of genus $g(\geqq 3)$ and let $F=\Gamma \times \Gamma$. Let $\mathfrak{A}$ be a Γ-disisor of degree d and let Δ be the diagonal. Let us put $X=\Gamma \times \mathfrak{A}+\Delta$. Then

$$(X^2)=2(\Delta.(\Gamma\times\mathfrak{A}))+\Delta^2=2(d+1-g)$$
$$(X.\Delta)=(\Delta.(\Gamma\times\mathfrak{A}))+\Delta^2=d+2(1-g)$$

Hence if $d=g$, $X^2=2$, $X.\Delta=2-g<0$. Thus Δ must be a fixed component of $|tX|$ for any t.

Let C be an irreducible curve on F. We shall say that C is a *fundamental curve* for the linear system if C is mapped onto a point by a rational map associated with that linear system. The following proposition is seen immediately.

PROPOSITION 4. *Let C be an irreducible curve of F. Then C is a fundamental curve of the linear system N if, and only if, $(C.X)=0$ for a variable member X of the linear system N.*

From Th. 3 and Prop. 4 we can assert the following

THEOREM 4. *Let X be a (p)-divisor, then $|tX|$ has no fundamental curve for large t.*

Let N be a linear system on F. We shall say that N is *simple* if the rational map associated with N gives the birational correspondence of F into a projective space.

THEOREM 5. *Let X be a divisor on F such that $(X^2)>0$, and assume that $|tX|$ has no fixed component for a sufficiently large t. Then the linear system $|tX|$ is simple for t large enough. In particular $|tX|$ is an irreducible system.*

PROOF. Let π_t be the rational map associated with the linear system $|tX|$ and let V_t be the image of F under π_t. We shall show first the dim $V_t=2$, if t is large enough.

Assume that $\dim V_t=1$, and let $d=\dim |tX|$. Then V_t is a curve in a projective space P of d-dimension, and it is not contained in any hyperplane of P. Let n be the projective degree of V_t, then as is easily seen, $n\geqq d$. Let H be a generic hyperplane in P over the field of reference and we shall put $V_t.H=\sum_{i=1}^{n} P_i$. Let us put $D_i=\pi_t^{-1}(P_i)$, then $D_i\equiv D_1$ $(i=2,\cdots,n)$. Since $|tX|$ has no fixed component we have $tX\sim\sum_{i=1}^{n} D_i$. Taking the intersection product with X we get $t(X^2)=\sum_{i=1}^{n}(X.D_i)=n(X.D_1)$. Since $(X^2)>0$, $(X.D_1)$ is also positive. Thus we get the relation

$$1\leqq(X.D_1)=(t/n)(X^2)$$

On the other hand,

$$n\geqq d\geqq 1+p(F)+(1/2)[t^2(X^2)-t(K.X)]$$

Hence $\lim_{t\to\infty} (t/n)=0$ on account of $(X^2)>0$. Thus we have arrived at a contradiction.

Let t be an integer such that dim $V_t=2$ and let $\varphi_1,\cdots,\varphi_N$ be a base of $L(tX)$. Let k be a common field of definition for V over which the divisor X is rational. Then we can assume that the functions φ_i's are all defined over k. Let P be a generic point of V over k and let us put $K=k(P)$, and $K_0=k(\varphi_1(P),\cdots,\varphi_N(P))$. Then K is finite over K_0. Let A be the affine ring $k[\varphi_1(P),\cdots,\varphi_N(P)]$ and let $\bar{A}$ be the integral closure of A in K. Then we can find a finite number of elements $\xi_1,\cdots,\xi_s$ in $\bar{A}$ such that $K=K_0(\xi_1,\cdots,\xi_s)$. Let ψ_i be the functions on V defined over k by $\psi_i(P)=\xi_i$ $(i=1,\cdots,s)$. Then there exists an integer l such that $(\psi_i)+lX>0$ for $i=1,2,\cdots,s$. This means that for such l, the module of functions $L(lX)$ generates the entire function field K of V over k. Thus the rational map associated with the linear system $|lX|$ gives a birational correspondence of F. Hene the linear system $|tX|$ is, still more, simple for any $t\geqq l$.

DEFINITION 2. *Let V be a normal variety and let X be a divisor on V. We shall say that X is a non-degenerate divisor if the linear system $|mX|$ is ample and has no fixed component for a sufficiently large m.*

Combining the results in Theorems 1–5, we get the following.

THEOREM 9. *Let F be a non-singular surface and let X be a divisor on F. Then X is non-degenerate if, and only if, X is a (p)-divisor.*

"Only if" part of the theorem is a direct consequences of the theorem of Bézout in a projective space.

COROLLARY. *Let X be a (p)-divisor on a non-singular surface, then the linear system $|tX|$ is regular for a sufficiently large t,* i.e. *$\omega(tX)=0$ for a large t.*

This is a consequence of Th. 6 and the general theory on the sheaf on a projective variety (Cf. [4]).

Biobliography

[1] Lang, S. Introductions to algebraic geometry, Interscience tracts in pure and applied mathematics, No. 5.

[2] Morikawa, H. On abelian varieties. Nagoya Math. J., **6** (1953), pp. 151–170.

[3] Rosenlicht, M. Equivalence relations on algebraic curves, Ann. of Math., **56** (1952), pp. 169–191.

[4] Serre, J. P. Faisceaux algébriqués cohérents, Ann. of Math. **56** (1955), pp. 197–278.

[5] ———— Groupes algébriques et corps de classes, Act. Sci. et Ind. 1264, Hermann, Paris.

[6] Weil, A. On the projective embedding of Abelian varieties, Volume in honour of S. Lefschetz, Princeton, 1957.

[7] Zariski, O. Intérprétations Algébro-géométriqués du Quatorziéme Problème de Hilbert, Bull. Sci, Math. de France, 2 ser., **78** (1954), pp. 155–168.

[8] ———— Algebraic sheaf theory, Scientific Report on the second summer Institute, Part III, Bull. of Amer. Math. Soc., **62** (1956), pp. 117–141.